Galois category and Riemann existence theorem

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Exodromy seminar

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Outline

Goal

- Define the Galois category of a scheme X (via stratified shape theory).
- $\mathrm{Gal}(X)$ can recover the étale homotopy type of X.
- (Riemann existence theorem) The analytic and algebraic version can be compared.

Galois 1-category of a scheme

X a coherent i.e qcqs scheme $\rightsquigarrow \mathrm{Gal}(X)$:

- Object x: geometric points $x \to X$.
- Morphism $x \to y$: étale specialization $y \rightsquigarrow x$ i.e a lift of y to the strict localization $X_{(x)} = \mathrm{Spec}(O^{sh}_{X,x_0}) \to X$.

Galois 1-category of a scheme

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X^{Zar} is a poset: $x_0 \leq y_0$ if and only if $x_0 \in \overline{\{y_0\}}$.
$\rightsquigarrow$ a functor $\mathrm{Gal}(X) \to X^{Zar} : x \mapsto x_0$, fiber $BG_{\kappa(x_0)}$ over x_0.
$\mathrm{Gal}(X)$ globalizes absolute Galois groups of points of X.

Profinite topology on Gal(X)

Gal(X) has a topology, like the profinite topology on $G_{\kappa(x_0)}$.
Idea: use finite level points $u \to X$.
An open basis of Gal(X): $y \rightsquigarrow x$ lying over a given specialization $v \rightsquigarrow u$.
Can be precise using pyknotic/condensed math.

Reconstruction

Theorem

Topological categorry Gal(X) can recover the étale homopotopy type of X (up to protruncation), hence $\pi_*^{et}(X, x)$.

Idea: Stratified profinite shape can recover the profinite shape by inverting all morphisms.

∞-topos theory

∞-category: ...
Topos: the category of sheaves on a site.
∞-topos: an ∞-category X satisfying ∞-Giraud's axiom.
Geometric morphism: a pair of adjoints $(f^*, f_*) : X \to Y$ s.t f^* is exact.
$\mathbf{S}$: the ∞-category of spaces (animas).
$\mathbf{Top}_\infty$: the ∞-category of ∞-topos.
the ∞-category $\mathrm{Pt}(X) := \mathrm{Fun}^*(\mathbf{S}, X_{et})$ of points of X: geometric morphisms $\mathbf{S} \to X$.
For us, let X_{et} be the ∞-topos of étale sheaves valued in $\mathbf{S}$ on the 1-site X^{et} of étale X-schemes. X_{et} is 1-localic.

π-finite spaces $\mathbf{S}_\pi$

In ∞-topos theory, the category of finite sets is replaced by the ∞-category of π-finite spaces $\mathbf{S}_\pi$.
A lisse object $F \in X$= a locally constant sheaf of π-finite spaces that can be trivialized on a finite cover $Y \to X$.
$X^{\text{lisse}} \subseteq X$: full subcategory of lisse objects, which is a bounded ∞-pretopos.
Constructible = lisse over a stratification of X.

Shape theory

Given an ∞-topos $X \in \mathbf{Top}_\infty$, Lurie constructed a pro-∞-groupoid $\Pi_\infty(X) \in \mathrm{Pro}(\mathbf{S})$ called the shape of X. If X is from a nice topological space, $\Pi_\infty(X)$ is the ∞-fundamental groupoid of X.

∞-Stone duality and profinite shapes

Stone duality: profinite sets = totally disconnected compact Hausdorff topological spaces.
∞-Stone duality: $\mathbf{S}_\pi^\wedge := \mathrm{Pro}(\mathbf{S}_\pi) \to \mathbf{Top}_\infty$ is fully faithful, with a left adjoint $\widehat{\Pi}_\infty : \mathbf{Top}_\infty \to \mathrm{Pro}(\mathbf{S}_\pi)$ (profinite shape).
Essential images are called Stone ∞-topoi.

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Construction of $\widehat{\Pi}_\infty$: a "profinite" completion.
For a π-finite space X, $X \simeq \widehat{\Pi}_\infty(X)$ e.g $\mathbb{RP}^\infty \simeq B(\mathbb{Z}/2)$.
By design, any quasi-equivalence $X \to Y$ is a shape-equivalence.

Étale homotopy type of a scheme

X is a locally noetherian scheme.
Artin–Mazur defined the étale homotopy type of $X \in \mathrm{Pro}(h_1\mathbf{S})$.
Friedlander refined it to étale topological type of $X \in \mathrm{Pro}(\mathbf{S})$.
$\widehat{\Pi}^{\mathrm{et}}_{\infty}(X)$:=the profinite étale topological type.
(Hoyois) $\widehat{\Pi}^{\mathrm{et}}_{\infty}(X) \simeq \widehat{\Pi}_{\infty}(X_{\mathrm{et}})$.

- $\widehat{\Pi}^{\mathrm{et}}_{\infty}(\mathrm{Spec}(k)) = BG_k$.
- $\widehat{\Pi}^{\mathrm{et}}_{\infty}(\mathbb{CP}^1) = (S^2)^{\wedge}_{\pi}$.

Monodromy

Bounded coherent ∞-topoi can be classified via ∞-pretopoi.
[SAG, Theorem E.2.3.2] For any ∞-topos X,
$\mathrm{Sh}_{\mathrm{eff}}(X^{\mathrm{lisse}}) \in \mathbf{Top}_{\infty}^{\mathrm{Stone}}$ (effective epimorphism topology) is called Stone reflection of X, $\mathrm{Sh}_{\mathrm{eff}}(X^{\mathrm{lisse}}) \leftrightarrow \widehat{\Pi}_{\infty}(X)$.
∞-Stone duality $\rightsquigarrow \mathrm{Fun}\left(\widehat{\Pi}_{\infty}(X), \mathbf{S}_{\pi}\right) \simeq X^{\mathrm{lisse}}$.
In particular for qcqs noetherian scheme X,

$$\mathrm{Fun}\left(\widehat{\Pi}_{\infty}^{\mathrm{et}}(X), \mathbf{S}_{\pi}\right) \simeq X_{et}^{\mathrm{lisse}}.$$

Next step: define a stratified version of $\widehat{\Pi}_{\infty}\left(X_{\mathrm{et}}\right)$.

π-finite stratified spaces $\mathbf{Str}_\pi$

P a finite poset.
A P-stratified space X= an ∞-category X with a conservative functor $X \to P$.
Hochster duality: profinite posets = spectral topological spaces.
$\rightsquigarrow$ S-stratified spaces for any spectral topological space S.
$\mathbf{Str}_\pi$= the ∞-category of π-finite stratified spaces.
S-stratified ∞-topos = an ∞-topos X equipped with a geometric morphism $X \to \mathrm{Sh}(S)$ to the ∞-topos of sheaves of spaces on S.

∞-Hochster duality

Theorem

$\mathrm{Pro}(\mathbf{Str}_\pi)_S \hookrightarrow \mathbf{StrTop}_{\infty,S}$ extending

$$[\Pi \to P] \mapsto [\mathrm{Fun}(\Pi, \mathbf{S}) \to \mathrm{Fun}(P, \mathbf{S})]$$

is fully faithful, with a left adjoint $\widehat{\Pi}^S_{(\infty,1)} : \mathbf{StrTop}_{\infty,S} \to \mathrm{Pro}(\mathbf{Str}_\pi)_S$ (profinite S-stratified shape).

Essential images are called spectral ∞-topoi.
Similar to Stone reflection, there is a spectralification functor

$$\mathbf{StrTop}_{\infty,S} \to \mathbf{StrTop}^{\mathrm{spec}}_{\infty,S} \overset{\widehat{\Pi}^S_{(\infty,1)}}{\simeq} \mathrm{Pro}(\mathbf{Str}_\pi)_S, X \mapsto \mathrm{Sh}_{\mathrm{eff}}(X^{\text{S-cons}}).$$

Exodromy

For any S-stratified ∞-topos X, adjunction gives a natural equivalence:

Exodromy

$$\mathrm{Fun}\left(\widehat{\Pi}^{S}_{(\infty,1)}(X), \mathbf{S}_{\pi}\right) \simeq X^{S-\mathrm{cons}}.$$

The ∞-category of representations of $\widehat{\Pi}^{S}_{(\infty,1)}(X)$ valued in π-finite spaces = S-constructible sheaves on X.

$\mathrm{Gal}(X)$

Return to the coherent scheme X, $S := X^{Zar}$, $\rightsquigarrow$ stratified ∞-topos $X^{et} \to X^{Zar}$. It's a spectral ∞-topos.
Profinite stratified étale homotopy type $\widehat{\Pi}^{et}_{(\infty,1)}(X) := \widehat{\Pi}^{X^{\mathrm{Zar}}}_{(\infty,1)}(X_{\mathrm{et}})$.

Theorem

$$\mathrm{Gal}(X) \simeq \widehat{\Pi}^{X^{\mathrm{Zar}}}_{(\infty,1)}(X_{\mathrm{et}})\,.$$

Corollary

$$\mathrm{Fun}\left(\mathrm{Gal}(X), \mathbf{S}_\pi\right) \simeq X^{\mathrm{cons}}_{\text{ét}}\,.$$

Reconstruction

Idea: A constructible sheaf $\mathcal{F}$ is lisse iff all specializations of $\mathcal{F}$ are isomorphisms.

Homotopy theorem

For any spectral S-stratified ∞-topos X, The profinite classifying space of $\widehat{\Pi}^S_{(\infty,1)}(X)$ is precisely $\widehat{\Pi}_\infty(X)$.

In paritcular, there is an equivalence $\theta_X : \widehat{\Pi}^{\text{ét}}_\infty(X) \to \varepsilon(\mathrm{Gal}(X))$. This finishes reconstruction theorem, let's see some examples.

An example

We use the language of spatial décollages.
$X = \mathbb{A}^1_{\mathbb{C}}$, $P = [0 \to \infty, 1 \to \infty]$, a stratification $X \to P$ given by $X(0) = Z_0 = \{0\}$, $X(1) = Z_1 = \{1\}$, $X(\infty) = U = \mathbb{A}^1_{\mathbb{C}} - \{0, 1\}$.
$\mathrm{Gal}^P(X) \to P$.

- $\mathrm{Gal}^P(X)(0) = \widehat{\Pi}_\infty(X(0)) = B\{*\}$.
  $\mathrm{Gal}^P(X)(1) = \widehat{\Pi}_\infty(X(1)) = B\{*\}$.
- $\mathrm{Gal}^P(X)(\infty) = \widehat{\Pi}_\infty(X(\infty)) = B\widehat{F(x_0, x_1)}$ the classifying groupoid for profinite completion of the free group of two variables.
- $\mathrm{Gal}^P(X)(0 \to \infty) = \widehat{\Pi}_\infty(X_{(x_0)} \backslash \{x_0\}) = B\widehat{\mathbb{Z}}$.
- $\mathrm{Gal}^P(X)(0) \leftarrow \mathrm{Gal}^P(X)(0 \to \infty) \to \mathrm{Gal}^P(X)(\infty)$.

Another example

Let (A, K, k) be a DVR, $S = \mathrm{Spec}A$, $s = \mathrm{Spec}k, \eta = \mathrm{Spec}K$. S_{et} is a naturally [1]-stratified spectral ∞-topos, with closed stratum s_{et} and open stratum η_{et}.
$s_{et} \overleftarrow{\times}_{S_{et}} S_{et} = S^h_{et}$. $s_{et} \overleftarrow{\times}_{S_{et}} \eta_{et} = \eta^h_{et}$.

Example

$$\widehat{\Pi}^{\text{ét}}_{\infty}(\eta) \simeq \mathrm{BG}_K, \widehat{\Pi}^{\text{ét}}_{\infty}\left(\eta^{\mathrm{h}}\right) \simeq \mathrm{BD}_A,$$

$$\widehat{\Pi}^{\text{ét}}_{\infty}\left(\eta^{\mathrm{sh}}\right) \simeq \mathrm{BI}_A, \widehat{\Pi}^{\text{ét}}_{\infty}\left(S^{\mathrm{h}}\right) \simeq \mathrm{BG}_k.$$

$BG_k \leftarrow BD_A \rightarrow BG_K$.

Yet another example

Let K be a number field, and write O_K be the ring of integers of K.
$\mathrm{Gal}(O_K)$ has objects (up to iso) the prime ideals of O_K.
The profinite stratified etale shape of $\mathrm{Spec} O_K$ is stratified by the various closed strata, each of which is an embedded profinite "circle" $BG_{k(\mathbf{p})} \cong \widehat{\mathbb{Z}}$ i.e a knot.
Enveloping each knot is a tubular neighborhood, given by $\mathrm{Gal}(\mathrm{Spec} O_{\mathbf{p}}^{sh})$. And the deleted tubular neighborhood is given by $BG_{K_{\mathbf{p}}}$.

Riemann existence theorem

X a finite type $\mathbb{C}$-scheme.
X^{an}= complex points of X with analytic topology.
SGA4 $\rightsquigarrow$ a geometric morphism of 1-localic ∞-topoi

$$\varepsilon_{X,*} : X_{\text{an}} \to X_{\text{et}}$$

s.t for any $f : X \to Y$, we have $f_*^{et}\varepsilon_{X,*} \simeq \varepsilon_{Y,*} f_*^{an}$.

Riemann Existence Theorem

$\varepsilon_{X,*}$ restricts to an equivalence $X_{\text{ét}}^{\text{lisse}} \simeq X_{\text{an}}^{\text{lisse}}$ between ∞-categories of lisse sheaves.

Equivalently, it induces an equivalence of profinite spaces

$$(X^{\text{an}})_{\pi}^{\wedge} = \widehat{\Pi}_{\infty}(X_{\text{an}}) \simeq \widehat{\Pi}_{\infty}(X_{\text{et}}).$$

Stratified version

Note $\varepsilon_{X,*} : X_{\text{an}} \to X_{\text{et}}$ is over $S = X^{Zar}$ i.e S-stratified, the pullback functor $\varepsilon^{X,*}$ restricts to a morphism of ∞-pretopoi:

$$\varepsilon^{X,*} : X_{\text{et}}^{S-\text{cons}} \to X_{\text{an}}^{S-\text{cons}}.$$

$(X/S)_{\text{an}} := \text{Sh}_{\text{eff}}\left(X_{\text{an}}^{S-\text{cons}}\right)$, $(X/S)_{\text{et}} := \text{Sh}_{\text{eff}}\left(X_{\text{et}}^{S-\text{cons}}\right)$.
$\rightsquigarrow \varepsilon_{X,*} : (X/S)_{\text{an}} \to (X/S)_{\text{et}}$.

Proposition 12.6.4 in [Exo]

The pullback functor $\varepsilon^{X,*}$ restricts to an equivalence on constructible sheaves.

Proof by reduction

Idea: reduce to lisse version by gluing. Do induction for dimension of X. If dim$=0$, then constructible=lisse, done. Write X^{Zar} as limits of $S = Z^{Zar} \cup \{\infty\}$.

$$(Z/Z^{Zar})_{\text{an}} \overset{i_*}{\hookrightarrow} (X/S)_{\text{an}} \overset{j_*}{\hookleftarrow} (U/\infty)_{\text{an}}.$$

$$Z_{\text{et}} \overset{i_*}{\hookrightarrow} (X/S)_{\text{et}} \overset{j_*}{\hookleftarrow} (U/\infty)_{\text{et}}.$$

An ∞-topos X can be recovered from a closed subtopos Z, its open complement U, and the gluing information in the deleted tubular neighborhood W of Z in U. $W = Z \overleftarrow{\times}_X U$ (oriented fiber product).

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ϵ is natural, i.e $f_*^{an} \varepsilon^{X,*} F \simeq \varepsilon^{Y,*} f_*^{et} F$ holds for any constructible sheaf $F \in X_{et}$.

$\rightsquigarrow$ the gluing data are also matched, we're done.

Van Kampen Theorem

If $X = Z \cup^{\phi} U$ is a bounded coherent constructible [1]-stratified ∞-topos. Then the pushout of the morphisms $\widehat{\Pi}_\infty(Z \overleftarrow{\times}_X U) \to \widehat{\Pi}_\infty(Z), \widehat{\Pi}_\infty(Z \overleftarrow{\times}_X U) \to \widehat{\Pi}_\infty(U)$ is exactly $\widehat{\Pi}_\infty(X)$.

Stratified Riemann Existence

$\rightsquigarrow$ The natural morphism $\epsilon : \mathrm{Gal}_{an}(X) \to \mathrm{Gal}(X)$ is an equivalence.
$\mathrm{Gal}_{an}(X)$ is related to the exit path category of X^{an} in topology.

An anabelian application

Let k be a finitely generated field of characteristic 0.
Then a normal k-variety X can be reconstructed from the stratified homotopy type of $(X \otimes_k \bar{k})^{an}$ with its action of G_k.

Thank you!