The Galois action on KSp and on CM abelian varieties

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Outline

Goal

- Study principal polarized abelian varieties (PPAV) A over $\mathbb{C}$ with CM by the maximal order in $E = \mathbb{Q}[\mu_q]$ ($q = p^n$ odd).

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Why?
$C_q \curvearrowright A \rightsquigarrow$ a map $\mathrm{B}(\mathbb{Z}/q) \to \mathcal{A}_g(\mathbb{C})(\to \mathcal{SP}(\mathbb{Z}))$ between groupoids $\rightsquigarrow$ a map $\mathbb{Z}/q[\beta] \to \mathrm{KSp}_*(\mathbb{Z};\mathbb{Z}/q)$.
Bott element
$\beta \in \pi_2^s(\mathrm{B}C_q;\mathbb{Z}/q) \cong H_2(\mathrm{B}C_q;\mathbb{Z}/q) \hookrightarrow H_1(\mathrm{B}C_q;\mathbb{Z}/q) = \mathbb{Z}/q$.
CM classes := image of powers of β, which generate all of $\mathrm{KSp}_*(\mathbb{Z};\mathbb{Z}/q)$.

Galois action

If we understand the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on these PPAVs
$\rightsquigarrow$ on CM classes by **Galois equivariance**.
$\rightsquigarrow$ on $\mathrm{KSp}_*(\mathbb{Z}; \mathbb{Z}/q)$ / $\mathrm{KSp}_*(\mathbb{Z}; \mathbb{Z}_p)$.
Although $\mathrm{KSp}_*(\mathbb{Z}; \mathbb{Z}/q)$ is small, they may contain interesting torsion information.
It's time to do computation.

Galois equivariance

$\mathrm{Aut}(\mathbb{C}) \curvearrowright \mathrm{KSp}$, $\mathrm{Aut}(\mathbb{C}) \curvearrowright \mathcal{A}_g(\mathbb{C})$.
$\mathcal{A}_g(\mathbb{C})$ is just a groupoid in sets, not $\mathcal{A}_g^{\mathrm{an}}(\mathbb{C})$.
The map $\mathcal{A}_g(\mathbb{C}) \to \mathcal{SP}(\mathbb{Z})$ given by $A \mapsto H_1(A(\mathbb{C}); \mathbb{Z})$
$\rightsquigarrow$ a group homomorphism
$\pi^s_{4k-2}(|\mathcal{A}_g(\mathbb{C})|; \mathbb{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q)$ which is equivariant for the action of $\mathrm{Aut}(\mathbb{C})$.
For $g = \varphi(q) = p^{n-1}(p-1)$, it's surjective.

CM types

Let E be a CM field of degree $2g$.
If A is a PPAV of dim g over $\mathbb{C}$ with CM by O_E, then Lie$A$ is an $E \otimes \mathbb{R}$ module.
$\rightsquigarrow$ an $\mathbb{R}$-linear isomorphism $\Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g$.
A CM type of E= an isomorphism $\Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g$
$=$ a subset $\Phi \subseteq \mathrm{Hom}(E, \mathbb{C})$ such that $\mathrm{Hom}(E, \mathbb{C}) = \Phi \coprod c(\Phi)$.

Main theorem of CM

Fix a CM type $\Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g$.

- $\mathfrak{a} \mapsto \mathbb{C}^g/\mathfrak{a}$ induces a bijection:
  $\pi_0\,(\mathrm{Pic}(O_E))$ " $\cong$ "$\{$PPAV $/\mathbb{C}$ with CM by O_E of type $\Phi\}/\sim$.
- They are all defined over H, the Hilbert class field of $E \rightsquigarrow$ the Galois action factors through $\mathrm{Gal}(H/\mathbb{Q})$.
- There exists an ideal $\mathfrak{a}_\sigma$, such that $\sigma(\mathbb{C}^g/\mathfrak{a}) \cong \mathbb{C}^g/(\mathfrak{a} \otimes \mathfrak{a}_\sigma)$ for all $\mathfrak{a}$.
- A precise formula for $\mathfrak{a}_\sigma$ under Artin isomorphism:
  $\pi_0\,(\mathrm{Pic}(\mathcal{O}_E)) \xrightarrow{\mathrm{Art}} \mathrm{Gal}(H/E)$.

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Polarizations

Here we omit polarizations for PPAV.
To be precise, we need to consider an ideal $\mathfrak{a}$ with a skew-Hermitian form $(x, y) \mapsto \mathrm{Tr}_E(xu\bar{y})$ on $\mathfrak{a}$, where u is a purely imaginary element in E.
This can be explained using some groupoids.

Linear algebra

Ring O with an involution $x \to \bar{x}$.
ω any rank 1 projective O-module, with an O-linear involution $\iota : \omega \to \bar{\omega}$.
The groupoid $\mathcal{P}(O, \omega, \iota)$:

- Objects: pairs (L, b) where L is a rank 1 projective O-module and an isomorphism $b : L \otimes_O \bar{L} \to \omega$ s.t $b(x \otimes y) = \iota(b(y \otimes x))$.
- Morphisms: O-linear isomorphisms preserving Hermitian forms.

Linear algebra

$\mathcal{P}_E^+ = \mathcal{P}(O_E, \omega = O_E, c)$ is the groupoid of Hermitian forms on O_E.
$\mathcal{P}_E^- = \mathcal{P}(O_E, \omega = \delta_E^{-1}, -c)$ is the groupoid of skew-Hermitian forms on $\mathcal{O}_E$ valued in the inverse different.
$\mathcal{P}_{E\otimes\mathbb{R}}^-$ is the groupoid of skew-Hermitian forms on $E \otimes \mathbb{R}$.
Objects of $\mathcal{P}_{E\otimes\mathbb{R}}^-$ are classified by CM types $\Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g$.

There is a map $\pi_0(\mathcal{P}_E^-) \stackrel{-\otimes\mathbb{R}}{\to} \pi_0(\mathcal{P}_{E\otimes\mathbb{R}}^-)$, sending (L, b) to its CM type $\Phi_{(L,b)}$.

Shimura-Taniyama map

$\mathfrak{a} \mapsto \Phi(O_E \otimes \mathbb{R})/\mathfrak{a}$ updates to

$$\mathrm{ST} : \mathcal{P}_E^- \to \mathcal{A}_g(\mathbb{C}), (L, b) \mapsto L_\mathbb{R}/L_\mathbb{Z}$$

with the **perfect symplectic pairing**
$L_\mathbb{Z} \times L_\mathbb{Z} \mapsto \mathbb{Z}, (x, y) \mapsto -\mathrm{Tr}_{E/\mathbb{Q}} b(x, y)$.
$\Phi_{(L,b)}$ is the unique CM type Φ such that the Hermitian form $\langle x, y\rangle := -2\sqrt{-1}\sum_{j\in\Phi} j(b(x, y))$ is positive definite.
Perfectness by design: $\delta_E^{-1} := \{a \in E | \mathrm{Tr}_{E/\mathbb{Q}}(ax) \in \mathbb{Z}, \forall x \in O_E\}$.

A tensoring bifunctor $\mathcal{P}_E^- \times \mathcal{P}_E^+ \to \mathcal{P}_E^-$

We can twist PPAV with CM by O_E by any ideal $\mathfrak{a}_1$ in O_E, $\mathbb{C}^g/\mathfrak{a} \to \mathbb{C}^g/(\mathfrak{a}_1 \otimes \mathfrak{a})$.
To be precise, such twisting is given by a tensoring bifunctor $\mathcal{P}_E^- \times \mathcal{P}_E^+ \to \mathcal{P}_E^-$:

$$(L', b') := (L, b) \otimes (X, q) = (L \otimes_{O_E} X, b \otimes q) .$$

Serre's tensor construction

$\mathcal{P}_E^{+\text{pos.def.}} \subset \mathcal{P}_E^+ =$ the full subgroupoid on the positive definite (X, q) i.e i.e $q(x, x) > 0, \forall x \neq 0 \in X$.
The action by positive definite (X, q) will not change CM types, and can be described via Serre's tensor construction:

$$\mathrm{ST}\,(L', b') \cong X \otimes_{\mathcal{O}} \mathrm{ST}(L, b).$$

Recall for any projective O-module X, and AV A with O-action, $X \otimes_O A$ is the functor $R \mapsto X \otimes_O A(R)$.

Action of complex conjugation c

Tensoring with $(X, q) = (O_E, -1)$, where -1 is the form $x \otimes y \mapsto -x\bar{y}$?
It sends $A = \mathrm{ST}(L, b)$ to its complex conjugate variety $\bar{A}$, changing the CM type Φ_A to its complement $c(\Phi_A)$.
$\bar{A} \cong A$ as AVs, not as PPAVs with O_E-action.
Explanation: $\mathcal{A}_g(\mathbb{C}) \simeq \mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})$, but complex conjugation c doesn't preserve $\mathbb{H}_g$.
How about general $\sigma \in \mathrm{Aut}(\mathbb{C})$?

Galois theory

For simplicity, assume $E/\mathbb{Q}$ Galois e.g $E = K_q = \mathbb{Q}[\zeta_p]$. Then $H/\mathbb{Q}$ is also Galois.
Choose for each $\tau \in \mathrm{Hom}(E, \mathbb{C})$ an extension $w_\tau : H \to \mathbb{C}$ to a complex embedding of H, such that

$$w_{\tau c} = w_{c\tau} = c w_\tau.$$

Then for each $\sigma \in \mathrm{Gal}(H/\mathbb{Q})$ and $\tau \in \mathrm{Hom}(E, \mathbb{C})$, $w_{\sigma\tau}^{-1} \sigma w_\tau \in \mathrm{Gal}(H/E)$.
Note if $\sigma \in \mathbf{Gal}(H/E)$, then $w_{\sigma\tau} = w_\tau$. $(w_{\sigma\tau}^{-1} \sigma w_\tau|_H)$ is just σ under an different embedding.

Main theorem today

$\sigma \rightsquigarrow F_\sigma : \mathcal{P}_E^- \to \mathcal{P}_E^-$.

Theorem

- *On each fiber of* $\pi_0(\mathcal{P}_E^-) \to \pi_0(\mathcal{P}_{E\otimes\mathbb{R}}^-)$ *(i.e fixing* $\Phi = \Phi_{(L,b)}$*),* $\pi_0(F_\sigma)$ *is given by tensoring certain* $[(X,q)] \in \pi_0(\mathcal{P}_E^+)$*, determined by* σ *and the CM type* Φ*.*
- $\mathrm{Art}([X]) = \left[\sum_{\tau\in\Phi} w_{\sigma\tau}^{-1}\sigma w_\tau\right]$ *in* $\mathrm{Gal}(H/E)$.
- *If* $E = K_q$*, then* $\pi_*(F_\sigma) : \pi_*^s\left(\left|\mathcal{P}_E^-\right|, \mathbb{Z}/q\right) \to \pi_*^s\left(\left|\mathcal{P}_E^-\right|, \mathbb{Z}/q\right)$ *is* $\mathbb{Z}/q[\beta]$*-linear.*

Slogan: Galois action is always given by twisting certain ideals, and they match under Artin isomorphism (up to summation under different embeddings).

Proof by ST formula

Part (1) is a reformulation of the ST formula in [Mil07, Theorem 4.2].

Linearity

Serre tensor construction commutes with Galois action, so for each positive definite $(X, q) \in \mathcal{P}_E^+$ we have

$$F_\sigma((L, b) \otimes (X, q)) \cong F_\sigma(L, b) \otimes (X, q),$$

naturally in (L, b) and (X, q).
So $\pi_*(F_\sigma)$ is linear over the graded ring $\pi_*^s(|\mathcal{P}_E^{+\text{pos.def.}}|; \mathbb{Z}/q)$.

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So $\pi_*(F_\sigma)$ is linear over the graded ring $\pi_*^s(|\mathcal{P}_E^{+\text{pos.def.}}|; \mathbb{Z}/q)$.
The map $\mathbb{Z}/q[\beta] \to \pi_*^s\left(\left|\mathcal{P}_E^-\right|, \mathbb{Z}/q\right)$ factors through
$\mathbb{Z}/q[\beta] \to \pi_*^s(|BC_q|; \mathbb{Z}/q) \to \pi_*^s(|\mathcal{P}_E^{+\text{pos.def.}}|; \mathbb{Z}/q) \to \pi_*^s\left(\left|\mathcal{P}_E^-\right|, \mathbb{Z}/q\right)$
as $\mathbb{Z}/q = U_1(O_q)$ is the automorphism group of $(X_0, q_0) = (O_q, 1)$.
Hence it's also $\mathbb{Z}/q[\beta]$-linear.

References

- Section 6 of T. Feng, S. Galatius, A. Venkatesh, The Galois action on symplectic K-theory, arxiv:2007.15078.
- Section 2 of J. Milne, The Fundamental Theorem of Complex Multiplication, `http://jmilne.org/math/articles/2007c.pdf`, 2007.

Thank you!