A rough introduction to Lubin-Tate spaces

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December 2018

1 Explicit Local Class Field Theory

The 1965 paper "formal complex multiplication in local fields" by Lubin and Tate constructs explicit local class field theory using formal modules. Let K be a local field, O be its integer ring, π be an uniformizer, $k = O/\pi$ be its residue field with q elements, and $F_\pi = \{f \in O[[X]] | f = X^q \bmod \pi, f = \pi X + O(2)\}$.

Lemma 1.1. *For $f, g \in F_\pi$, $a_1, \ldots, a_n \in O$, there exists an unique $F \in O[[X_1, \ldots, X_n]]$ s.t $f \circ F = F \circ g, F(X) = \sum_{i=1}^{n} a_i X_i + O(2)$.*

Proof. Construct F_f step by step like the proof of Hensel lemma. □

Corollary 1.2. *For $f \in F_\pi$, there exists an unique formal O module F_f over O s.t $f = [\pi] \in End(F_f)$. Moreover, any two $f_1, f_2 \in F_\pi$ give isomorphic formal O modules.*

Let $A = (m_{K^{alg}}, F_f)$ be $m_{K^{alg}}$ with the O module structure given by F_f, $K_n = K(A[\pi^n])$ and K_∞ be the p-adic completion of $K(A[\pi^\infty])$. By looking at newton polygon, we see $[\pi] : A \to A$ is surjective.

Proposition 1.3. *$A[\pi^n] \cong \pi^{-n}O/O$; K_n/K is totally ramified; $Gal(K_n/K) \cong (O/\pi^n)^\times$ and $\pi \in Norm(K_n)$.*

Proof. WLOG $f(X) = X^q + \pi X$, elements in $A[\pi^n]$ are the same as roots of $f^{(n)}$ hence $\#A[\pi] \leq q$, but $A[\pi]$ is an O/π module so $A[\pi] \cong \pi^{-1}O/O$. Consider the exact sequence

$$0 \to A[\pi] \to A[\pi^{n+1}] \xrightarrow{\pi} A[\pi^n] \to 0$$

we see $A[\pi^n] \cong \pi^{-n}O/O$ for any n. Moreover, the Galois action on A is compatible with the O module structure so there is an injection $Gal(K_n/K) \hookrightarrow Aut_O(A[\pi^n]) \cong (O/\pi^n)^\times$. But $f^{(n)}/f^{(n-1)} = (f^{(n-1)})^{q-1} + \pi$ is Eisenstein hence irreducible we see $\#Gal(K_n/K) \geq \#(O/\pi^n)^\times$ so above injection must be an isomorphism, hence K_n/K is totally ramified. Note the constant term of $f^{(n)}/f^{(n-1)}$ is π so $\pi \in Norm(K_n)$. □

As $\pi \in Norm(K_n)$, composing with the maximal unramified extension, this gives an explicit construction of the maximal abelian extension of K. A key observation is that $[\pi] : A \to A$ is surjective, which may motivate the definition of p-divisible groups.

From a modern point of view, choose $z_n \in m_{K^{alg}}$ inductively such that z_n is a generator of $A[\pi^n]$ and $[\pi](z_{n+1}) = z_n$, from above we know $K_n = K(z_n)$ and $O_{K_n} = O[z_n]$. As $f = X^q \bmod \pi \Rightarrow z_{n+1}^q = z_n \bmod \pi$, we see the Frobeniu on O_{K_∞}/p is surjective so K_∞ is perfectoid.

2 Lazard Ring

We see that formal group law is a useful tool, then a naive question is to describe all formal group laws over a ring. If we only concern about 1-dimensional commutative ones, this is solved by Lazard in "Sur les groupes de Lie formels à un paramètre" (1955).

Lazard's key lemma is to consider truncated formal group laws:

Lemma 2.1. *Let $n \in \mathbb{Z}_{>0}$. Let F be a truncated at the order n formal group law over R that can be extended to a truncated at the order $n+1$ formal group law. Then the set of such extensions G to a truncated at the order $n+1$ formal group law is is a principal homogeneous space under R via $\forall a \in R, \forall G, a.G := G + aC_n(X,Y)$. Where*

$$C_n(X,Y) = \begin{cases} (X+Y)^n - X^n - Y^n & \text{if } n \text{ is not a power of a prime,} \\ \frac{(X+Y)^n - X^n - Y^n}{p} & \text{if } n = p^a,\ p \text{ prime.} \end{cases} \tag{1}$$

Proof. Fix an extension G, consider $G + \Gamma$ as another potential candicate, only need to solve

$$\Gamma(X,Y) + \Gamma(X+Y,Z) = \Gamma(Y,Z) + \Gamma(X,Y+Z)$$

where $\Gamma(X,Y) = \sum_{i+j=n,i,j>0} a_{ij} X^i Y^j$, this is a not hard linear algebra problem. □

Example 2.1. For instance, write a formal group law as $X + Y + aX^2 + bXY + aY^2 + O(3)$ then from the associativity condition we see $a = 0$.

To move further, we recall some general facts. An elementary and useful lemma is:

Lemma 2.2. *For $f \in R[[X]]$, there exists $g \in R[[X]]$ s.t $fg = 1$ iff $f(0) \in R^\times$; Assume $f(0) = 0$, there exists $g \in R[[X]]$ s.t $f \circ g = id$ iff $f'(0) \in R^\times$.*

Definition 2.1. $w(F) :=$ {F-invariant differentials}, it's a free $R-$module of rank 1 generated by $w_F = \frac{1}{\partial_X F(0,T)} dT$.

Corollary 2.3. *Given $f : F_1 \to F_2$, then $f^* w_F = f'(0) w_G$. And f is an isomorphism iff $f'(0) \in R^\times$*

Proof. By above lemma and uniqueness of invariant differentials after normalization. □

Definition 2.2. If R is a $\mathbb{Q}$-algebra, define $log_F = \int_0^T w_F$, it gives an isomorphism $F \cong \mathbb{G}_a$ (it's a homomorphism because of invariance and an isomorphism by considering tangent map).

Proposition 2.4. *If $p = 0$ in R, then there exists an unique maximal $h \in \mathbb{Z}_{>0}$ such that $[p]$ factors through $X \mapsto X^{p^h}$, or $[p] = 0$. The maximal h is called the height of F.*

Proof. $[p]'(0) = 0$ by definition, note $[p]'(T)\partial_X F(0,[p](T))dT = [p]^* w = [p]'(0) w = 0$ and $\partial_X F(0,T)$ is invertible by lemma 2.2 so $[p]' = 0$. □

The functor sending a ring to the set of formal group laws over it is represented by $L = \mathbb{Z}[a_{ij}]/I$ where I is generated by axioms of formal group laws. If we put X, Y both by -1 and a_{ij} by degree $i+j-1$, then $F_{univ} = \sum_{i,j} a_{ij}X^iY^j$ preserves the degree so $L = \oplus L_k$ is a graded ring.

Applying lemma 2.1 on L, there exists $t_1 \in L$ such that $F_{univ} = X + Y + t_1C_2(X,Y) + o(2)$ so $\deg(t_1) = 1$. Applying above lemma on L/t_1 and lifting, there exists $t_2 \in L$ such that $F_{univ} = X + Y + t_2C_3(X,Y) + o(3) \mod t_1$. Comparing degree, we can assume $\deg(t_2) = 2$. Keep doing, we find $t_k \in L$ with $\deg(t_k) = k$ and

$$F_{univ} = X + Y + t_kC_{k+1}(X+Y) + o(k+2) \mod t_1, \ldots, t_{k-1} \tag{2}$$

In other words, A truncated at the order $n+1$ formal group law over R corresponds to a morphism $\oplus_{0\le k\le n-1} L_k \to R$ so we get the primitive part of L_k for any k is a free abelian groups of rank 1 by lemma 2.1.

Theorem 2.5. *(Lazard) The natural construction above gives an isomorphism of graded rings*

$$\phi : \mathbb{Z}[t_k]_{k\ge1} \cong L \quad (\text{here } \deg t_k = k)$$

Therefore, any truncated FGL can be lifted.

Proof. ϕ preserves the degree. Note the $k-$th primitive part of L is generated by $\{a_{ij}\}_{i+j=k+1}$ hence generated by t_k by (2), so ϕ is surjective on every primitive part hence surjective by induction. To prove ϕ is injective we can tensor $\mathbb{Q}$, and $L_{\mathbb{Q}} \cong \mathbb{Q}[b_k]_{k\ge1}$ by logarithm. We know $\phi_{\mathbb{Q}} : \mathbb{Q}[t_k]_{k\ge1} \to \mathbb{Q}[b_k]_{k\ge1}$ is surjective, and the dimension of each graded part is the same on each side, so the surjectivity forces injectivity. □

So there are plenty of formal group laws, the remaining problem is classifying them up to isomorphism. Assume $p = 0$ in R in the rest of this section. We have the key lemma about the behavior of $[p]_F$ when deforming F.

Lemma 2.6. $G_1 = G_2 + aC_{p^i}(X,Y) \bmod \deg(p^i+1)$*, then* $[p]_{G_1} = [p]_{G_2} - aT^{p^i} \bmod \deg(p^i+1)$

Proof. Consider $[n]$ and do induction. For instance, if F is a formal group law such that $F(X,Y) = X + Y + \sum_{i+j=m,i,j>0} a_{ij}X^iY^j + o(m+1)$, we can assume $[n]X = a_nX + b_nX^m + o(m+1)$ by induction, and then finds the relation $a_{n+1} = a_n + 1, b_{n+1} = b_n + \sum a_{ij}n^i$. □

Corollary 2.7. *(infinite height = additive)* $[p] = 0 \Leftrightarrow G \cong \mathbb{G}_a$.

Proof. Suppose $G = X + Y + o(n)$, write $G = X + Y + aC_n(X,Y) + o(n+1), a \in R$. If n is not a power of p then consider $h(T) = T - aT^n$ and replace G by hGh^{-1}, or n is a power of p then $a = 0$ by above lemma. □

To study the finite height case, we assume R is a field and consider normalized formal group law (i.e $[p] = T^{p^h}$). If R is separably closed then Artin-Schreier equations are always solvable and one could prove any formal group law with finite height is isomorphic a normalized one. Using lemma 2.6 again, one proves normalized formal group laws are isomorphic iff their height agree.

Example 2.2. $f(T) = \sum_{k\geq 0} \frac{T^{p^{kh}}}{p^k} \in \mathbb{Q}[[T]], F(X,Y) = f^{(-1)}(f(X)+f(Y)) \in \mathbb{Z}_{(p)}[[X,Y]]$ then $F \bmod p$ is a formal group law of height h over $\mathbb{F}_p$.
Another example is $L \to \mathbb{F}_p$ with $t_i \mapsto 0$ if $i \neq p^h - 1$ and $t_{p^h-1} \mapsto 1$. To see why they have height h, one could use the argument in lemma 2.6.

Example 2.3. We can also consider Lubin-Tate O_L module F_f over O_L with $f(X) = X^{p^h} + pX$ and modulo p to get a FGL H of height h, where L is the unramified extension of $\mathbb{Q}_p$ of degree h. Therefore, $O_L \subseteq End(H)$, and note Frobenius $\Phi(X) = X^p$ lies in $End(H)$ such that $\Phi^h = p$, one can show $O_L[\Phi] = End_k(H)$.

So we get

Theorem 2.8. *The isomorphism classes of one dimensional formal group laws over a separably closed field of char* $= p > 0$ *are in bijection with* $\mathbb{Z}_{>0} \cup \{\infty\}$*, the bijection being given by the height.*

Remark 2.9. *Another proof is to use Diedonne theory to classify crystals of rank* 1 *and height* h*. Besides,* $End(G_h) = O_D$ *where* D *is a division algebra over* $\mathbb{Q}_p$ *with invariant* $1/n$ *and* O_D *is its maximal order.*

Remark 2.10. *There are lots of things about formal group laws in Hazewinkel's book, for example theorem* (1.6.7) *in the book shows every one dimensional formal group law over a reduced ring is commutative.*

3 Lubin-Tate spaces

Reference: Fargues's course note.

From above, one knows how to classify formal group laws on any fields (at least for algebraically closed fields). A natural question is how to classify them over a general ring such as a DVR. One could fix the $\bmod p$ part and consider the deformation problem. This motivates the notion of Lubin-Tate space which is introduced in the paper "Formal moduli for one-parameter formal Lie groups" (1966) by Lubin and Tate.

Let k be an algebraically closed field of char $p > 0$, F_0 be a FGL over k with height $h < \infty$.

Definition 3.1. Let C be the category of Artin local $W(k)$-algebras (A, m) with residue field k, morphisms being local ring morphisms inducing the identity on k.

1. A formal group law G over A is called a deformation of F_0 if $G = F_0 \bmod \mathfrak{m}$
2. Two deformations G_1, G_2 are isomorphic if there is an isomorphism $f : G_1 \to G_2$ such that $f = Id \bmod \mathfrak{m}$.

Let $\mathbb{M}_0$ be the associated functor of isomorphism classes of deformations of G_0 on C.

Remark 3.1. $\mathbb{M}_0$ *is essentially the same as the ususal functor* $\mathcal{M}_0 : \mathcal{C} \to$ *Sets that associates* $A \in \mathcal{C}$ *to the set of isomorphic classes of* (G, ρ) *where* G *is a FGL over* A *and* $\rho : G \otimes_A A/m_A \cong F_0$*. Here* $\mathcal{C}$ *is the category of complete Noetherian local* $W(k)$*-algebras* (A, m) *with residue field* k*.*

To attack such deformation problem, one can formulate the problem in a general setting. Let $F : C \longrightarrow Sets$ be a covariant functor such that $F(k)$ is a point. We can define it's tangent space $TF(k) := F(k[x]/(x^2))$. In order for F to be prorepresentable i.e by a complete noetherian local $W(k)$-algebra with residue field k, one important neccesary condition is the Mayer-Vietoris property i.e F preserves push-outs. A key observation is

Lemma 3.2. *Let $A \in C$ with with maximal ideal $\mathfrak{m}$ and let I be an ideal of A s.t. $m.I = 0$. Then*

$$A \times_{A/I} A \cong A \times_k (k \oplus I)$$

$$(a_1, a_2) \mapsto (a_1, \bar{a}_1 \oplus (a_1 - a_2))$$

So if F satisfies Mayer-Vietoris propety then $F(A) \to F(A/I)$ is a $F(k \oplus I)$-torsor.

Proof. The isomorphism can be checked directly. Note $F(k \oplus I)$ is a k-vector space and the action on $F(A)$ follows from the isomorphism $F(A) \times F(k \oplus I) \cong F(A) \times_{F(A/I)} F(A)$. □

Therefore we can glue everything from the bottom, and plus formal smoothness we get

Theorem 3.3. *Let k be a perfect field Suppose F satisfies the Mayer-Vietoris property, is formally smooth and $F(k)$ is a point, and $dim_k TF(k) = n < \infty$. Then F is prorepresentable by $W(k)[[T_1, \dots, T_n]]$.*

Proof. Choose a basis of $TF(k)$, it gives an element in $F(k[T_1, \dots, T_n]/(T_i^2)) \cong \prod_{i=1}^n TF(k)$ (by MV property), which can be lifted to an element in $F(W(k)[[T_i]])$ by formal smoothness i.e a morphism

$$f : G = \mathrm{Spf}(W(k)[[T_1, ..., T_n]] \to F$$

Note f is an isomorphism on tangent space, hence on any $F(k \oplus M) \cong TF(k) \otimes_k M$ where M is a finite dimensional k linear space. With notations in previous lemma and formal smoothness of F and G, we find if f is an isomophism on A/I then it's also an isomorphism on A. As any object in C can be filtered into such situation (as they are all artin local rings), we know f is a isomorphism. □

Remark 3.4. *Above lemma is a special case of Schlessinger's deformation criterion (all 4 conditions).*

From Lazard's result, we know $\mathbb{M}_0$ is formally smooth, and the MV property clearly holds. So we only need to compute the tangent space of $\mathbb{M}_0$. Consider a formal group laws F' over $k[\epsilon]/\epsilon^2$ which is a deformation of F_0, and write $F'(X, Y) = F_0(X, Y) + \epsilon\phi(X, Y)h(F_0(X, Y))$ where $\omega_{F_0} = h(T)dT$ is the normalized generator of invariant differentials of F_0. Unraveling the definition, we get

Lemma 3.5. *The tangent space of $\mathbb{M}_0$ is identified with the middle cohomology of the complex*

$$Tk[[T]] \overset{\partial_0}{\to} XYk[[X, Y]]^{S_2} \overset{\partial_1}{\to} k[[X, Y, Z]]$$

where

$$\partial_0\phi(T) = \phi(X +_{F_0} Y) - \phi(X) - \phi(Y)$$

$$\partial_1\psi(X, Y) = \psi(Y, Z) - \psi(X +_{F_0} Y, Z) + \psi(X, Y +_{F_0} Z) - \psi(X, Y).$$

Remark 3.6. *In the paper of Gross-Hopkins, this is called the symmetric cohomology.*

To do computation, we specify our F_0 as $L \to \mathbb{F}_p$ with $t_j \mapsto 0$ if $j \neq p^n - 1$ and $t_{p^n-1} \mapsto 1$, the same in example 2.2. For any $1 \le i \le n-1$,Let ψ_i be the cocycle corresponding to the deformation $L \to \mathbb{F}_p$ with $t_j \mapsto 0$ if $j < p^i - 1$, $t_{p^i-1} \mapsto \epsilon$, $t_j \to 0$ if $p^i - 1 < j < p^n - 1$, $t_{p^n-1} \to 1$ and $t_j \to 0$ if $j > p^n - 1$.

Lemma 3.7. *ψ_i $(1 \le i \le n-1)$ form a basis for the tangent space of $\mathbb{M}_0$.*

Proof. Note $\psi = o(k) \Rightarrow \psi = aC_k(X,Y) + o(k+1)$ and ∂T^k is known so we could do it step by step. Details are omitted. □

From above computation, we get

Theorem 3.8.

$$\mathbb{M}_0 \cong \mathrm{Spf}(W(\overline{\mathbb{F}_p})[[t_1, \ldots, t_{n-1}]])$$

In particular, it's coordinate ring is a regular local ring of dimension n.

Remark 3.9. *From the proof, one sees that we can choose the universal deformation FGL F on $R_0 = W(k)[[x_1, \ldots, x_{h-1}]]$ such that on R_0*

$$F(X,Y) = X + Y + x_i C_{p^i}(X,Y) \mod (x_1, \ldots, x_{i-1}, \deg >= p^i + 1)$$

and there exists $u_i \in R_0[[T]]^\times$ $(i = 0, \ldots, h)$ such that

$$[p](T) = pu_0 T + x_1 u_1 T^p + \ldots + x_{h-1} u_{h-1} T^{p^{h-1}} + u_h T^{p^h}.$$

Remark 3.10. *Another universal formal group law: the logarithm is $f(T) \in \mathbb{Q}[v_1, \ldots, v_{n-1}]$ be the unique solution of Hazewinkel's functional equation:*

$$f(X) = X + \sum_{i=1}^{n-1} \frac{f(v_i X^{p^i})}{p} + \frac{f(X^{p^n})}{p}$$

All above can be generalized to formal O modules, which is done by Drinfeld.

4 p-divisible groups and Dieudonne modules

In 1967, the fundamental paper "p-divisible groups" by Tate was published , with the motivation to study elliptic curves or more general abelian varieties. Given an elliptic curve E over an algebraically closed field k of char $p > 0$, completion at origin gives a formal group law which is however not sufficient to recover the information of E, neither does the Tate module. Moregenerally, one could consider $E[p^\infty]$ which is a p-divisible group and there is a theorem by Sere-Tate identifying the deformation of E and $E[p^\infty]$. Slogans (assume p is locally (topological) nilpotent on the base):

1. p-divisible groups = inductive system of finite commutative flat p-group schemes s.t $[p]$ is epimorphism.

2. Formal p-divisible groups = formal lie groups (+ fixed coordinates = formal group laws) .

3. Etale p-divisible groups (e.g $\mathbb{Q}_p/\mathbb{Z}_p$)= finite free $\mathbb{Z}_p$ representations of the fundamental group.
4. Every p-divisble group is the extension of an etale one by a formal one.
5. Cartier duality = exchange of multi and comulti of hopf algebras (duality of F gives V).
6. $\text{height}(G) = \dim(G) + \dim(G^D)$

Dieudonne theory classifies p-divisible groups over a perfect field using semi-linear datas i.e crystal (for general base there is a theory of displays which couldn't be presented here at present).

Definition 4.1. Over a char $p > 0$ perfect field k, an F-crystal is a free module M of finite rank over the ring W of Witt vectors of k, together with a σ-linear injective endomorphism of M. An F-isocrystal is defined in the same way, except that M is a module for the quotient field K of W.

Example 4.1. Let G be a p-divisible group over k, $\mathbb{D}(G) = LieE(G)$ is a F-crystal.

Theorem 4.1. *(Dieudonne) $\mathbb{D}$ is a fully faithful functor:*

$$\{p\text{-divisible groups over } k\} \to \{\text{crystals over } k\}$$

$$\{p\text{-divisible groups over } k \text{ up to isogeny }\} \to \{\text{isocrystals over } k\}$$

where $\operatorname{rank} \mathbb{D}(G) = \text{height}(G)$, $\dim G = \dim_{\mathrm{k}} \mathbb{D}(G)/F\mathbb{D}(G)$.

Proof. It's sufficient to prove the analogs in the finite flat group scheme case, and by local-etale exact sequence which splits in the perfect field case one only concern about the local-local case. (I wrote a proof in some old notes in Chinese so it's omitted in this note). □

Remark 4.2. *Note the category of finite flat group schemes over a field is an abelian category, by Mitchell's embedding theorem it's a full subcategory of a module category. So Dieudonne theory is more or less possible.*

Now the classifiction reduces to a semi-linear algebra problem which is done by Dieudonné (1955) and Manin (1963).

Theorem 4.3. *(Dieudonne-Manin classification)*
Assume k is algebraically closed. The category of F-isocrystals over k is abelian and semisimple. The simple F-isocrystals are the modules $E_{\frac{d}{h}}$ where d and h are coprime integers with $h > 0$. $E_{\frac{d}{h}}$ has a basis over K of the form $v, Fv, F^2v, \ldots, F^{h-1}v$ for some element v, and $F^h v = p^d v$. The rational number $\frac{d}{h}$ is called the ***slope*** *of the F-isocrystal.*
Over a general perfect field k, an F-isocrystal can still be written as a direct sum of subcrystals that are isoclinic, where an F-crystal is called isoclinic if over the algebraic closure of k it is a sum of F-isocrystals of the same slope.

Proposition 4.4. *The functor $G \to \mathbb{D}(G)_{\mathbb{Q}}$ induces an equivalence between p-divisible groups over k up to isogeny and isocrystals whose slope lie bewteen 0 and 1. G is etale iff the isocrystal is isoclinic with slope 0, G is formal p-divisible iff the isocrystal doesn't have zero slope.*

Proof. $\text{height}(G) \geq \dim(G)$. If G is etale, $\operatorname{Lie} G = 0$. □

An important tool to study isocrystals is the newton polygon (there is an analog with vector bundle on algebraic curves and Harder-Narasimhan filtration).

Definition 4.2. If $\lambda_1, ..., \lambda_r$ are the slopes of $\mathbb{D}(G)$ with multiplicity $(a_1, ..., a_r)$ where $\lambda_i = \frac{d_i}{h_i}$ with $(d_i, h_i) = 1$. Then the **Newton polygon** attached to the data (λ_i, a_i) is the unique convex polygon whose breakpoints are in $\mathbb{Z}^2$ that begins at $(0, 0)$ and whose slopes are the λ_i each with multiplicity $a_i h_i$. Therefore, it ends at $(\text{height}(G), \dim(G))$.

Example 4.2. We can use p-divisible groups to distinguish elliptic curves over $\mathbb{F}_p^{alg}$. The newton polygon of the LHS is the supersingular one ($E[p]$ is a non-split extension of α_p by α_p), the RHS is the ordinary one ($\mu_{p^\infty} \oplus \mathbb{Q}_p/\mathbb{Z}_p$):

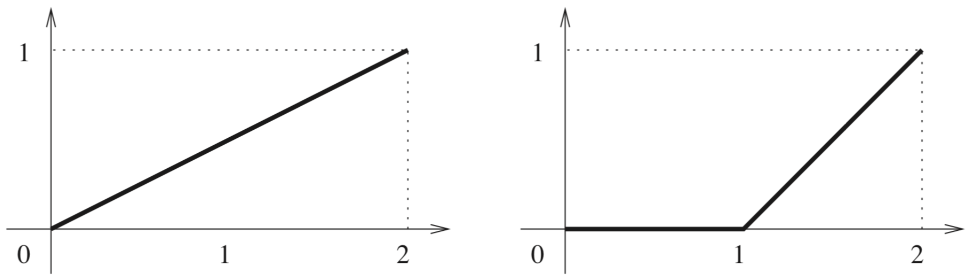

Figure 1: Newton polygon of p-divisible groups from elliptic curves.

In fact, for such an elliptic curve we can assume it's defined over $\mathbb{F}_q$ for some q. Consider the (geometric) Frobenius action on $H^1_{et}(\bar{E}, \mathbb{Q}_l)$ with it's characterestic polynomial $P_1(t) = 1 - (\alpha + \beta)t + qt^2$ (α, β are the eigenvalues, they are algebraic integer and $\alpha\beta = q$). In that case, the newton polygon corresponds to the newton polygon of the poylnomial $P_1(t)$ i.e the lower convex polygon on the coeffecients. Note the elliptic curve is ordinary iff one of α, β is a p-adic unit iff $E[p^\infty](\overline{\mathbb{F}_p}) \neq 0$, so we get above pictures.

Example 4.3. Using the classification one can reprove the classification of one dimensional formal p-divisible groups over $\overline{\mathbb{F}_p}$ by their height. Let H be a formal p-divisible groups of height h (unique up to iso), the crystal is of rank n with Frobenius action given by the matrix F (under a special basis e_i):

$$\begin{bmatrix} 0 & \dots & 0 & p \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix}$$

Therefore $End(H) = End_(W(k), F)(\mathbb{D}(H)) = O_D$ where D is the unique division algebra over $\mathbb{Q}_p$ with invariant $\frac{1}{n}$. More concretely,

$$D = \mathbb{Q}_{p^n}(u) \text{ s.t } u^n = p, ux = x^\sigma u \text{ for any } \; x \in \mathbb{Q}_{p^n}; O_D = \mathbb{Z}_{p^n}[u]$$

where $x \in \mathbb{Z}_{p^n}$ acts on $\mathbb{D}(H)$ by $x.e_i = \sigma^i(x)e_i$ and u acts by F.

5 Serre-Tate theorem, Grothendieck-Messing theory, Canonical lifting

The story of Serre-Tate theorem began in a letter of from Tate to Serre in 1964, while the simplest proof is given by Drinfeld at 1976 using the rigid lemma. The Grothendieck-Messing theory concerns about deformation of p-divisible groups as well as the crystalline feature of $\mathbb{D} = \mathrm{Lie}\, E(\cdot)$. Here are the main theorems:

Theorem 5.1. *Let k be a char $p > 0$ perfect field. Let H_0 be a dimension d and height h p-divisible group over $\mathrm{Spec}(k)$, Let Def_{H_0} be the functor from artinian rings with residue field k to sets that associates to A the isomorphism classes of couples (H, ρ) where H is a p-divisible group over A and $\rho : H_0 \to H \otimes_A k$ is an isomorphism. Then Def_{H_0} is pro-representable:*

$$Def_{H_0} \cong \mathrm{Spf}\, W(k)[[T_1, ..., T_{d(h-d)}]]$$

Theorem 5.2. *(Serre-Tate)*

$$Def_E = Def_{E[p^\infty]}.$$

Example 5.1. Here is an application to the geometry of modular curves. Let $N \geq 5$, so that the moduli problem $\Gamma_1(N)$ is representable by a scheme $Y_1(N)$ which is smooth over $\mathbb{Z}[1/N]$. Let the point $x \in Y_1(N)(\overline{\mathbb{F}_p})$ (p is prime to N) correspond to the pair $(E_0/\overline{\mathbb{F}_p}, P_0)$. Then by Serre-Tate, we know the deformation problem of $E[p^\infty]$ is representable by $\hat{O}_{Y_1(N),x}$, in particular it's isomorphic to $W(\overline{\mathbb{F}_p})[[x]]$ by theorem 5.1.

To prove Serre-Tate theorem, one key tool is Drinfeld's rigidity lemma of quasi-isogenies:

Lemma 5.3. *(**Rigidity lemma**) Let $i : S_0 \hookrightarrow S$ be an immersion defined by a localy nilpotent ideal, and p is localy nilpotent on S. Let G, H be two p-divisible groups over S and G_0, S_0 their reduction to S_0. Then the reduction map induces an injection of torsion free $\mathbb{Z}_p$-modules (torsion-free as $[p]$ is epimorphism)*

$$Hom_S(G, H) \hookrightarrow Hom_S(G_0, H_0).$$

and if moreover S is quasi-compact there exists $N \in \mathbb{N}$ s.t.

$$p^N Hom_{S_0}(G_0, H_0) \subseteq Hom_S(G, H).$$

Proof. we can reduce to the case the ideal sheaf I of i has zero square, then regarding p-divisible groups as fppf sheaves we have

$$Hom_{S_0}(G_0, H_0) = Hom_S(G, i_* i^* H).$$

Note p acts nilpotently on $K = \mathrm{Ker}(H \to i_* i^* H) \cong i_* \underline{Hom}(w_{H_0}, I)$ and is an epimorphism on G hence $Hom_S(G, K) = 0$. Assume $p^N = 0$ on S then $p^N K = 0$ so $\mathrm{Ext}^1(G, K)$ is killed by p^N. The results follow from the long exact sequence induced by $0 \to K \to H \to i_* i^* H \to 0$. $\square$

6 Gross-Hopkins period map

It's good to study the Lubin-Tate space M_0 more explicitly with respect to the action of $O_D^\times$. In the 1994 paper "Equivariant vector bundles on the Lubin-Tate moduli space", Hopkins and Gross construct a $O_D^\times$ **equivariant** period map from the generic fiber of $\mathbb{M}_0$ to a projective space (more precisely the Severi-Brauer variety of D) and show it's surjective.

Let $\mathbb{H}$ be a height n one dimensional formal p-divisible group over $k = \overline{\mathbb{F}_p}$, the associated Lunbin-Tate space $\mathbb{M}_0 \cong \mathrm{Spf}(W(k)[[x_1, \ldots, x_{n-1}]])$, (H, ρ) be the universal deformation where $\rho : \mathbb{H} \cong H \bmod \mathfrak{m} = (p, x_1, \ldots, x_{n-1})$. $\mathbb{D}(\mathbb{H})$ and $\mathbb{M}_0$ are equipped with natural action of $Aut(\mathbb{H}) \cong O_D^\times$. Note the action is continuous on $\mathbb{M}_0$:

Lemma 6.1. *For any $\ell \in \mathbb{Z}_{>0}$, there exists an open compact subgroup of $O_D^\times$ acts trivially on $\mathbb{M}_0 \bmod p^\ell$.*

Proof. By Drinfeld rigidity lemma, there exists $N \in \mathbb{N}$ s.t. for every $g \in O_D^\times$, $p^N g$ lifts to $End(H \bmod \mathfrak{m}^k)$ so $Id + p^N g \in Aut(H \bmod \mathfrak{m}^k)$. □

Let $K = W(k)[1/p]$, $\mathbb{M}_0^{rig}$ is the generic fiber of $\mathbb{M}_0$ as a rigid space over K

$$\mathbb{M}_0^{rig} = \{(x_1, \cdots, x_{n-1}) \in \mathbb{A}^n | v(x_i) > 0, \forall i\} = \bigcup_{a \in \mathbb{Z}_{>0}} B(0, p^{-\frac{1}{a}}) = \bigcup_{a \in \mathbb{Z}_{>0}} Sp(A_a[\frac{1}{p}])$$

where A_a is the affinoid algebra

$$A_a = O_K < X_1, \ldots, X_{n-1}, T_1, \ldots, T_{n-1} > /(X_i^a - pT_i)$$

with $Sp(A_a[1/p]) \hookrightarrow Sp(A_b[1/p])$ for any $a \leq b$ is given by $X_i \to X_i$ and $T_i \to X_i^{b-a} T_i$. To construct the equivariant period map, we need to construct an equivariant vector bundle on $\mathbb{M}_0^{rig}$. Let $E(H)$ be the universal vector extension of H on $\mathbb{M}_0$ and consider $\mathbb{D}(\mathbb{H}) = \mathrm{Lie}\, E(H)$ which is free $O_{\mathbb{M}_0}$ of rank n (as $\mathrm{Ext}^1(H, \mathbb{G}_a)$ is free of rank $n-1$)and there is a short exact sequence of free $O_{\mathbb{M}_0}$ modules

$$0 \to w_{H^D} \to LieE(H) \to w_H^* \to 0$$

Let $\mathrm{Lie}(E(H))^{rig}$ be the $O_D^\times$-equivariant rigid analytic vector bundle over $\mathbb{M}_0^{rig}$.

Theorem 6.2. $\mathrm{Lie}(E(H))^{rig}$ *is a flat $O_D^\times$-vector bundle with an equivavariant isomorphism*

$$\mathrm{Lie}(E(H))^{rig} \cong \mathbb{D}(\mathbb{H})_{\mathbb{Q}} \otimes_K O_{\mathbb{M}_0^{rig}}$$

Proof. This follows from Drinfeld rigidity lemma and the crystalline structure of $\mathbb{D}(H)$, and everything became isomoprhic after inverting p. More precisely, we pull back H by $Sp(A_a) \hookrightarrow \mathbb{M}_0$ to get H_a and an isomorphism

$$\rho_a : \mathbb{H} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(\overline{\mathbb{F}_p}(T_1, \ldots, T_n) \cong H_a \bmod (p, X_1, \ldots, X_{n-1})$$

note the ideal $(X_1, \ldots, X_{n-1})$ is nilpotent in A_a/pA_a so we can lift the isomorphism to quasi-isogenies between $\mathbb{H} \times_{\mathrm{Spec}(\overline{F}_p)} \mathrm{Spec}(A_a/p)$ and $H_a \times_{\mathrm{Spec}(A_a)} \mathrm{Spec}(A_a/p)$, by the crystalline nature of $\mathbb{D}$ we get an isomorphism

$$\mathbb{D}(\mathbb{H}) \otimes_{W(k)} A_a[\frac{1}{p}] \cong \mathrm{Lie}\, E(H_a)[\frac{1}{p}]$$

It's compatible with different a and the continuous action of $O_D^\times$, taking the limit we get

$$\mathbb{D}(\mathbb{H})_{\mathbb{Q}} \otimes_K O_{\mathbb{M}_0^{rig}} \cong \mathrm{Lie}\, E(H)^{rig}$$

□

Definition 6.1. The equivariant vector bundle $\mathbb{D}(\mathbb{H})_{\mathbb{Q}} \otimes_K O_{\mathbb{M}_0^{rig}} \cong \operatorname{Lie} E(H)^{rig}$ induced a $O_D^{\times}$-equivariant rigid analytic morphism $\widehat{\pi} : \mathbb{M}_0^{rig} \to \mathbb{P}(\mathbb{D}(\mathbb{H}))$, which is called the Gross-Hopkins period map. It's etale by Grothendieck-Messing deformation theory.

Furthermore, one can show $\widehat{\pi}$ is surjective and describe the fibers.

Proposition 6.3. *Let $x = (H, \rho) \in \mathbb{M}_0^{rig}(K^{alg}) = \mathbb{M}_0(O_{K^{alg}})$, then the fiber $\widehat{\pi}^{-1}(\widehat{\pi}(x))$ is in bijection with $GL_n(\mathbb{Q}_p)^1/GL_n(\mathbb{Z}_p)$, here $GL_n(\mathbb{Q}_p)^1/$ consists of those matrices whose determinants are p-adic unit.*

Example 6.1. The GL_2 case.

The period map is very useful, it can be used to construct rigid etale covering of projective line, and has some applications in algebraic topology.

7 Lubin-Tate tower, Drinfeld level structure

Summary: Motivation for Drinfeld level structure, representability by complete local rings, regularity (R_n by R_1, R_1 by L_r), flatness anf finiteness of $\mathcal{M}_{n+1} \to \mathcal{M}_n$, étaleness on the generic fiber, Galois group $\operatorname{Aut}(M_n \to M_0) = GL_h(\mathbb{Z}/p^n\mathbb{Z})$.

One hopes to define the level structures like the modular curve case. For a FGL F over $A \in \mathcal{C}$, the p^n-torsion points $F[p^n](A) := (m_A, +_F)[p^n]$. So by definition $F[p^n](k) = 0$, we can't expect the level structure map always be an isomorphism. What if we just require there is a group homomorphism $\eta : (p^{-n}\mathbb{Z}/\mathbb{Z})^h \to F(A)[p^n]$? By definition it's representable by $B_n = R_0[[T_1]]/[p^n](T_1) \otimes_{R_0} \ldots \otimes_{R_0} R_0[[T_h]]/[p^n](T_h)$ which is a complete local ring and finite over R_0 (here we use Weierstrass division theorem). The problem is that B_n is not an integral domain in general so does not have good ring properties.

Example 7.1. If $h = 1$, then $R_0 = W(k)$ and one can take $\mathbb{G}_m$ as the universal deformation on R_0, then $B_n = W(k)[T]/(T+1)^{p^n} - 1$. So we hope to get something like adding p-power roots of unity. $B_1 = W(k)[T]/(T+1)^p - 1$ is not the correct choice, but we want $R_1 = W(k)[\zeta_p] = W(k)[\zeta_p - 1] = W(k)[T]/\frac{(T+1)^p-1}{T} = B_1/\frac{(T+1)^p-1}{T}$ which is indeed a DVR hence a regular local ring. Similiarly, R_{n+1}=$W(k)[\zeta_{p^n}]$ or more inductively $R_{n+1} = R_n[T]/([p](T) - (\zeta_{p^{n-1}} - 1))$ for $n \geq 1$.

The example suggests we shall use some conditions to cut out B_k and get the correct R_k. By definition, we have $[p^n](\eta(x)) = 0$ for any x by definition, so $(T - \eta(x))|[p^n](T)$ in $A[[T]]$. A Drinfeld level structure just requires η is surjective even when counting multiplicity, namely we require $\prod_{x \in (p^{-n}\mathbb{Z}/\mathbb{Z})^h}(T - \eta(x))|[p^n](T)$ in $A[[T]]$.

Definition 7.1. $\mathcal{M}_n$ is the functor from $\mathcal{C}$ to **Sets**:

$$A \in \mathcal{C} \mapsto \{(F, \rho, \eta)|(F, \rho) \in \mathcal{M}_0(A), \eta : (p^{-n}\mathbb{Z}/\mathbb{Z})^h \to F(A)[p^n] \text{ a Drinfeld level } n \text{ structure } \}$$

Proposition 7.1. *$\mathcal{M}_n$ is prorepresentatble by a complete local ring R_n finite over R_0 with residue field k.*

Proof. By above discussion, we know $\mathcal{M}_n = \operatorname{Spf} R_n$ where $R_n = B_n/\sim$ where $\sim$ means the ideal generated by all the coefficients of the residue term in applying Weierstrass division to $[p^n](T)$ by $\prod_{x\in(p^{-n}\mathbb{Z}/\mathbb{Z})^h}(T-\eta_{B_n}(x))$ in $B_n[[T]]$, here η_{B_n} is the universal level structure map on B_n. The claim on R_n follows from that B_n is a complete local ring finite over R_0. $\square$

We firstly study R_1. For $0 \le r \le h$, consider the functor Φ_r which associates to each $A \in \mathcal{C}$ the set of homomorphisms $\eta : (p^{-n}\mathbb{Z}/\mathbb{Z})^r \to F(A)[p^n]$ such that $\prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^r}(T-\eta(x))|[p](T)$.

Proposition 7.2. *Φ_r is represented by a complete local ring L_r with residue field k such that*

1. *L_r is regular with $\dim L_r = h$, and the image of $(p^{-1}e_i)(1 \le i \le r)$ under the universal η over L_r along with $x_j \in R_0$ $(r \le j \le h-1)$ form a system of local parameters for L_r.*
2. *The natural forgetful map gives a finite flat (hence injective) morphism $L_r \to L_{r+1}$.*
3. *The universal level map $\phi_r : (p^{-n}\mathbb{Z}/\mathbb{Z})^r \to F(L_{r-1})[p^n]$ on L_r is injective.*

Proof. Proof by induction on r, $r = 0$ is already known. Assume it's true for $r-1$, we set $\theta_i = \phi_{r-1}(p^{-1}e_i)$ and

$$g_{r-1}(T) = \frac{[p](T)}{\prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^{r-1}}(T-\phi_{r-1}(x))},$$

here we use the notation $(p^{-1}\mathbb{Z}/\mathbb{Z})^0 = 0$ so $g_0(T) = \frac{[p](T)}{T}$, and note that $g_{r-1}(T)$ lies in $L_{r-1}[[T]]$ because the induction case $r-1$ (We use the fact that regular local ring is a domain, and ϕ_{r-1} is injective).
Now we set $L_r := L_{r-1}[[\theta_r]]/(g_{r-1}(\theta_r))$, so $L_{r-1} \hookrightarrow L_r$ is finite flat by Weierstrass division theorem hence $\dim L_r = \dim L_{r-1} = h$. Recall the universal deformation group law on $R_0 = W(k)[[x_1, \ldots, x_{h-1}]]$ satisfies

$$[p](T) = pu_0T + x_1u_1T^p + \ldots + x_{h-1}u_{h-1}T^{p^{h-1}} + u_hT^{p^h}, u_i \in R_0[[T]]^\times.$$

Besides, on the ring L_{r-1}

$$[p](T) = \prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^{r-1}}(T-\phi_{r-1}(x))g_{r-1}(T)$$

Combining these two formulas on the ring $\overline{L_r} := L_r/(\theta_1, \ldots, \theta_r, x_r, \ldots x_{h-1})$ (note all $\phi_{r-1}(x)$ and one root θ_r of g_{r-1} is zero on $\overline{L_r}$ so $T^{p^{r-1}+1}|[p](T)$ on $\overline{L_r}$), we find $p, x_1, \ldots, x_{r-1}$ also become zero in $\overline{L_r}$. Hence $\theta_1, \ldots, \theta_r, x_r, \ldots x_{h-1}$ generate the maximal ideal of L_r so L_r is a complete regular local ring of dimension h, in particular an integral domain and $\theta_r \neq 0$ in L_r (Note we don't know whether θ_r is nonzero on L_r in previous construction).
Now we define $\phi_r : (p^{-n}\mathbb{Z}/\mathbb{Z})^r \to F(L_r)[p^n]$ by $\phi_r(p^{-1}e_i) := \phi_{r-1}(p^{-1}e_i) = \theta_i$ for $1 \le i \le r-1$ and $\phi_r(p^{-1}e_r) = \theta_r$. If $\phi_r(\sum_{i=1}^h a_ip^{-1}e_i) = \sum_{i=1}^r [a_i]\theta_i = 0$ on L_r for some $a_i \in \mathbb{Z}$, we get $\sum_{i=1}^{r-1}[a_i]\theta_i = 0$ on $L_{r-1} = L_r/\theta_r$. As ϕ_{r-1} is injective by induction, this implies $p|a_i$ for i less than r. So $[a_r]\theta_r = 0$, but we already know $[p]\theta_r = 0$ and $\theta_r \neq 0$ on L_r therefore $p|a_r$. In conclusion, ϕ_r is injective.
Finally, we prove L_r represents Φ_i and the universal map is just ϕ_r. Note $[p](\phi_r(x)) = 0$ for every x and ϕ_r is injective, so $\phi_r(x)$ are different roots of $[p](T) \in L_r[[T]]$ for different x. Moreover, L_r is a domain so

$$\prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^r}(T-\phi_r(x))|[p](T) \text{ in } L_r[[T]]$$

therefore Φ_k is representable by L_r. $\square$

In particular $R_1 = L_h$ is a regular local ring. Then everything becomes easy, as one characterize Drinfeld level n structure only using the p-torsion part:

Lemma 7.3. *For any group homomorphism $\eta : (p^{-n}\mathbb{Z}/\mathbb{Z})^h \to F(A)[p^n]$, η is a a Drinfeld level structure iff $\prod_{x\in(p^{-1}\mathbb{Z}/\mathbb{Z})^h}(T-\eta(x))|[p]_A(T)$.*

Proof. $(X -_F Y) = (X - Y) \times (unit)$. □

By above lemma, one sees that $R_n = R_1[[T_1, \dots, T_h]]/([p^{k-1}]T_1 - \eta(p^{-1}e_1), \dots, [p^{k-1}]T_h - \eta(p^{-1}e_h))$, where η is the universal Drinfeld level 1 strucutre. So $R_1 \hookrightarrow R_k$ is finite and flat so $\dim R_n = \dim R_1 = h$, and one sees that $R_n/(T_i) = R_1/(\eta(p^{-1}e_i)) = k$, so maximal ideal of R_n can be generated by $\dim R_n = h$ elements hence R_n is regular. So we get

Theorem 7.4. *1. $\mathcal{M}_n = \mathrm{Spf}(R_n)$, where R_n is a regular local ring with $\dim R_n = h$ and the image of a basis under the universal level structure map form a system of local parameters.*

2. The natural embedding $(p^{-n}\mathbb{Z}/\mathbb{Z})^h \hookrightarrow (p^{-n-1}\mathbb{Z}/\mathbb{Z})^h$ gives a finite flat morphism $\mathcal{M}_{n+1} \to \mathcal{M}_n$.

Now we pass to the generic fiber i.e to consider $R_n[1/p]$ over $K = W(k)[1/p]$, then

Proposition 7.5. *$R_0[1/p] \to R_n[1/p]$ is finite etale with Galois group $GL_h(\mathbb{Z}/p^n\mathbb{Z})$.*

Proof. The case $h = 1$ is clear, see [3, Theorem 2.1.2] for a brief discussion. □

Intuitively, Drinfeld level structure coincides with the naive level structure (requiring an isomorphism with $(\mathbb{Z}/p^n\mathbb{Z})^h$) on the generic fiber, hence $(M_n)_\eta$ is a $\mathrm{Aut}((\mathbb{Z}/p^n\mathbb{Z})^h)$ torsor over M_0.

Remark 7.6. *The structure of Galois groups motivates what we have done for the integral models: Note the map $GL_h(\mathbb{Z}/p^{n+1}\mathbb{Z}) \to GL_h(\mathbb{Z}/p^n\mathbb{Z})$ is surjective with kernel isomorphic to a $\mathbb{F}_p$-vector space of dimension p^{h^2}, which is a hint that R_n is determined by R_1 by just adding "[p]-th roots". Also $GL_h(\mathbb{Z}/p\mathbb{Z})$ has order $(p^h-1)(p^h-p)\dots(p^h-p^{h-1})$, which is a hint that we could define L_r and $R_1 = L_h$.*

8 Lubin-Tate Tower at infinity level is perfectoid

Scholze-Weinstein give a classification of p-divisible group in terms of linear algebra objects. Using the embedding into products of universal covers of p-divisible groups, they show that Lubin-Tate Tower at infinity level is perfectoid in some sense.

9 Canonical and quasi-canonical liftings

The theory of ccanonical and quasi-canonical liftings describe the endormophism ring of reduction mod p^n of quasi-canonical liftings. Therefore, it's a very useful tool to compute length of deformation spaces. We mention one application.

Definition 9.1. Let $j = j(\tau)$ be the elliptic modular function on the upper half-plane. For $m \geq 1$ let $\phi_m \in \mathbb{Z}[j, j']$ be the classical modular polynomials, defined by

$$\phi_m(j, j') = \phi_m(j(\tau), j'(\tau')) = \prod_{\det A = m, A \in SL_2(\mathbb{Z})M_2(\mathbb{Z})} (j(\tau) - j(A\tau'))$$

Theorem 9.1. *Let $S = \operatorname{Spec} Z[j, j'] \cong \mathbb{A}^2_{\mathbb{Z}}$ and T_m be the arithmetic divisor defined by $\phi_m = 0$. Then the cycles T_{m_1}, T_{m_2} and T_{m_3} intersect properly on S if and only if there is no positive definite binary quadratic form over $\mathbb{Z}$ which represents the three integers m_1, m_2, m_3. In this case the intersection $T_{m_1} \times_S T_{m_2} \times_S T_{m_3}$ lies over the locus in S corresponding to pairs (E_1, E_2) of elliptic curves which are supersingular in some characteristic p with $p < 4m_1m_2m_3$. The arithmetic intersection number is equal to*

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) = \sum_{p < 4m_1m_2m_3} n(p) \log p$$

where

$$n(p) = \frac{1}{2} \sum_{Q} \left(\prod_{\ell | \frac{1}{2} \det Q, \ell \neq p} \beta_\ell(Q) \right) \cdot \alpha_p(Q)$$

Here the sum is the taken over all positive definite integral ternary quadratic forms Q with diagonal (m_1, m_2, m_3) which are isotropic over $\mathbb{Q}_\ell$ for all $\ell \neq p$. Furthermore $\beta_\ell(Q)$ is a normalized representation density of Q by the $\mathbb{Z}_\ell$-lattice $M_2(\mathbb{Z}_l)$ with its norm form. Finally, $\alpha_p(Q)$ is the length of a certain local deformation space of isogenies of formal groups in characteristic p. The main ingredient of the proof is the determination of the quantity $\alpha_p(Q)$, and has some connection with Lubin-Tate spaces, see [6] for details.

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