p-divisible groups over O_C

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1 Complex story

Can we classify abelian varieties over $\mathbb{C}$?

Definition 1. A complex torus is a connected compact complex Lie group T (which must be commutative). Any complex abelian variety A gives an example $A(\mathbb{C})$, which gives a fully faithful embedding of categories by GAGA theorem.

For any complex torus T over $\mathbb{C}$, the exponential map $\mathrm{Lie}\, T \to T$ is surjective with kernel lattice Λ. $T \cong \mathbb{C}^g/\Lambda$, the lattice is naturally determined as $\Lambda \cong H_1(\mathbb{C}^g/\Lambda, \mathbb{Z})$.

Theorem 1. The functor $T \mapsto (H_1(T, \mathbb{Z}), \mathrm{Lie}(T))$ gives an equivalence between category of complex tori and the category of pairs (Λ, V) where V is a finite-dimensional $\mathbb{C}$-linear subspace, and $\Lambda \subseteq V$ is a lattice.

Noting (Λ, V) is determined by $(\Lambda, W := \mathrm{Ker}(\Lambda \otimes \mathbb{C} \to V))$ and vice versa, the latter pair precisely means a $\mathbb{Z}$-Hodge structure of weight -1 and type $(-1, 0), (0, -1)$ i.e a finite free $\mathbb{Z}$-module Λ together with a $\mathbb{C}$-linear subspace $W \hookrightarrow \Lambda \otimes \mathbb{C}$, such that $W \oplus \overline{W} = \Lambda \otimes \mathbb{C}$.

Remark: For an abelian variety, $\Lambda \otimes \mathbb{C} = \mathrm{Lie}\, EA$, $W = (\mathrm{Lie}\, A^*)^*$.

When does a complex torus come from an abelian variety? Riemann's theorem tells us iff the Hodge structure is polarizable. Here a polarization on a $\mathbb{Z}$-Hodge structure (Λ, W) of weight -1 is an alternating form $\psi : \Lambda \otimes \Lambda \to 2\pi i\mathbb{Z}$ such that $\psi(x, Cy)$ is a symmetric positive definite form on $\Lambda \otimes \mathbb{R}$, where C is Weil's operator on $W \oplus \overline{W} = \Lambda \otimes \mathbb{C}$, acting as i on W , and as $-i$ on $\overline{W}$.

Idea: Choose an embedding $A \hookrightarrow \mathbb{P}^n$ with n minimal, then the line bundle $O(1)$ defines a class in $H^2(A, \mathbb{Z}) = \mathrm{Hom}(\wedge^2\Lambda, \mathbb{Z})$, this is the Riemann form.

Conclusion: a complex abelian variety is determined by its singular homology together with the Hodge filtration. The category of complex abelian varieties is equivalent to the category of polarizable $\mathbb{Z}$-Hodge Structures of weight -1 with type $(-1, 0), (0, -1)$. A useful application:

Corollary 1. The moduli of principle polarized abelian varieties $\mathcal{A}_g$ has a complex uniformization (analytically):

$$\mathcal{A}_g(\mathbb{C}) \cong Sp_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$$

Example 1. Every complex elliptic curve $E \cong \mathbb{C}/\Lambda$, choose any basis $\Lambda = \mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau_2$, get $\tau_1/\tau_2 \in \mathbb{C}-\mathbb{R}$ well-defined up to $GL_2(\mathbb{Z})$ action, hence a point in $GL_2(\mathbb{Z})\backslash\mathbb{C}-\mathbb{R} = SL_2(\mathbb{Z})\backslash\mathbb{H}$.

It's the classification of complex tori that uses analytical property of $\mathbb{C}$, and we understand algebraic objects e.g abelian varieties using GAGA theorem.

Exercise: classification over $\mathbb{R}$.

2 p-adic story

What analytic objects are analogues of complex tori in p-adic world? In good reduction case, a good answer may be p-divisible groups over O_C viewed as formal schemes.

Definition 2. Let Nil_O be the category of O_C-algebras R on which p is nilpotent. For a p-divisible group G over O_C, regard it as a functor on Nil_O and extend it to the category of p-adic complete O_C-algebras R by $G(R) := \varprojlim_n G(R/p^n)$.

Fact: this recovers G. So from now on we still denote this formal scheme over O_C by G.

While $\mu_{p^k}(O_C) = \varprojlim_n \mu_{p^k}(O_C/p^n)$ in the classical sense, for any local p-adic complete O_C-algebra R

$$\varprojlim_n(\mu_{p^\infty}(R/p^n)) = \varprojlim_n((1+m_R)/p^n) = 1 + m_R$$

so we shall think the formal scheme μ_{p^∞} as **the open unit ball** $D_C = \{z \in C \mid |z-1| < 1\}$.

p-divisible groups are relatively easier to understand than abelian varieties, for instance we can hope a classification in terms of (semi-)linear algebra objects as above. A good analogue for singular homology H_1 is the Tate module, and we can also define Lie algebra (only as a O_C-module).

Remark 1. For G over O_C, we say G is connected iff $G[p^n]$ are connected as schemes. By proper base change for H^0, this is equivalent to that the special fiber of G is connected. We have connected-étale sequence over O_C/p^n, and it lifts to a short exact sequence over O_C.

Theorem 2. (Scholze-Weinstein) The functor $G \mapsto (T_pG, \mathrm{Lie}\, G \otimes_{O_C} C)$ gives an equivalence between category of p-divisible groups over O_C and the category of pairs (T, W), where T is a finite free $\mathbb{Z}_p$-module, and $W \subseteq T \otimes C$ is a C-linear subspace.

A useful application:

Corollary 2. PEL type Rapoport-Zink space (deformation space of p-divisible groups with PEL structure) has a perfectoid uniformization: over C at infinite level, they are perfectoid spaces (in a weak sense).

Example 2. Every 1-dimensional height 2 connected p-divisible groups over O_C gives (T, W), choose $T \cong \mathbb{Z}_p^2$, get a point $[W] \in \Omega^1 = \mathbb{P}^1(C) - \mathbb{P}^1(\mathbb{Q}_p)$ well-defined up to $GL_2(\mathbb{Z}_p)$ action (the only non-connected one is $\mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$, which can be shown using connected-étale sequence and rigidity of deformation of étale ones and multiplicative ones), here Drinfeld half space Ω^1 is the analogue of complex half plane in p-adic world.

To see why the theorem is true, let's see some examples. By definition $\mathrm{Hom}_{O_C}(\mathbb{Q}_p/\mathbb{Z}_p, G)$ is T_pG. Taking duality, $\mathrm{Hom}_{O_C}(G, \mu_{p^\infty})$ is $T_pG^\vee$, as $T_pG^\vee = \mathrm{Hom}(T_pG, \mathbb{Z}_p)$ (check this) we see

$$\mathrm{Hom}_{O_C}(\mathbb{Q}_p/\mathbb{Z}_p, G) = \mathrm{Hom}((\mathbb{Z}_p, 0), (T_pG, \mathrm{Lie}\, G \otimes_{O_C} C))$$

$$\mathrm{Hom}_{O_C}(G, \mu_{p^\infty}) = \mathrm{Hom}((T_pG, \mathrm{Lie}\, G \otimes_{O_C} C), (\mathbb{Z}_p, C))$$

So fully faithfulness is known in two baby examples, which can be thought of as the "building blocks" of p-divisible groups over O_C.

We haven't describe the map $W = \mathrm{Lie}\, G \otimes C \to T_pG \otimes C$ in the theorem. One has a tautological map $\mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} T_pG^\vee \to G^\vee$ between p-divisible groups over O_C, taking dual we get a universal map $G \to H$, where $H = T_p(G) \otimes \mu_{p^\infty}$. By definition, the induced map $T(G) \to T(H)$ is an isomorphism, and on Lie algebra (inverting p) it's $W \hookrightarrow T_pG \otimes C$. This is the desired map.

Exercise: this is injective ($\mathrm{Lie}\, A \hookrightarrow H_1(A, \mathbb{C})$ is injective by Hodge decomposition).

Let's prove the theorem. What can we reconstruct form (T, W)? **First of all, we want to reconstruct the $\mathbb{Z}_p$-module** $G(O_C)$. $G(O_C)[p^\infty]$ is easy to understand.

Proposition 1. $G(O_C)[p^\infty] = \mathbb{Q}_p/\mathbb{Z}_p \otimes T_pG$

Proof. $G(O_C)[p^k] = \varprojlim_n G(O_C/p^n)[p^k] = \varprojlim_n G[p^k](O_C/p^n) = G[p^k](O_C)$. Then apply the usual Pontryagin duality as $T_pG = \varprojlim_n G[p^k](O_C)$. ☐

How about $G(O_C)$? In complex story, exponential map is an important tool. In p-adic world, it's not good because the convergence locus is too small. Instead, we can define the logarithm map which is more suitable.

Theorem 3. For any p-adically complete and separated flat $\mathbb{Z}_p$-algebra R, and p-divisible group G over R, there is a natural $\mathbb{Z}_p$-linear logarithm map $\log_G : G(R) \to \mathrm{Lie}\, G \otimes R[1/p]$. We have a short exact sequence

$$0 \to G(R)\,[p^\infty] \to G(R) \to \mathrm{Lie}\, G \otimes R[1/p]$$

Proof. By Grothendieck-Messing theory ([3] Lemma 2.2.5), we have

$$\log_G : \ker\left(G(R) \to G\left(R/p^2\right)\right) \overset{\cong}{\to} p^2\,\mathrm{Lie}\, G$$

As multiplication by p is topologically nilpotent on $G(R)$, any section $x \in G(R)$ will have $[p^n](x)$ in the kernel for some large n, define it's image as $\log_G([p^n](x))/p^n$. The exactness is obvious from this construction. ☐

Example 3. (**Key**) For $G = \mu_{p^\infty}$, the short exact sequence is $0 \to (1 + \mathfrak{m}_C)\,[p^\infty] \to 1 + \mathfrak{m}_C \xrightarrow{\log_p} C \to 0$ and the log map $D_C \to C$ is defined by the usual power series

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{-(-x)^n}{n}$$

3 Proof of main theorem

The fully faithfulness part is known before by Fargues, and we will explain how to construct $G(O_C)$. For the tautological map $G \to H \cong \mu_{p^\infty}^h$, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & G(O_C)[p^\infty] & \longrightarrow & G(O_C) & \longrightarrow & \mathrm{Lie}\, G \otimes C \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & H(O_C)[p^\infty] & \longrightarrow & H(O_C) & \longrightarrow & \mathrm{Lie}\, H \otimes C \end{array}$$

The leftmost vertical morphism is an isomorphism (as Tate modules are the same). So the middle f is an injection.

The right square is cartesian: if $x \in H(O_C)$ maps to $\mathrm{Lie}\, G[1/p]$, as $\log_H$ is isomorphism on a small neighborhood of 0 we see $p^n x \in G(O_C)$ for large n, as $G(O_C)$ **is p-divisible** (Lemma 4.3.5. in [4]) we can find $y \in G(O_C)$ s.t $p^n y = p^n x$ then $y - x \in H(O_C)[p^n] = G(O_C)[p^n]$ so $x \in G(O_C)$.

This recovers $G(O_C)$. Doing the procedure for any "good" (R, R^+) over (C, O_C), We reconstruct G. In the language of adic spaces, this means the generic fiber determines the adic space by

$$G = \coprod_{Y \subset G_\eta^{\mathrm{ad}}} \mathrm{Spf}\, H^0\left(Y, \mathcal{O}_Y^+\right)$$

where Y runs through the connected components of G_η^{ad}. This shows the fully faithfulness.

Remark 2. The language of adic generic fiber allows us to really view functors as analytic spaces over C (the original ring can be thought as ring of functions on it), and remembers the integral structure at the same time (the original integral ring becoms ring of functions with norm not bigger than 1 on it). Then doing above things for any complete affinoid (C, O_C)-algebra (R, R^+) over (C, O_C) really recovers the space.

Definition 3. For any p-divisible group G over O_C, the generic fiber of G is (fppf) sheafification of the functor on $\mathrm{CAff}_{\mathrm{Spa}(C,O_C)}$ i.e the category of complete affinoid (C, O_C)-algebras given by

$$\left(S, S^+\right) \mapsto \varinjlim_{S_0 \subset S^+} G\left(S_0\right) = \varinjlim_{S_0 \subset S^+} \varprojlim_{n} G\left(S_0 / p^n\right)$$

where S_0 runs over all open and bounded subrings. we denote it by G_η^{ad} (which is representable as an adic space).

We define the Tate algebra $C\left\langle T_1, \ldots, T_n\right\rangle$ to be the C-algebra of power series in $C[[T_1, \ldots, T_n]]$ whose coefficients tend to zero equipped with Gauss norm i.e maximum of absolute values of coefficients. A complete affinoid (C, O_C)-algebra (S, S^+) means S is a quotient of $C\left\langle T_1, \ldots, T_n\right\rangle$ and the subring S^+ is power bounded.

The diagram updates to

$$\begin{tikzcd}
0 \arrow[r] & G_\eta^{\mathrm{ad}}[p^\infty] \arrow[r] \arrow[d] & G_\eta^{\mathrm{ad}} \arrow[r] \arrow[d] & \operatorname{Lie} G \otimes \mathbf{G}_a \arrow[d] \\
0 \arrow[r] & H_\eta^{\mathrm{ad}}[p^\infty] \arrow[r] & H_\eta^{\mathrm{ad}} \arrow[r] & \operatorname{Lie} H \otimes \mathbf{G}_a
\end{tikzcd}$$

where $\operatorname{Lie} G \otimes \mathbb{G}_a$ is the sheaf associated to $(S, S^+) \mapsto \operatorname{Lie} G \otimes_R S$.

Essential surjectivity: starting from (T, W), we can just reverse the above procedure, define H as $T_pG \otimes \mu_{p^\infty}$ and G_η^{ad} as

$$\begin{tikzcd}
G_\eta^{\mathrm{ad}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Lie} G \otimes \mathbf{G}_a \arrow[d] \\
H_\eta^{\mathrm{ad}} \arrow[r] & \operatorname{Lie} H \otimes \mathbf{G}_a
\end{tikzcd}$$

Fargues shows it's a p-divisible rigid-analytic group, and if one can verify the connected component of G_η^{ad} is isomorphic to the open unit ball, then one can construct a p-divisible G from G_η^{ad} i.e showing the formal group structure is p-divisible. By basic group scheme theory, one can show that it is an increasing union of closed balls. If C was spherical complete ($\mathbb{C}_p$ is not), we're done. Then one bypasses non-spherical completeness using Rapoport-Zink spaces.

4 Universal cover

For an elliptic curve, the universal cover of it is the vector space in the classification. We also have a notion of universal covering (which is a vector space) for p-divisible groups.

Definition 4. For a p-divisible group G over a p-adically complete $\mathbb{Z}_p$-algebras R, define the universal cover of G as a sheaf of $\mathbb{Q}_p$-vector space on Nilp_R^{op}

$$\tilde{G}(R) := \varprojlim_{[p]:G \to G} G(R)$$

For instance, $\widetilde{\mathbb{Q}_p/\mathbb{Z}_p} = \mathbb{Q}_p$. Note $\tilde{G} = \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)[p^{-1}]$ only depends on isogeny class of G. And we have a short exact sequence (Pontryagin duality)

$$0 \to T_pG \to \tilde{G}(O_C) \to G(O_C) \to 0$$

Take inverse limit along multiplication by $[p]$ of the log exact sequence $0 \to G(O_C)[p^\infty] \to G(O_C) \to \operatorname{Lie} G \otimes C \to 0$, we get

Proposition 2. There is a short exact sequence of $\mathbb{Q}_p$-vector spaces over $\mathrm{CAff}_{\mathrm{Spa}(C,O_C)}$

$$0 \to V(G)^{\mathrm{ad}}_\eta \to \tilde{G}^{\mathrm{ad}}_\eta \to \mathrm{Lie}\, G \otimes \mathbf{G}_a \to 0$$

in particular

$$0 \to V(G) \to \tilde{G}(O_C) \to \mathrm{Lie}\, G \otimes C \to 0$$

This short exact sequence is very useful e.g one can use it to compute

Conclusion: similar to the complex story, $G(O_C)$ is quotient of the universal cover by a lattice. And one can ask for which $\mathbb{Z}_p$-lattice in the universal cover, one can form the quotient to get a p-divisible group. Let's look at the universal covering more closely.

Proposition 3. (rigidity of quasi-isogeny) Let $S \to R$ be a surjection with nilpotent kernel, then the categories of p-divisible groups over R and S up to isogeny are equivalent.

In particular, for any p-divisible group G over R, the universal cover $\tilde{G}$ lifts canonically to $\tilde{G}_S$ over S with $\tilde{G}_S(S) = \tilde{G}(R)$, so it can be considered as a crystal on the infinitesimal site of R.

Corollary 3. $\tilde{G}(O_C) = \tilde{G}(O_C/p)$ (like $\varprojlim_{x\mapsto x^p} O_C/p = \varprojlim_{x\mapsto x^p} O_C$).

This motivates our interest on p-divisible group over O_C/p. Such consideration will (in the end) give the following theorem.

Theorem 4. For any height, there are only finitely many possibilities for the universal cover of p-divisible group over O_C.

This is analogous to that all complex tori with fixed dimension have same universal cover.

Remark 3. By embedding into products of univer cover of the base p-divisible group, one can show Rapoport-Zink spaces at infinite level is perfectoid.

Remark 4. Tate has already described p-divisible groups over O_K using Galois representation of the Tate module where K is a p-adic local field, see Theorem 4 and Corollary 1 in [1].

5 Dieudonne theory, classification over O_C/p

Now we want to understand the O_C/p story, where $p = 0$. Let's recall the classical theory for char p rings.

Let k be a perfect field of char $p > 0$.

Theorem 5. (Classification, Dieudonne theory)

$$\{\text{p-div gps over } k\} \cong \{\text{Dieudonné module}/W(k)\}$$

$\dim G = dim M(G)/VM(G)$, $\mathrm{ht} G = \mathrm{rank} M(G)$

Here a Diedonné module over k is a finite free $W(k)$-module M equipped with a φ-linear isomorphism $\varphi_M : M[\frac{1}{p}] \cong M[\frac{1}{p}]$ and $pM \subseteq \varphi_M(M) \subseteq M$.

Example 4. $M(\mathbb{Q}_p/\mathbb{Z}_p) = (W(k), F = p\varphi)$, $M(\mu_{p^\infty}) = (W(k), F = \varphi)$.

Example 5. $k = \overline{\mathbb{F}_p}$, we have Dieudonné-Manin classification for isocrystals, in particular we know a simple p-divisible group over k is determined up to isogeny by it's height and dimension.

Scholze-Weinstein established Dieudonne theory over O_C/p:

Theorem 6. Let $R = O_C/p$ (in general R can be quotient of a perfect ring S by a regular ideal $J \subseteq S$). The category of p-divisible groups over R (resp. up to isogeny) is equivalent to the category of finite projective $A_{crys}(R)$ (resp. $B^+_{crys}(R)$)-modules equipped with Frobenius and Verschiebung maps. And this equivalence is compatible with base change.

Here we define $R^\flat := \varprojlim_{(x\mapsto x^p)} R$, and theta map $\theta : W(R^\flat) \to R$ by $\sum_{i=0}^\infty p^i [x_i] \mapsto \sum_{i=0}^\infty p^i x_i^\#$ with $x^\# = x_0$ (the projection from $R^\flat$ to R) whose kernel is generated by a single element ζ , and $A_{crys}(R)$ to be the p-adic completion of the PD hull of the surjection $W(R^\flat) \to R$ i.e the p-adic completion of $A_{\mathrm{crys}}(R) = W\left(R^\flat\right)\left[\frac{\xi^m}{m!}\right]_{m\geq 1} \subset W\left(R^\flat\right)\left[\frac{1}{p}\right]$. And $B^+_{crys} = A_{crys}[1/p]$. Note if R is perfect, then $A_{crys}(R) = W(R)$.

Example 6. $M(\mathbb{Q}_p/\mathbb{Z}_p) = (A_{crys}(R), F = p\varphi)$, $M(\mu_{p^\infty}) = (A_{crys}(R), F = \varphi)$.

Key example: the case $\mathbb{Q}_p/\mathbb{Z}_p \to G$. Recall $\mathrm{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, G)[p^{-1}] = \tilde{G}(R)$ which is a crystal on the infinitesimal site. So the fully faithfulness in this case claims

$$M(G)^{\varphi=p} = \tilde{G}(R)$$

e.g $B^+_{crys}(O_C/p)^{\varphi=1} = \widetilde{\mathbb{Q}_p/\mathbb{Z}_p}(O_C) = \mathbb{Q}_p$, $B^+_{crys}(O_C/p)^{\varphi=p} = \widetilde{\mu_{p^\infty}}(O_C) = \varprojlim_{x\mapsto x^p} 1 + m_C$.

Remark 5. We see the category of p-divisible group over O_C is not an abelian category (but up to isogeny it is), the natural question is whether we can enlarge it into an abelian category. This motivates the category of Banach-Colmez spaces (roughly it's an extension of a finite dimensional C-vector space by a finite dimensional $\mathbb{Q}_p$-vector space). For a p-divisible group G over O_C, the short exact sequence $0 \to V(G) \to \tilde{G}(O_C) \to \mathrm{Lie}\, G \otimes C \to 0$ shows universal covers of p-divisible groups over O_C are Banach-Colmez spaces. For $G = \mu_{p^\infty}$, the short exact sequence becomes the fundamental one in p-adic Hodge theory:

$$0 \to \mathbb{Q}_p \to (B^+_{crys})^{\varphi=p} \to C \to 0$$

6 The curve

An interesting question is how the classifications over O_C and k interact. That is, we have a diagram

$$\{\text{p-divisible groups } /\mathcal{O}_C\} \longrightarrow \{(T, W)\}$$

$$\{\text{p-divisible groups }/k\} \longrightarrow \{\text{ Dieudonné modules }\}$$

What is the functor on the right side corresponding to reduction functor on the left? This functor can be described using the Fargues-Fontaine curve.

Definition 5. The Fargues-Fontaine curve over C is the regular noetherian 1-dimensional scheme $X = \mathrm{Proj}(P)$ where $P = \bigoplus_{d\geq 0} \left(B^+_{\text{crys}}\right)^{\varphi=p^d}$, $B^+_{\text{crys}} = B^+_{\text{crys}}(O_C/p)$.

The real motivation for the definition is that closed points of X classifies untilts of $C^\flat$ up to Frobenius twist.

There is a special point $\infty \in X$ corresponding to the homomorphism $\theta : B_{crys} \to C$. We write $i_\infty : \{\infty\} \to X$ for the inclusion.

The curve shares some common properties with the usual projective line. For example, the classification of vector bundles on this curve is similar and important (but harder).

Theorem 7. For any isocrystal M over $k = \overline{\mathbb{F}_p}$, one gets a vector bundle on X asscoiated to $\bigoplus_{d\geq 0} \left(M \otimes B^+_{\text{crys}}\right)^{\varphi=p^d}$. All vector bundles arise in this way. Every vector bundle is direct sum of some $O(\lambda)$ ($\lambda \in \mathbb{Q}$).

Another analogue is the geometric interpretation of Hodge structures (twistor theory).

Complex story (Simpson, Ree's construction): A twistor structure is a (holomorphic) vector bundle on $\mathbb{P}^1_{\mathbb{C}}$. Complex Hodge structure = $\mathbb{C}^\times$-equivariant twistor structure.

Moreover, we have Simpson's Meta theorem which expect that Hodge theory can be reinterpreted in twistor theory.

Real story (Simpson): Real Hodge structures can be regarded as modification of vector bundles on the twisted real projective line $\widetilde{\mathbb{P}^1_{\mathbb{R}}} := \mathbb{P}^1_{\mathbb{C}}/z \sim -\frac{1}{\bar{z}}$.

p-adic story: the Fargues-Fontaine curve is the p-adic analogue of $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$, p-divisible groups over O_C are examples of p-adic Hodge structures, it's not hard to believe one can regard p-divisible group over O_C as modification of vector bundles on the Fargues-Fontaine curve.

This along with the classification of vector bundles will (in the end) give us the following theorem:

Corollary 4. p-divisible groups over O_C are isotrivial i.e there exists a p-divisible group H over O_C and a quasi-isogeny

$$\rho : H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p \to G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$$

So $\tilde{G}(\mathcal{O}_C) \cong \tilde{H}(\mathcal{O}_C/p) \cong \left(M(H) \otimes B^+_{\text{crys}}\right)^{\varphi=p}$, and the rational log exact sequence updates to exact sequence of coherent sheaves on X:

$$0 \to \mathcal{F} := T_pG \otimes_{\mathbb{Z}_p} O_X \to \mathcal{E} = \mathcal{E}(H) \to i_{\infty *}(\mathrm{Lie}\, G \otimes C) \to 0$$

7 Complements

Moreover, there is a Hodge-Tate exact sequence $0 \to \mathrm{Lie}\, G(1)[1/p] \to T_pG \otimes \mathbb{C}_p \to \omega_{G^\vee}[1/p] \to 0$, which has no good analogue over complex number.

Questions:

1. How to reconstruct the lattice $\mathrm{Lie}\, G$ from (T, W)?
2. When does a p-divisible group over O_C come from an abelian variety?
3. What information of the abelian variety over O_C can the p-divisible group recover?
4. How about abelian varieties over C with bad reduction?
5. Given a map $G_1 \to G_2$, how to determine whether it's surjective/ injective (as fppf sheaves) from (T_i, W_i)?
6. Can we tell when G is connected? Is it always isogenous to connected times étale?
7. When does a p-divisible group over O_C/p lift to O_C/p^n and O_C? Classification over more general rings?

2. Prop 14.8.4 Lecture XIV in Berkeley lecture note. For example, if $h = 2, d = 1$ it's always from an elliptic curve. Necessary and sufficient condition to come from a formal abelian variety: G is of height $2d$ and dim d, the Newton polygon is symmetric. Idea: this problem only depends on its isogeny class (if $A[p^\infty] \to H$ is an isogeny then take A' be A quotient by the kernel), and we know over k they are from abelian varieties (as the Newton polygon is symmetric). Any G over O_C is isotrivial, so G over O_C/p is from AV. By Serre-Tate, A lifts over O_C/p^n as p-divisible group lifts, hence a formal one. We want to algebraize it, this is related to whether the polarization lifts. And one may define a polarization condition on (T, W) to see those coming from abelian varieties.

3. For instance, by classification we see an elliptic curve E has ordinary reduction iff $E[p^\infty]$ is the unique non-connected p-div group of height 2 and dim 1, this explains why the preimage of $P^1(\mathbb{Q}_p)$ under Hodge-Tate period map is the ordinary locus, while the preimage of Drinfeld upper plane is the supersingular locus. And the Hodge-Tate period map is etale on supersingular locus hence $E[p^\infty]$ can determine E (up to finitely many choices) in this case.

4. The Neron model will still give a p-divisible group over O_C. This is used in [5]: $\mathrm{Lie}\, A \subseteq TA \otimes C$ is a $\mathbb{Q}_p$-rational subspace if and only if (the abelian part of the reduction of) A is ordinary. If the Hodge-Tate filtration is close to a $\mathbb{Q}_p$- rational point, then A lies in a small neighborhood of the ordinary locus (and the converse hold). As under the action of $GSp_{2g}(\mathbb{Q}_p)$, any filtration can be mapped to one that is close to any given $\mathbb{Q}_p$-rational point.

5-6. Direct application of classification results. It's connected iff W is in the Drinfeld half space (choose a basis of T).

7. Using prismatic formalism and Scholze-Weinstein's result, recently people have developed prismatic Dieudonne theory to classify p-divisible groups over quasi-syntomic rings e.g p-complete locally complete intersection rings and perfectoid rings.

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