Perfectoidization and perfect prismatic complex $\Delta_{X/A,\mathrm{perf}}$

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STAGE seminar

May 14th, 2021

Outline

1. What is perfection in char p..?
2. Construction via prismatic cohomology
3. Universal property, applications

Perfection in char p

k perfect field of char $p > 0$. $\{\text{perfect ring over } k\} \overset{forget}{\to} \{\text{rings over } k\}$ has a left adjoint:

The perfection of a k-algebra R

$R_{\text{perf}} := \text{colim}(R \overset{\phi}{\to} R \overset{\phi}{\to} R \overset{\phi}{\to} ..)$, where $\phi : x \to x^p$.

Perfect algebraic Geometry

Observations (**to be generalized later**):

- R_{perf} is a R-algebra via first term of colim, or by adjointness.
- R_{perf} is independent of k.
- $R \to R_{\text{perf}}$ is the universal map from R to a perfect k-algebra.
- Zariski closed=Strongly Zariski closed: $R_1 \twoheadrightarrow R$ a surjective map with perfect R_1, then $R \to R_{\text{perf}}$ is surjective.
- perfect rings are reduced. Frobenius is zero on higher $\pi_i, i > 0$.

Reconstruction via derived de Rham cohomology

$\mathrm{dR}_{-/k}$ = left Kan extension of de Rham complex $\Omega^*_{-/k}$ on polynomial k-algebras.

ϕ on $R \rightsquigarrow \phi_k$-semilinear endomorphism $\phi_R : \mathrm{dR}_{R/k} \to \mathrm{dR}_{R/k}$.

The perfection of $\mathrm{dR}_{R/k}$

$\mathrm{dR}_{R/k,perf} := \mathrm{colim}(\mathrm{dR}_{R/k} \stackrel{\phi_R}{\to} \mathrm{dR}_{R/k} \stackrel{\phi_R}{\to} \mathrm{dR}_{R/k} \stackrel{\phi_R}{\to} ..)$.

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The projection $\mathrm{dR}_{R/k} \to R$ gives $\mathrm{dR}_{R/k,perf} \cong R_{perf}$: we reduce to the case R is a polynomial algebra, then $d(x^p) = px^{p-1}dx = 0$, so colimits of $\Omega^i_{R/k}$ $(i > 0)$ under ϕ_R is zero.

Reconstruction via derived prismatic cohomology

$(A, I) = (W(k), p)$ the perfect prism corresponding to k. R a $A/I = k$ algebra. $\Delta_{R/A} \in D(A)$, with $R \to \overline{\Delta}_{R/A}$. $I = (p)$, $\phi(p) \subseteq (p)$, ϕ still acts on $\overline{\Delta}_{R/A}$.

Proposition

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WLOG R is a polynomial algebra. By Hodge-Tate comparison $gr_i^{HT}(\overline{\Delta}_{R/A}) = \Omega^i_{R/k}$, only need to check $gr_i^{HT}(\phi) = 0, i > 0$, $gr_0^{HT}(\phi) = \phi_R, i > 0$. WLOG $R = k[x]$. By crystalline comparison, reduce to de Rham cohomology of $\mathbb{A}^1_W$ over W, where mod p Hodge-Tate filtraion $=$ canonical filtration.

Main players

S p-complete ring, $X = \mathrm{Spf}(S)$.

Choose a perfect prism (A, I), with $A/I \to S$.

$\rightsquigarrow \Delta_{X/A} \in D(A)$, $\Delta_{X/A,\mathrm{perf}} := (\mathrm{colim}_\phi \Delta_{X/A})^\wedge \in D_{(p,I)-comp}(A)$.

The (derived) "perfectoidization" of S

$S_{\mathrm{perfd}} := \Delta_{X/A,\mathrm{perf}} \otimes_A^L A/I \in D_{p-\mathrm{comp}}(S)$.

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Why the name? Is S_{perfd} (derived) perfectoid / independent of (A, I) / the classical perfection in char p case? Is S_{perfd} **universal** ?

Why is S_{perfd} nice?

- HT comparison + goodness of $L_{X/(A/I)}$ + derived Nakayama $\rightsquigarrow$ control $\Delta_{X/A}$ hence S_{perfd}, get descent and base change.
- Study general S via reduction to nice $A/I \to S$. For nice S (e.g quasiregular semi-perfectoids, in particular perfectoids), show S_{perfd} is the classical universal perfectoid over S.

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- Study general S via reduction to nice $A/I \to S$. For nice S (e.g quasiregular semi-perfectoids, in particular perfectoids), show S_{perfd} is the classical universal perfectoid over S.
- Universality of S_{perfd} comes from **derived "functorial" universality** of $\Delta_{X/A}$, at least if $\Delta_{X/A} \otimes_A^L A/I$ is discrete.

$\Delta_{X/A}$ has a derived δ-ring (even "derived prism") structure (we need simplicial tools to see this). Therefore $\Delta_{X/A,\text{perf}}$ is a derived perfect δ-ring, and $S_{\text{perfd}} := \Delta_{X/A,\text{perf}} \otimes_A^L A/I$ is "derived perfectoid".

3 levels of universality in category theory

For $X \in \mathcal{C}$, weakly initial < "functorial" initial < initial (uniqueness).

"Functorial": for any $Y \in \mathcal{C}$, There is a map $X \to Y$, functorial on Y.

Initial object × something can be 'functorial' initial.

- By design, cohomology of a site $R\Gamma(-, \mathcal{O}) = R\lim(*)$ e.g $\Delta_{X/A}$, has good **"functorial" universality** (provided it's discrete and in the site). But the true universality i.e uniqueness is subtle: the conjecture in [BS] Lemma 7.7 seems open in general.

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- But via computations on simple examples and descent, still get universality of $S \to S_{\text{perfd}}$ in good cases, and prove things needed for comparison theorems.

Applications

Why is perfectoidization powerful? Many applications related to descent:

- Zariski closed=Strongly Zariski closed: If S is semiperfectoid, then $S \to S_{\text{perfd}}$ is surjective and universal among $S \to$ perfds .
- $\Delta_{X/A,\text{perf}} \cong R\Gamma((X/A)^{\text{perf}}_{\Delta}, O_{\Delta})$, and arc descent for $S \to S_{\text{perfd}}$ (next time).
- The étale comparison $R\Gamma_{et}(X_\eta, \mathbb{Z}/p^n) \cong (\Delta_{X/A}[1/d]/p^n)^{\phi=1}$ for perfect prisms, can be proved via descent to the perfectoidization (next time).

Now we do some recollections.

Prismatic cohomology

Conceptual leap: we don't need a Frobenius on R, only Frobenius on the test objects.

S p-complete ring, $X = \mathrm{Spf}(S)$. Choose a perfect prism (A, I), with $A/I \to S$.

The prismatic site of X over A/I

$(X/A)_\Delta$ is the opposite of the category of prisms (B, J) with a map $(A, I) \to (B, J)$ and a map $\mathrm{Sp}(B/J) \to X$ over $\mathrm{Sp}(A/I)$.

$\mathcal{O}_\Delta : (B, J) \mapsto B$. $\overline{\mathcal{O}_\Delta} : (B, J) \mapsto B/J = B/IB$.

$\rightsquigarrow \Delta_{X/A} = R\Gamma((X/A)_\Delta, \mathcal{O}_\Delta) \in D(A)$, with a natural map $S \to \overline{\Delta}_{X/A}$.

The Hodge-Tate filtration

Proposition

$\overline{\Delta}_{X/A} := \Delta_{X/A} \otimes^{L}_{A} A/I$ admits a natural increasing $\mathbb{N}$-indexed filtration, with i-th graded piece given by the derived p-completion of $\wedge^i L_{S/(A/I)}\{-i\}[-i]$.

Idea: Universal property of the de Rham complex $\rightsquigarrow$ the comparison map from de Rham to Hodge-Tate cohomology.

Reminder of Hodge-Tate in char p

k perfect field char $p > 0$. X over k smooth, relative Frobenius $F : X \to X^{(p)}$. Two filtration on de Rham complex $\Omega^*_{X/k}$:

- Hodge filtration=stupid filtration $\rightsquigarrow$ Hodge spectral sequence $E_1^{pq} = H^q(X, \Omega^p_{X/k})$.
- Conjugate filtraion=canonical filtration $\rightsquigarrow$ Hodge-Tate spectral sequence $E_2^{pq} = H^p(X, H^q(\Omega^*_{X/k}))$.

Hodge-Tate filtration is a generalization of conjugate filtration, via Cartier isomorphism.

Perfection in mixed characteristic

$\Delta_{X/A,\text{perf}} := (\text{colim}_\phi \Delta_{X/A})^\wedge \in D_{(p,I)-comp}(A)$.

The (derived) "perfectoidization" of S

$S_{\text{perfd}} := \Delta_{X/A,\text{perf}} \otimes^L_A A/I \in D_{p\text{-comp}}(S)$ a commutative algebra object.

The universal classical perfectoid of S

$S_{\text{perfd}'}$:= the universal classical perfectoid over S (**if it exists**), i.e $S_{\text{perfd}'}$ is perfectoid, and for any $S \to R$ with R perfectoid, there is a **unique** map $S_{\text{perfd}'} \to R$ extending it.

Independence of base

Proposition

Let $(A, I) \to (B, J)$ be a map of perfect prisms, and let S be a p-complete B/J-algebra. Then the natural map gives an isomorphism $\Delta_{S/A} \cong \Delta_{S/B}$. In particular, $\Delta_{S/A,\mathrm{perf}} \cong \Delta_{S/B,\mathrm{perf}}$, S_{perfd} is independent of (A, I).

Proof.

Use HT comparison, we're reduced to show $L^{\wedge}_{(B/J)/(A/I)} = 0$, which follows from that A/I and B/J are both perfectoid. $\square$

Base change

The formation of $\Delta_{X/A}$ commutes with base change in the sense that for any map of bounded prisms $(A, I) \to (B, J)$, $\Delta_{X_B/B} = B \otimes_A^L \Delta_{X/A}$. We can check directly that $S \to S_{\mathrm{perfd}'}$ also commutes with base change of the perfect prism.

$S_{\text{perfd}} = S = S_{\text{perfd}'}$ for perfectoid S

So if S perfectoid, $(S/A)_{\Delta}$ has an object $(A_{inf}(S), \text{Ker}\theta_S)$, hence a map $\Delta_{S/A} \to A_{inf}(S)$, it's an isomorphism: apply derived Nakayama and HT, done by $L^{\wedge}_{S/(A/I)} = 0$. We see $S_{\text{perfd}} = S$, in particular discrete. Note $(A_{inf}(S), \text{Ker}\theta_S)$ is also the initial object in $(S/A)_{\Delta}$:

Proposition

Let (A, I) be a perfect prism corresponding to a perfectoid ring $R = A/I$. Then for any prism (B, J), any map $A/I \to B/J$ of commutative rings lifts uniquely to a map $(A, I) \to (B, J)$ of prisms.

Proof: use the relation between deformation theory and cotangent complex, we're done by $L^{\wedge}_{A/\mathbb{Z}_p} = 0$.

The case S is semiperfectoid

Proposition

If S is semiperfectoid i.e there is a surjection $R \to S$ with R perfectoid, then $S_{\mathrm{perfd}'}$ exists.

Proof.

We can cut out the perfect prism for $S_{\mathrm{perfd}'}$ inside the perfect prism $(A_{inf}(R), d)$ for R (we know $R_{\mathrm{perfd}'} = R = R_{\mathrm{perfd}}$), using the kernel of $A_{inf}(R) \to S$. We need to do transfinite induction, to make it both d-torsion free and derived complete. □

Derived "functorial" universality of $\Delta_{X/A}$

Proposition

Assume $\overline{\Delta}_{X/A}$ is concentrated in degree zero. Then the pair $(\Delta_{X/A}, I\Delta_{X/A})$ gives a prism over (A, I), with a map $R \to \overline{\Delta}_{X/A}$. For any prism (B, J) over (A, I) equipped with a map $R \to B/J$, there is a map $(\Delta_{X/A}, I\Delta_{X/A}) \to (B, J)$, functorial on (B, J).

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Proof.

$\Delta_{X/A} = R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}) = R\lim_{(B,J)}(B, J)$. Use Cech-Alexander complexes, and the canonical simplicial resolution of X, we see the existence of a derived δ-structure i.e a section of $W_2(-) \to (-)$ on $\Delta_{X/A}$. $\Delta_{X/A}$ is discrete by assumption, so $(\Delta_{X/A}, I\Delta_{X/A})$ gives a prism over (A, I). The universality is clear by definition of $R\Gamma$. □

Discreteness of perfection when S is semiperfectoid

For any X, $\Delta_{X/A,perf}$ always lies in $D^{\geq 0}$ i.e $H^i(\Delta_{X/A,perf}) = 0, i < 0$, because Frobenius is zero on higher homotopy groups $\pi_i, i > 0$.

If S is semiperfectoid, then $\Omega^1_{S/(A/I)} = 0$. By HT comparison, this implies $L_{X/A}[-1], \Delta_{X/A,\mathrm{perf}} \in D^{\leq 0}$. So $\Delta_{X/A,\mathrm{perf}}$ is discrete. By previous proposition, it's a classical perfect δ-ring and d-torsion free. Hence S_{perfd} is discrete and perfectoid. By equivalence of perfectoid rings and perfect prisms, we see

Proposition

If S is semiperfectoid, then $S \to S_{\mathrm{perfd}}$ satisfies "functorial" universality among $S \to$ perfds, in particular there is a section $S_{\mathrm{perfd}} \to S_{\mathrm{perfd}'}$ to the map $S_{\mathrm{perfd}'} \to S_{\mathrm{perfd}}$.

Andre's Flatness lemma

Proposition

Let R be a perfectoid ring. For any set $\{f_s \in R\}_{s\in I}$ of elements of R, there exists a p-completely faithfully flat map $R \to R_\infty$ of perfectoid rings such that each f_s admits a compatible system of p-power roots in R_∞. In other words, the map $\# : R^\flat \to R$ is surjective locally for the p-completely flat topology.

Andre's Flatness lemma

WLOG $\#I = 1$. Let S be the p-adic completion of $R[x^{1/p^\infty}]/(x - f)$, so $R \to S$ is p-completely faithfully flat. We reduce the problem to S, but S is not perfectoid in general. We only know S is semiperfectoid.

Let (A, I) be the perfect prism corresponding to R. We shall show that S_{perfd} solves the problem. We know discreteness and perfectoidness, and only need to check S_{perfd} is p-completely faithfully flat over R.

Andre's Flatness lemma

It suffices to show $A \to \Delta_{S/A,\mathrm{perf}}$ is (p, I)-completely faithfully flat, which is implied by that $A \to \Delta_{S/A}$ is (p, I)-completely faithfully flat. So we only need to check $\overline{\Delta}_{S/A}$ is p-completely faithfully flat over R. Use HT filtration, we see it's faithful as the zero graded piece $gr_0 = R$. We only need to check $L_{S/R}[-1]$ (noting $\wedge^i L_{S/R}[-i] = \wedge^i (L_{S/R}[-1])$) is p-complete flat over R.

Andre's Flatness lemma

Consider $R \to R[x^{1/p^\infty}] = R' \twoheadrightarrow S$. $L^\wedge_{R'/R} = 0$ by perfectoidness. We only need to show $\wedge^i L_{S/R'}[-i]$ is p-completely faithfully flat over R. But S is the quotient of R' by non-zero divisor $x - f$, so $L_{S/R'}[-1]$ is simply isomorphic to S, hence p-completely flat over R. We're done.

Zariski closed=Strongly Zariski closed

Proposition

Let R be a perfectoid ring, and let $S = R/J$ be a p-complete quotient (so S is semiperfectoid). Then there is a universal map $S \to S'$ with S' being a perfectoid ring. Moreover, this map is surjective.

We just need to check $S \to S_{\mathrm{perfd}}$ is surjective.

Zariski closed=Strongly Zariski closed

Assume first that the kernel $J \subseteq R$ of $R \to S$ is the p-completion of an ideal generated by a set $\{x_i\}$ of elements that lie in the image of the map $\# : R^\flat \to R$.

In this case, if J_∞ denotes the p-completion of the ideal generated by $x^{1/p^n \#}$, then check directly that the R/J_∞ is perfectoid, and $S \cong R/J \to R/J_\infty$ is the universal map from S to a perfectoid ring.

Zariski closed=Strongly Zariski closed

This proves the assertion in this case. In general, as the surjectivity of a map of p-complete R-modules can be detected after p-completely faithfully flat base change (and S_{perfd} commutes with base change), we reduce to previous case by Andre's flatness lemma.

Completely flatness

A complex M of A-modules is I-completely flat if for any I-torsion A-module N, the derived tensor product $M \otimes^L AN$ is concentrated in degree 0. This implies in particular that $M \otimes_A^L A/I$ is concentrated in degree 0, and is a flat A/I-module.

Proposition

Let (A, I) be a bounded prism. For any (p, I)-completely flat A-complex $M \in D(A)$. Then M is discrete and classically (p, I)-complete.

True universality of $S \to S_{\text{perfd}}$

In fact, one can deduce the following thing from section $7-8$ of [BS]

Proposition

Consider any p-complete ring S over a perfectoid ring A/I where (A, I) is a perfect prism.

- If $S_{\text{perfd}'}$ exists, then S_{perfd} is discrete and agrees with $S_{\text{perfd}'}$
- If S_{perfd} is discrete, then $S_{\text{perfd}'}$ exists and agrees with S_{perfd}.

True universality of $S \to S_{\text{perfd}}$

The uniqueness part in the universality can be also deduced from the compatiblity between $(-)_{\text{perfd}}$ and derived tensor product $- \otimes^{L} -$: assume S_{perfd} is discrete and R is a perfectoid ring. Let $f_1, f_2 : S_{\text{perfd}} \to R$ be two maps over S, then they induce a map $(f_1, f_2) : S_{\text{perfd}} \otimes^{L}_{S} S_{\text{perfd}} \to R$, which factors through $(S_{\text{perfd}} \otimes^{L}_{S} S_{\text{perfd}})_{\text{perfd}} = S_{\text{perfd}}$.

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Thank you!