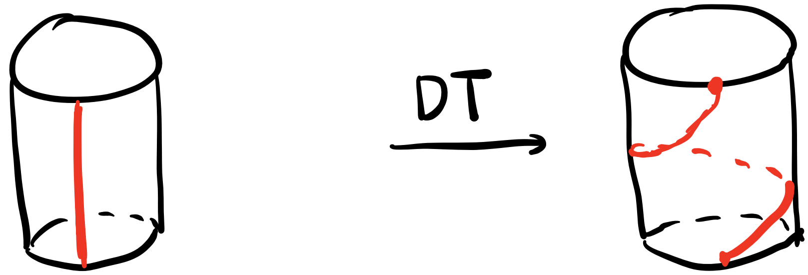


full monodromy for KP family

$\overline{\mathrm{Im}\,\pi_1} \supset Sp(H_1(X_y))$ $\qquad G = \mathrm{Aff}(q) \curvearrowright \mathbb{F}_q$

$q \geq 3$ $\quad (q=3,\ G=S_3)$

§1 Overview

§2 MCG, DT $\curvearrowright H^1$

§3 lifting DT, normal form of singly ram G-cover $\quad$ ($\star$)-cover

§1 Recall

$Y/\mathbb{C}$ the curve $g \geq 2$

$$Z \xrightarrow{\text{rel curve}} Y' \xrightarrow{\text{finite}} Y$$

fiber of y' is a Ram G-cover of Y branched at y

fiber of y $= \{\text{Ram } G\text{-cover of } Y \text{ branched at } y\}/\sim$

$\rightsquigarrow \quad X \xrightarrow{\text{rel AV}} Y' \longrightarrow Y \qquad \deg = q$

fiber of y' $\underset{\sim}{\text{isog}}$ $\mathrm{Prym}(Z'_{y'} := Z_{y'} \times_G \mathbb{F}_q \to Y)$

$$H_1(X_{y'}, \mathbb{Q}) \cong \mathrm{Ker}(H_1(Z'_{y'}) \longrightarrow H_1(Y))$$

always

So Full monodromy$/\mathbb{C}$ for $\pi_1(Y, y_0) \curvearrowright H^1(X_{y_0})$

$$= \bigoplus_{y' \to y_0} H^1(X_{y'})$$

is implied by

Thm 8.1 let Y be cpt (oriented) surface

$Z_1, \ldots, Z_N \to Y$ $\boxed{\text{deg } q \text{ singly ram } G\text{-cover at } y_0}$

$(*)$ - cover

Then Mon : $\pi_1(Y, y_0)_0 \to \prod_{i=1}^{N} Sp(H_1^{pr}(Z_i, Y))$

has Zar dense image

where • $H_1^{pr} = \ker((-)_* : H_1(Z) \to H_1(Y))$

$\qquad\qquad\qquad\qquad H_1(Z) = H_1(Z - Z_0), \quad H_1(Y) = H_1(Y - y_0)$

• $\pi_1(\ )_0 \leq \pi_1(\ )$ index $< +\infty$ (later)

Idea: (i) Sp almost simple gp WLOG $N=1$

$\mathrm{Im}\,\pi_1(\ )_0$ not in center : easy | Goursat's lem

extends Mon along $\pi_1(\ )_0 \triangleleft\ ?$

If $\overline{\mathrm{Im}\,?} = Sp$ then $\overline{\mathrm{Im}\,\pi_1(\ )_0} = Sp$

(ii) $? = (\text{mapping class gp of } Y - y_0)_0$

s.t. $? \curvearrowright (Z \to Y)$ hence on H^1_{pr}

Birman exact seq: if $\chi(Y) < 0$

$$0 \to \pi_1(Y, y_0) \to MCG(Y - y_0) \to MCG(Y) \to 0$$

enough elements in MCG

$e : S^1 \hookrightarrow Y - y_0$ simply closed curve

injection

$\rightsquigarrow DT_e : Y - y_0 \simeq Y - y_0$

$\curvearrowright H^1(Y - y_0) : x \mapsto x + (e, x)e$

(iii) lift DT_e^2 to $MCG(Y - y_0)_0 \hookrightarrow MCG(Z)$

for liftable curve e (lifting DT to Z is easy)

Show enough such e using normal form of ($\divideontimes$)-cover

$\rightsquigarrow \overline{\mathrm{Im}\ ?} = Sp$

Rek (iii) is a relative version of classical $MCG(\Sigma) \twoheadrightarrow Sp(H^1(\Sigma))$ for closed surface Σ

Rek Picard-Lefschetz

= "Dehn twist" of vanishing cycles computes local monodromy $\curvearrowright H^*$

42. S (connected, oriented) surface

$$MCG(S) := \pi_0(\mathrm{Hom}^+(S, \partial S))$$

so $\phi : S \simeq S$ $\quad \phi|_{\partial S} = \mathrm{id} \rightsquigarrow [\phi] \in MCG(S)$

$[\phi_0] = [\phi_1] \iff \exists\ \phi_t : S \simeq S \quad t \in [0,1]$

isotopy $\approx$ homotopy

$MCG(S) \curvearrowright H_1(S) \quad \curvearrowright$ simply closed curves in S

Ex. $\mathrm{MCG}($  $) = *$

$\mathrm{MCG}($  $) \cong Sp(H^1(T^2)) \simeq SL_2(\mathbb{Z})$

Dehn twist $\quad e : S^1 \hookrightarrow S$ simple

Obs $\quad A = S^1 \times [0,1] \qquad (\theta, t)$

twist $T : A \simeq A \qquad (\theta, t) \to (\theta + 2\pi t, t)$

$\exists \ N \subseteq S$ tubular neighborhood of e

$$\begin{array}{ccc} N & \simeq & A \\ \cup\!\vert & & \vert\!\cup \\ e & \simeq & \{(\theta, t_0)\} \end{array}$$

$$DT_e : S \xrightarrow{\sim} S = \begin{cases} \mathrm{id} & S \setminus N \\ T & N \end{cases}$$

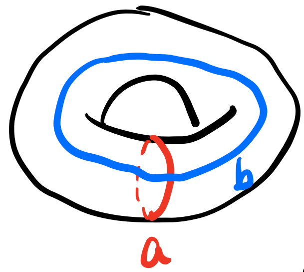
$DT_e(e) \cong e$

$DT_e(\{(0,t)\}) = \{(2\pi t, t)\} \cong e + \{(0,t)\}$

$DT_e \in \mathrm{MCG}(S)$ well-defined

Dehn twist for $[e] \in \pi_1(S)$

Ex



$DT_a : \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

$DT_b : \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$

Facts a, b (oriented) isotopy class of simple curves in S

- $DT_a = DT_b \iff a = b$
- $(a,b) = 0 \iff DT_a DT_b = DT_b DT_a$
- $(a,b) = 1 \implies DT_a DT_b DT_a = DT_b DT_a DT_b$

$\star$ • $DT_b(a) = a + (b,a)\, b$

same as [LV]
differ by a sign in [FM]

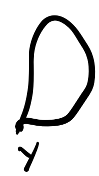
- (Birman) if $\chi(S) < 0$

$$0 \to \pi_1(S,y) \xrightarrow{\text{push}} MCG(S-y) \xrightarrow{\text{forget}} MCG(S) \to 0$$

$\qquad\qquad\qquad\qquad\qquad\qquad \psi \quad \phi_1$

$\phi_1 \to 0 \iff \exists\ \phi_t : S \simeq S \quad \phi_0 = id$

then $\phi_t(y)$ is a path



$\in \pi_1(S,y)$

$\leadsto \text{push}(\phi_t(y)) = \phi_1$.

- $\Psi : MCG(S) \twoheadrightarrow Sp(H_1(S,\mathbb{Z}))$ if S closed

pf: Dehn twist.

- $MCG(\Sigma_g) = \langle 2g+1 \text{ Dehn twists} \rangle$

§3. Return to Y

$$\{\text{finite cover } Z \to Y\} \simeq \{\text{finite set } F \text{ with } \pi_1(Y)\text{-action}\}$$

$$\# \pi^{-1}(y) = \# F$$
$$\# \pi_0(Z) = \# \pi_1\text{-orbit in } F$$

$$\mathrm{Mon} = \mathrm{Cov} : \pi_1 \to \mathrm{Aut}(F)$$

$$\begin{tikzcd} Z_1 \arrow[rr, "f"] \arrow[dr] & & Z_2 \arrow[dl] \\ & Y & \end{tikzcd} \quad \leftrightarrow \quad S_1 \xrightarrow{f} S_2 \text{ as } \pi_1\text{-sets}$$

so deg q G-cover of Y

$$:= \pi : Z \to Y \text{ cover}$$
$$\text{s.t} \begin{cases} \pi_1(Y, y) \curvearrowright \pi^{-1}(y) \simeq \mathbb{F}_q \\ \text{with } \mathrm{Im}(\mathrm{Cov}) = G \end{cases}$$

$(*)$-cover of $Y := \pi : Z \to Y$ finite

$$\text{s.t} \begin{cases} \pi : Z \setminus z_0 \to Y \setminus y_0 \text{ deg } q \ G\text{-cover} \\ \text{monodromy at } y_0 \text{ non-trivial} \end{cases}$$

$*$ $\boxed{y_0 \text{ fixed}}$

$MCG(Y - y_0) \curvearrowright \{(\star)\text{-cover of } Y\}$

$\quad \searrow Out(\pi_1) \curvearrowright$

$MCG(Y - y_0)_Z := Stab(\pi : Z \to Y)$

$$\alpha \in MCG(Y - y_0)_Z \Leftrightarrow \exists! \begin{tikzcd} Z \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, "\alpha"] & Y \end{tikzcd}$$

$\Rightarrow \alpha$ acts on $H_1(Z) = \pi^* H_1(Y) \oplus \underset{= \ker \pi_*}{H_1^{pr}}$

$\rightsquigarrow$ $Mon : MCG(Y - y_0)_Z \to Sp(H_1^{pr}(Z, Y))$

$= H_1^{pr}(Z \setminus z_0, Y \setminus y_0)$

$$MCG(Y - y_0)_0 := \bigcap_{i=1}^{N} MCG(Y - y_0)_{Z_i}$$

extends Mon action of $\pi_1(Y, y_0)_0 := MCG(Y - y_0)_0 \cap \pi_1(Y, y_0)$

Thm 8.1 for $Mon\big|_{\pi_1(Y, y_0)_0}$, $\overline{Im} = \prod_{i=1}^{N} Sp$

Idea: ① lift D_e^i to $Z \simeq Z$

for liftable curve $e : S^1 \hookrightarrow Y - y_0$

compute action on $H_1^{pr}(Z, Y)$

② normal form of $(\star)$-cover $\Rightarrow$ enough such e

② <u>Prop 8.5</u> $\forall$ $\begin{matrix} Z \\ \pi \downarrow \\ Y \end{matrix}$ ($*$)-cover

$\exists$ $Y = S_{g-1} \# T$ (T: torus; S_{g-1}: genus $= g-1$)

s.t

- $y_0 \in T^0$
- $\pi|_{S^0_{g-1}}$ split
- $\pi|_{T^0}$ has trivial monodromy at ∂T^0
- Under std basis β_1, β_2 for $\pi_1(T-y, *)$

$$\mathrm{Cov}(\beta_1) = \begin{pmatrix} *_1 & *_2 \\ & 1 \end{pmatrix} \quad \langle *_1 \rangle = \mathbb{F}_q^\times$$

$$\mathrm{Cov}(\beta_2) = \begin{pmatrix} 1 & * \neq 0 \\ & 1 \end{pmatrix}$$

Idea: $\mathrm{Cov}: \pi_1(Y - y_0) \twoheadrightarrow G$

$$\begin{tikzcd} \mathrm{Cov}^{ab} : H_1(Y) \arrow[r] \arrow[dr, dashed] & G^{ab} \\ & \mathbb{Z} \arrow[u, two heads] \end{tikzcd}$$

Poincare duality $\Rightarrow$

$\exists \ \alpha_1 : S^1 \hookrightarrow Y$ simple

$$\mathrm{Cov}^{ab} = (\alpha_1, -)$$

cut $Y^1 \longrightarrow Y - y_0$

s.t. $\forall \ \beta : S^1 \hookrightarrow Y$ simple

$$(\alpha_1, \beta) = 0$$

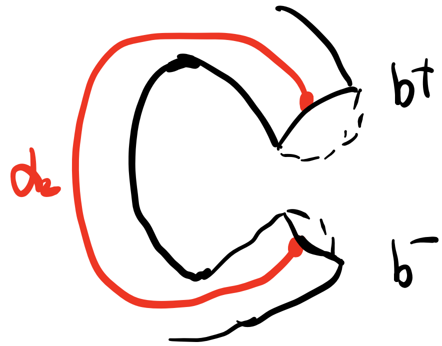
Then $\mathrm{Cov}|_{Y^1} : \pi_1 \longrightarrow [G, G] =: G_1$

do same thing for $G_1 \Rightarrow$ split over Y^2

$Y =$ 

Cut along $\alpha_1^+ \sqcup \alpha$

$Y^1 =$



$(\alpha^1)(b^+) = 1 \in \mathbb{F}_2^+$

cut along d_2

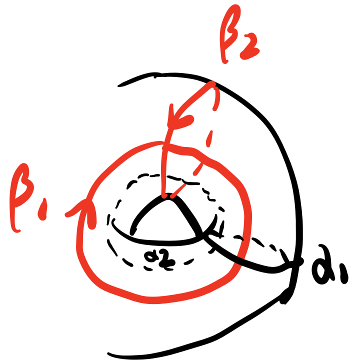
Poincaré duality

$d_2 \in H_1(Y', \partial Y', \mathbb{Z}) \times H_1(Y') \to \mathbb{Z}$

$Y^2 =$

d_2 can be made simple Lem 8.4

$S^0_{g-1} = (Y^2)^0$



$\beta_1 \cap d_2 = \phi$

$\beta_1 \cap d_1 = *$

$\beta_2 \cap d_1 = \phi$

$\beta_2 \cap d_2 = *$

$\square$

① $e : S^1 \hookrightarrow Y - y_0$

$$Cov(e) \in G \circlearrowright \mathbb{F}_q$$

e liftable if $Cov(e) = \begin{pmatrix} *_1 & *_2 \\ & 1 \end{pmatrix}$

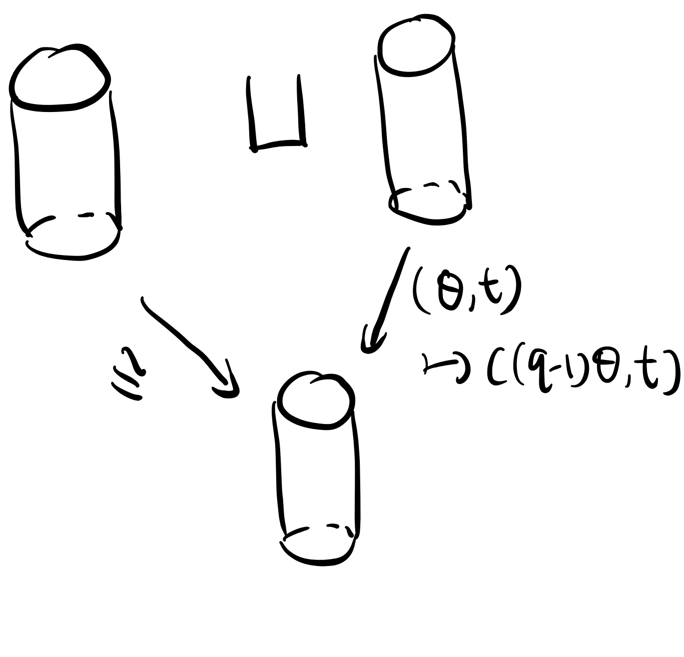
$$\langle *_1 \rangle = \mathbb{F}_q^\times$$

orbit of $Cov(e) \circlearrowright \mathbb{F}_q$ $=$ $\underbrace{\underbrace{\text{fix pt}}_{(\ )}}_{1}$ $\underbrace{\underbrace{\text{other pt}}_{(\ )}}_{q-1}$

$$\Rightarrow \quad \pi^{-1}(e) = e^+ \sqcup e^-$$

$$e^+ \xrightarrow{\cong} e, \qquad e^- \to e, \quad \theta \mapsto (q-1)\theta$$

$$\pi^* e = e^+ + e^-$$



So
$$\begin{tikzcd} DT_{e^+}^{q-1} DT_{e^-} : & Z \arrow[r, "\sim"] \arrow[d] & Z \arrow[d] \\ \text{lifts} \quad DT_e^{q-1} & Y \arrow[r, "\simeq"] & Y \end{tikzcd}$$

$$\rightsquigarrow \quad DT_e^{q-1} \in MCG(Y - y_0)_Z$$

Recall $\quad DT_{e^\pm} \curvearrowright H_1(Z)$

via $\quad x \mapsto x + (e^\pm, x)\, e^\pm$

so $\quad [DT_e^{q-1}] \curvearrowright H_1(Z)$

by
$$x \longmapsto x + (q-1)(e^+, x)e^+ + (e^-, x)e^-$$

projection $\sim : H_1(Z) \to \operatorname{Ker} \pi_* = H_1^{pr}$

$$\widetilde{e}^+ + \widetilde{e}^- = \widetilde{\pi^* e} = 0$$

$$\rightsquigarrow \quad [DT_e^{q-1}] \curvearrowright H_1^{pr}(Z, Y) \quad x \mapsto x + q(e^+, x)e^+$$

$\square$.

Rek, $\quad \pi^* : H_1(Y) \to H_1(Z)$

$$(\pi^* -, \pi^* -) = q(-,-)$$

$$\rightsquigarrow \quad H_1^{pr}(Z, Y) = (\pi^* H_1(Y))^\perp$$