Ultra Weil II

Zhiyu Zhang

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This is a learning note on Cadoret's work on ultraproduct Weil II.

1 Introduction

Let X_0 be a smooth projective variety over $k_0 = \mathbb{F}_q$ of dimension n, $k = \mathbb{F}_q^{alg}$, $X = X_0 \otimes_{k_0} k$.

Question: $\#X_0(\mathbb{F}_{q^m}) =?$

Example 1.

$$\#\mathbb{P}^n(\mathbb{F}_{q^m}) = 1 + q^m + q^{2m} + \ldots + q^{mn}$$

$$\#E(\mathbb{F}_{q^m}) = 1 + q^m - \alpha^m - \beta^m \ (E \text{ elliptic curve, } \alpha\beta = q)$$

and Weil's computation for Fermat hypersurface using Jacobi sums in his famous paper.

Theorem 1. (Weil conjecture) There exists algebraic integers $\alpha_{i,j} \in \overline{\mathbb{Z}}, 0 \leq i \leq 2n$ such that

1. $\#X_0(\mathbb{F}_{q^m}) = \sum (-1)^i \alpha_{ij}^m$ for all m.
2. $\{q^{2n}/\alpha_{i,j}\}$ is the same as $\{\alpha_{2n-i,j}\}$ as multisets.
3. $\forall \tau : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $|\tau(\alpha_{ij})| = q^{\frac{i}{2}}$.

Idea: $X(\mathbb{F}_{q^m})$ is the fixed point set of Frobenius map Fr_q^m on $X(k)$. If we have a cohomology theory for X that looks like singular cohomology, then first part will follow by Lefschetz trace formula:

$$\#X(\mathbb{F}_{q^m}) = \sum_{i \geq 0} (-1)^i \mathrm{tr}((Fr^*)^m | H^i(X)).$$

and second part will follow from Poincare duality.

To really count, the coefficient shall be char 0. Other important things include finiteness, cycle map.. conditions are formularized precisely by Weil. In other words, we need a Weil cohomology theory.

For any prime number $\ell \neq p$, Grothendieck developed étale cohomology theory $H^*(X)$ with coeffcents in $\mathbb{Z}/\ell^n$ (hence by passing to limit) $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$. It has a Frobenius action by functoriality.

Example 2. $\mathbb{C}^n$ is contractible so $H^i_{sing} = 0$ for $i > 0$, over k we know $H^i(\mathbb{A}^n, \mathbb{Q}_l) = 0$. $\mathbb{CP}^1 \cong S^2$ is the sphere with $H^0, H^2 \cong \mathbb{Z}$ and $H^i = 0$ else, over k we know $H^0(\mathbb{P}^1, \mathbb{Q}_l) = \mathbb{Q}_l$, $H^2(\mathbb{P}^1, \mathbb{Q}_l) = \mathbb{Q}_l(-1)$ and $H^i = 0$ else.

Example 3. If X can be lift to $\mathbb{C}$ (still smooth and projective), then we have $H^i_{et}(X, \mathbb{Q}_\ell) = H^i_{sing}(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Q}_\ell$ (Artin's comparison theorem).

How about the last part? It's related to Frobenius purity of cohomology and proved in Weil I, then Weil II developed the relative theory for general local systems. **Another question is: why $\alpha_{ij} \in \overline{\mathbb{Z}}$?**

Proof. (assuming purity) The L-function $L(X_0, s)$ is in $(1+\mathbb{Z}[[t]]) \cap \mathbb{Q}_\ell(t) \subseteq \mathbb{Q}(t)$. $Z(X_0, s) = \prod_i P_i(T)^{(-1)^i}$, $P_i(T) = \prod_i (1-\alpha_{i,j}T)$ are coprime and we can separate them by absolute values of roots by purity. Then by uniqueness of decomposition, $P_i(T) \in \mathbb{Z}[T]$, so $\alpha_{ij} \in \overline{\mathbb{Z}}$. □

Such ℓ-independence result can be explained by the dream of universal cohomology theory: "$H^i(X, \mathbb{Z})$" $\overset{\otimes \mathbb{Q}_\ell}{\to}$ $H^i(X, \mathbb{Q}_\ell)$.

But we don't have a theory with $\mathbb{Q}$-coefficient in general:

Example 4. (Serre) E supersingular elliptic curve over k, then $End(E) \to End(H^1(X))$. If H^* is of $\mathbb{Q}$-coefficient, then tensor with $\mathbb{R}$ we get $\mathbb{H} \to M_2(\mathbb{R})$, but such ring morphism does not exist because $\mathbb{H}$ is a division algebra and both side has dimension 4.

So it's not obvious why it's reasonable to believe the dream. Today we will talk about some evidences.

Example 5. The Euler characterestic for all Weil cohomology theories are the same, it's the self-intersection number of diagonal in $X \times X$ by Lefschetz trace formula.

Also, integral story is also missing as in the above. How about $H^k(-, \mathbb{Z}_\ell)$? Does it have any torsion?

Complex picture: X compact Riemann surface of genus g, then $H^*_{sing} = \mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$.

For a connected smooth projective curve over k, $H^0 \cong \mathbb{Z}_\ell$ hence $H^2 \cong \mathbb{Z}_\ell$ by Poincare duality. H^1 is the same as its Jacobian, so $H^1 \cong \mathbb{Z}_\ell^{2g}$ by standard computation on abelian varieties using group structure.

Proposition 1. Let A be an abelian variety of dimension g over k, then $H^i(A, \mathbb{Z}_\ell) = \wedge^i H^1(A, \mathbb{Z}_\ell)$ and $H^1(A, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell^{2g}$.

Proof. See Milne's book on abelian varieties. □

But there exists torsion example e.g Enriques surfaces ($\ell = 2, p > 2$) has $H^2_{tor} \cong \mathbb{Z}_2$. This is also true in the complex setting.

If A is a finitely generated abelian group, then $A \otimes \mathbb{Z}_\ell$ is torsion free for $\ell >> 0$. So by the dream, we expect the torsion phenomenon will disappear when we consider all $\mathbb{F}_\ell$ coefficients and let $\ell \to \infty$.

Theorem 2. (Gabber) If X is a smooth projective variety over k, then for all but finitely many ℓ, $H^i(X, \mathbb{Z}_\ell)$ is torsion free. In particular, $H^i(X, \mathbb{F}_\ell)$ is uniformly bounded with respect to ℓ.

Proof. It's non-trivial but elementary once one know the gcd theorem (the use of Lefschetz pencils). By universal coefficient theorem,

$$0 \to H^i(X, \mathbb{Z}_\ell) \otimes \mathbb{F}_\ell \to H^i(X, \mathbb{F}_\ell) \to H^{i+1}(X, \mathbb{Z}_\ell)[\ell] \to 0$$

we only need to show $\dim H^i(X, \mathbb{F}_\ell) = \dim H^i(X, \mathbb{Q}_\ell)$ for $\ell >> 0$. We can do induction on the dimension using Lefschetz pencil and weak Lefschetz for mod ℓ cohomology, see [1]. H^0 case is trivial, $H^1 \cong Hom(\pi_1, \mathbb{Z}_\ell)$ is torsion free by definition. □

Remark 1. Mod ℓ cohomology is still good e.g finiteness, Poincare duality, weak Lefschetz hold. Recently, Orgogozo proves uniformity bounds for stalks $Rf_*\mathbb{Z}/\ell$ where $f : X \to S$ a is proper morphism between Noetherian schemes when ℓ varies, see [4].

Gabber's result is the starting point of ultraproduct étale cohomology.

2 Ultraproduct Étale Cohomology

Ultraproduct coefficients is a systematic way to study the phenomenon that modulo ℓ cohomology behaves asymptotically like ℓ-adic cohomology.

Definition 1. Given an index set I, an ultrafilter on I is a set U consisting of subsets of I such that:

- $\varnothing \notin U$.
- For any $A \subseteq B \subseteq I$, if $A \in U$ then $B \in U$
- if $A, B \in U$ then $A \cap B \in U$.
- (ultra) For any $A \subseteq I$, $A \in U$ or $X - A \in U$.

Example 6. Given an element x in I, the collection of all subsets containing x is an ultrafilter. These are called principal ultrafilters.

Example 7. $I \cong \mathbb{N}$ is countable, U is the collection of all cofinite subsets. This is a non-principal filter (but not a ultrafilter), and is called Fréchet filter. But we can extend this to a ultrafilter by Zorn lemma. In fact, any non-principal ultrafilter must contain Fréchet filter.

Definition 2. Given a collection of sets M_i indexed by I and an ultrafilter U on I, the ultraproduct is defined as the quotient set $\prod_{i \in I} M_i$ by the equivalence relation $(x_i) \sim (y_i)$ iff $\{i | x_i = y_i\} \in U$.

Now choose any infinite set I of prime numbers not containing p and a non-principal ultrafilter U on I, denote $F = F_U = \prod_{\ell \in I} \mathbb{F}_\ell^{alg}/U$.

Proposition 2. F_U is an algebraically closed field of characteristic zero.

Proof. For any nonzero x in F, $\{i | x_i = 0\}$ is not in U, $\{i | x_i \neq 0\}$ is in U, so the inverse make sense i.e F is a field. It's not of positive characteristic as $1/p \in F$ for any prime p. It's algebraically closed because $\mathbb{F}_\ell^{alg}$ are. □

Now for we define ultraproduct cohomology as

$$H^*(X,F) := \prod_{\ell} H^*\left(X, \mathbb{F}_\ell^{alg}\right)/\mathcal{U}$$

Remark 2. The theory of ultrafilter can be explained using prime spectrum of $\prod_{\ell\in I}\mathbb{F}_\ell^{alg}$: $m_U = \{(x_\ell)|\{\ell|x_\ell = 0\} \in U\}$ is a closed point, and F is just the residue field. Any product of fields is an absolutely flat ring, in particular the quotient map $\prod_{\ell\in I}\mathbb{F}_\ell^{alg} \to F_U$ is flat. This is useful to transform information between individual ones and the ultra one, and it explains why we don't derive in the definition.

Theorem 3. The assignment $X \mapsto H^*(X) := H^*(X,F)$ is a Weil cohomology theory with coefficient in F: Finiteness, Poincare duality, Kunneth formula, cycle map, weak Lefschetz theorem, hard Lefschetz theorem...

Proof. See SGA 4 XIV, XVII, XVIII, where finiteness, weak Lefschetz (SGA5 VII thm 7.1.), Kunneth formula, Poincare duality are proved for mod ℓ cohomology. To prove finiteness for ultraproduct, notice that Gabber's result show dimensions of mod ℓ cohomology are uniformly bounded. Other results are proved in similar way, noting torsions (in $\ell-$adic cohomology, in kernel and cokernel of integral hard Lefschetz map) are uniformly bounded so large enough ℓ will kill these torsions. □

Ultra étale cohomology theory reflects some of the integral story, and is useful as the coefficient field is char zero.

3 Ultra Weil II

Serre's example will not work if we change $\mathbb{Q}$ to $\overline{\mathbb{Q}}$, therefore we have a guideline: For smooth projective variety, there shall exist a universal cohomology theory s.t "$H^i(X,\overline{\mathbb{Q}})$" $\overset{\otimes\overline{\mathbb{Q}}_\ell}{\to}$ $H^i(X,\overline{\mathbb{Q}}_\ell)$. And for any "good" $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}_\ell$, there shall exist "$H^i(X,\mathcal{F})$" $\overset{\otimes\overline{\mathbb{Q}}_\ell}{\to}$ $H^i(X,\mathcal{F}_\ell)$.

Motivated possibly by the dream, in Weil II Deligne proposes the companion conjecture (Weil II, 1.2.10)

Theorem 4. $\mathcal{F}_\ell$ irreducible $\overline{\mathbb{Q}}_\ell$-local system on X_0 with finite determinant, then

- (Purity) $\mathcal{F}_\ell$ is pure of weight 0 : for every $x_0 \in |X_0|$ the eigenvalues of Frobenius φ acting on the stalk $\mathcal{F}_\ell$ are algebraic and have absolute value 1 for every embedding to $\mathbb{C}$.
- (Companion) For every $\ell' \neq p$, there exists a (unique) semisimple $\overline{\mathbb{Q}}_{\ell'}$-local system $\mathcal{F}'_\ell$ on X_0 compatible with $\mathcal{F}_\ell$ i.e characteristic polynomial of Frobenius at stalks are the same $\forall x_0 \in |X_0|$.

It can be used to show every $\overline{\mathbb{Q}}_\ell$-local system on X is mixed (Weil II, 1.2.9).

The curve case is essential, and there is also a theory of crystalline companion. Cadoret's work shows some integral results are true for large enough ℓ (which provides another evidence to the dream):

Theorem 5. (rough version of main theorem) Given a compatible family $\mathcal{F}_\ell$ ($\ell \neq p$) of pure $\overline{\mathbb{Q}_\ell}$-local systems $\mathcal{F}_\ell$ on X_0, choose any torison-free integral model $\mathcal{H}_\ell$, and let $\mathcal{M}_\ell$ be the reduction. Then for $\ell >> 0$,

1. $H^i(X, \mathcal{H}_\ell)$ are torsion-free.
2. residual semisimplicity/irreducibility.
3. unicity of integral models.

Idea: introduce a category of étale local systems with ultraproduct coefficients on varieties over finite fields and develop a partial theory of Frobenius weights in this setting (as ultraproduct coefficient is of char zero), we can mimic the proof in Weil II.

Let $S_{lcc}(X_0)$ be the category of sheaves of $\prod_\ell \mathbb{F}_\ell^{alg}$-modules of locally constant constructible (lcc for short) sheaves on X_0. Its objects are direct products $\underline{\mathcal{M}} = \prod_{\ell \in I} \mathcal{M}_\ell$ where $\mathcal{M}_\ell$ are local systems of $\mathbb{F}_\ell^{alg}$-modules. Then we define almost U-tame local systems.

Definition 3. Let X_0 be a smooth and geometrically connected variety over k_0. For every ultrafilter U, let $S_U^t(X_0) \subseteq S_{lcc}(X_0)$ denote the full subcategory of almost U-tame sheaves of locally constant constructible that is of those $\underline{\mathcal{M}} = \prod_{\ell \in I} \mathcal{M}_\ell$ such that

1. $\mathcal{M}_{x,U} := \underline{\mathcal{M}}_x \otimes F_U$ has finite F_U dimension;
2. There exists a connected étale cover $X' \to X$ for which the set of primes $\ell \in I$ such that $\mathcal{M}_\ell|_{X'}$ that is curve-tame is in U. Here, "curve-tame" means that for every smooth curve C over k and morphism $C \to X'$, $\mathcal{M}_\ell|_C$ is tamely ramified in the usual sense (factors through the tame fundamental group which classify covers $C_1 \to C$ s.t all points in $\overline{C} - C$ (a compactification) is tamely ramified in the field extension $k(C_1)/k(C)$).

And the category of F_U-local systems on X_0 (which is denoted by $C(X_0, F_U)$) is the quotient of $S_U^t(X_0)$ by the full subcategory of all $\underline{\mathcal{M}}$ such that $\{\ell \in I | \mathcal{M}_\ell = 0\} \in U$.

The finite rank condition is for the need to define Frobenius weight (at every stalk): V a finite dimensional F-vector space with a linear operator φ on it, we can define the weight as usual by choosing an embedding $\tau : F \hookrightarrow \mathbb{C}$.

Tameness is to avoid some wild phenomenon, in particular the cohomology groups shall be uniform bounded.

Theorem 6. $H^i(X, \mathcal{M}_U)$ are finite dimensional.

Proof. We only need to show the mod ℓ cohomology can be uniformly bounded. H^0 is true by finite rank condition, and H^{2n} is true by duality. If X_0 is a curve, we can use Grothendieck-Ogg-Shafarevich formula (here we use tameness) to show Euler charactersitic is uniformly bounded. For higher dimension, we can use Lefschetz pencils and some ramification theory to do induction. $\square$

Remark 3. Another consequence of tameness is also frequently used in the paper: When X is smooth over k, the profinite C-tame fundamental group is topologically finitely generated. In particular, every finite index subgroup is open (Nikolov-Segal). Then one can really think almost U-tame local systems as F_U-representations of fundamental groups of X_0.

As cohomology groups are finite dimensional, we can again develop Lefschetz trace formula hence a cohomological interpretation of L-function.

Theorem 7. The L-function $L(\mathcal{M}_U, T) := \prod_{x_0 \in |X_0|} \det\left(Id - T^{\deg(x_0)}\varphi_{x_0}|\mathcal{M}_x\right)^{-1}$ is equal to

$$\chi(\mathcal{M}_U, T) := \prod_{i \geq 0} \det\left(1 - T\varphi|H^i_c(X, \mathcal{M})\right)^{(-1)^{i+1}}.$$

Then one follows classical Weil II to prove the following key theorem:

Theorem 8. (Analogue of Weil II theorem 2) Let X_0 be a smooth curve over k_0, $\underline{\mathcal{M}}$ is an almost U-tame local system on X_0. If $\mathcal{M}_U$ is pure of weight w, then $H^i_c(X, \mathcal{M}_U)$ has weights $\leq w + i$.

Combined with Bertini theorem and Lefschetz pencils, this will imply purity (Ultraproduct Weil II), and geometric semisimplicity, weak Chebotarev...

Remark 4. There is a general version of weak Chebotarev density theorem for semisimple local system over any algebraic variety on a finite field.

4 Applications to uniform results

And we discuss the main theorem about integral models. Proofs of uniform results via ultraproduct coefficients not only provide more general results but they are also more elementary than the previous ones.

Previous result is proved via working with mod ℓ coefficients directly (e.g Gabber's result), and one can reinterpret the result in terms of ultraproduct coefficients. But with a good formulation of arbitrary ultraproduct coefficients Frobenius weights theory, we can work directly with ultraproduct coefficient and then deduce some uniform results on curves by formal argument (as we can choose arbitary non-principal filters).

The set up is as before.

Key obeservation: $\mathcal{F}_\ell$ is pure, so $\mathcal{M}_U = (\prod_\ell \mathcal{M}_\ell) \otimes F_U$ is also pure because reduction will not change characteristic polynomial of Frobenius.

Firstly, we know torsion-freeness:

Corollary 1. $H^i(X, \mathcal{H}_\ell)$ is torsion free.

Proof. The motivic reason is that ultra product and ℓ-adic étale cohomology have same Betti numbers. To show this, by naturality of reduction, $\mathcal{M}_\ell$ has same characteristic polynomial of Frobenius with $\mathcal{F}_\ell$ so there is an equality of L-functions. We have Lefschetz trace formula for both cohomology groups, and H^i are pure, hence by weight reason we can separate each part, this shows they have same Betti numbers. Then we deduce the uniform version by formal argument. □

Secondly, we know residual semisimplicity and irreducibility:

Corollary 2. For $\ell >> 0$, we have

1. $\mathcal{M}_\ell|_X$ is semisimple. If $\mathcal{F}_\ell$ is semisimple / irreducible, $\mathcal{M}_\ell$ is semisimple / irreducible;
2. For every geometric point x on X_0, $H_{\ell,x}^{\pi_1(X,x)} \otimes \mathbb{F}_{\ell,x} \cong \mathcal{M}_\ell^{\pi_1(X,x)}$; If $\mathcal{F}_\ell$ is semisimple, the result holds on X.

Proof. Semi-simpleness: let $\mathcal{M}'_\ell$ be the maximal $\overline{\mathbb{F}_\ell}$-semisimple sub local system of $\mathcal{M}_\ell$ on X_0 and denote the quotient by $\mathcal{M}''_\ell$, form the ultra version $\mathcal{M}'_U$, $\mathcal{M}'_U$, which are pure as $\mathcal{F}_\ell$ is. Then $H^1(X, \mathcal{M}'_U \otimes \mathcal{M}''^{\vee}_U)^{\varphi=1} = 0$ as H^1 is pure of weight 1 by ultra Weil II. As U is arbitary, this shows $H^1(X, \mathcal{M}'_\ell \otimes \mathcal{M}''^{\vee}_\ell)^{\varphi=1} = 0$ for $\ell >> 0$, i.e the $\mathrm{Ext}^1_X(\mathcal{M}''_\ell, \mathcal{M}'_\ell)^{\varphi=1} = 0$. Hence the sequence splits over X, and $\mathcal{M}_\ell|_X$ is semisimple for large enough ℓ.

The irreducible result is a little harder. □

It remains to show the uniqueness of integral model for ℓ large enough.

Corollary 3. For $\ell >> 0$, every $\overline{\mathbb{Z}_\ell}$-model $\mathcal{H}_\ell$, $\mathcal{H}'_\ell$ of $\mathcal{F}_\ell$ will be isomorphic over X. If $\mathcal{F}$ is semisimple, then they are isomorphic (i.e over X_0).

Proof.

$$\mathrm{Hom}_{\pi_1(X,x)}\left(\mathcal{H}'_{\ell,x}, \mathcal{H}_{\ell,x}\right) \to \mathrm{Hom}_{\pi_1(X,x)}\left(\mathcal{M}'_{\ell,x}, \mathcal{M}_{\ell,x}\right)$$

is surjective for $\ell >> 0$ by applying above to $\mathcal{F}_\ell \otimes \mathcal{F}_\ell^\vee$. But $\mathcal{M}_{\ell,x}$ and $\mathcal{M}'_{\ell,x}$ are semisimple with same characteristic polynomial (by the naturality of reduction representations), so they are isomorphic by Chebotarev density theorem. Then we lift the isomorphism to integral models by the surjection, it's again an isomorphism by Nakayama lemma. □

Remark 5. If $\mathcal{F}_\ell$ is simple, then it can be reduced to a representation theory problem: G profinite group, $G \to GL_n(\mathbb{Q}_\ell)$ a compatible family of representations, are the invariant lattices unique up to scaling? This can be deduced from irreduciblity of mod ℓ reps.

Remark 6. Every higher-dimensional uniform result in the paper requires X being proper (no mixed version), It seems currently there is no good notion of general almost U-tame constructible sheaves. But compactly supported version is known for smooth curves.

5 Classical story

Most of the proof of ultra Weil II will generalize Delinge's approach to Weil II using Fourier transform. By standard reduction, it suffices to show

Theorem 9. $j_0 : X_0 \hookrightarrow \mathbb{A}^1$ open subscheme, and let $\underline{\mathcal{M}}$ be an almost U-tame local system τ-pure of weight 0. If $\mathcal{M}_U|_X$ is unramified at ∞ and non-constant irreducible, then $H^1_c(\mathbb{A}^1, j_*\mathcal{M}_U)$ is of τ-weights ≤ 1.

Let's review some classical ingredients.

$$\text{naive bounds for } \#X_0(\mathbb{F}_{q^m}) \Rightarrow \text{rough estimates of poles of L-function}$$

$$\text{rough estimates + trace formula} \overset{\text{tensor product trick}}{\Rightarrow} \text{semicontinuity of weights}$$

$$\text{semicontinuity of weights} \Rightarrow \text{weight-monodromy} \Rightarrow j_{0*}(\mathcal{M} \oplus \mathcal{M}^\vee) \text{ is } \tau\text{-real}$$

$$\tau\text{-real} \overset{\text{tensor product trick}}{\Rightarrow} \tau\text{-mixed}$$

$$\text{monodromy theorem+ Fourier transform} \Rightarrow \text{purity on curves}$$

$$\text{purity on curves} \overset{\text{Lefschetz pencil}}{\Rightarrow} \text{purity in general}$$

$$\text{purity} \Rightarrow \text{semisimple geometric monodromy} \Rightarrow \text{hard Lefschetz} \Rightarrow \text{gcd theorem}$$

gcd theorem is an important corollary of Weil II, recall the statement

Theorem 10. (gcd theorem, Weil II, Thm. 4.5.1) X smooth projective dim n over $k_0 = \mathbb{F}_q$, $P^i(X,t) = det(1 - tFr|H^i(X_{\mathbb{F}_q^{alg}})) \in \mathbb{Z}[t]$. Then for every integer $d \geq 2$, and every Lefschetz pencil $X_{t\{t\in\mathbb{P}^1\}}$ of hypersurface sections of degree d of X, the polynomial $P^{n-1}(X/\mathbb{F}_q, t)$ may be reconstructed as the least common multiple of all complex polynomials $f(T) = \prod(1 - \alpha_i T)$ such that whenever $t \in \mathbb{F}_{q^r}$ s.t X_t is smooth we have $f^{(r)}(T) = \prod(1 - \alpha_i^r T)$ divides $P^{n-1}(X_t/\mathbb{F}_{q^r}, t)$.

Proof. This is a corollary of hard Lefschetz in Weil II, Thm. 4.3.9. We omit the proof. □

It has many applications e.g Katz-Messing's result that any reasonable Weil cohomology theory over finite fields will have same Betti number, satisfy purity and hard Lefschetz, see [3].

6 Langlands correspondence with ultra-coefficients

Deligne's companion conjecture can also be extend to ultraproduct coefficients.

The original proof given by Drinfeld-Lafforgue using Langlands correspondence over function field. In a similar way, one hopes to develop a Langlands correspondance for ultraproduct coefficients. This will again have many applications to asymptotic results e.g Deligne's finiteness theorem for ℓ-adic local systems with bounded ramification. We omit the details.

References

[1] O. Gabber, Sur la torsion dans la cohomologie ℓ-adique d'une variété, C. R. Acad. Sci., Paris, Sér. I 297(1) (1983), p. 179-182.

[2] A. Cadoret, Ultraproduct Weil II for curves and integral models in compatible fa- milies of l-adic local systems, Preprint 2018, preprint available at `https://webusers.imj-prg.fr/anna.cadoret/Travaux.html`.

[3] N. Katz, Messing, W. Some consequences of the Riemann hypothesis for varieties over finite fields. Invent. Math. 23 (1974), 73–77.

[4] F. Orgogozo, Constructibilité et modération uniformes en cohomologie étale, Compositio Mathematica, 155(4) (2019), 711-757.