# P-ADIC HALF PLANE

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## 1. Introduction

The real half plane $\mathbb{H}_\infty = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) = \mathbb{C} - \mathbb{R}$ is a complex analytic space, equipped with the natural action by $GL_2(\mathbb{R}) \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Every point $\tau \in \mathbb{H}_\infty$ gives a lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$ in $\mathbb{C}$, hence an elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. In this way, $\mathbb{H}_\infty/\mathrm{GL}_2(\mathbb{Z})$ can be thought as the moduli of complex elliptic curves. In general, modular curves $X_0(N)$ admit complex uniformization as quotient of $\mathbb{H}_\infty$ by congruence subgroups of $GL_2(\mathbb{Z})$.

There is a $p$-adic analog of $\mathbb{H}_\infty$, namely the $p$-adic Drinfeld half plane. $\mathbb{H}_p = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p) = \mathbb{C}_p - \mathbb{Q}_p$ is ($\mathbb{C}_p$-points of) a p-adic analytic space, equipped with the natural action by $GL_2(\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$. In fact, $\Omega = \mathbb{P}^1_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$ is an analytic open subspace of $\mathbb{P}^1_{\mathbb{Q}_p}$ (it's not algebraic as $\#\mathbb{P}^1(\mathbb{Q}_p) = \infty$).

There is also a combinatoric $p$-adic analog $BT = BT(PGL_2(\mathbb{Q}_p))$, namely the Bruhat-Tits tree of $PGL_2(\mathbb{Q}_p)$. Its vertices correspond to $\{\mathbb{Z}_p\text{-lattices} \subseteq V_0 = \mathbb{Q}_p^2\}$ modulo $\mathbb{Q}_p^\times$-scaling. Two lattices $L_1, L_2$ form an edge iff after scaling $L_1, L_2$, the relation $pL_1 \subsetneq L_2 \subsetneq L_1$ holds. The natural action of $GL_2(\mathbb{Q}_p)$ on lattices.

These two $p$-adic analogs are closely related: there is a natural formal model $\widehat{\Omega}$ of $\Omega$ over $\mathbb{Z}_p$, extending the action of $GL_2(\mathbb{Q}_p)$. Its special fiber is union of projective lines indexed by vertices of $BT$. $\mathbb{P}_{L_1}$ and $\mathbb{P}_{L_2}$ intersects iff $L_1, L_2$ form an edge, in which case they intersect transversally at the $\mathbb{F}_p$-point $pt_{L_1L_2} = L_2/pL_1 \in \mathbb{P}_{L_1}(\mathbb{F}_p)$. Moreover, the "boundary" of $BT$ can also be identified with $\mathbb{P}^1(\mathbb{Q}_p)$ by degenerating a chain of lattices to a line.

Recall $\tau \in \mathbb{H}_\infty$ is called a special point / CM point if $\tau$ lies in some imaginary quadratic field $K \hookrightarrow \mathbb{C}$, equivalently if $E_\tau$ has complex multiplication.

Our main interest in this short note is to understand analogs of special points, and their intersection numbers on the integral model (local heights). The general principle predicts

*Date*: March 2, 2021.

that these local intersection numbers shall be related to periods on the local automorphic side.

The note is organized as follows. In section 2, we will construct the canonical formal model $\widehat{\Omega}$ over $\mathbb{Z}_p$, following ideas in [3].

In section 3, we will analyze the fixed point locus of regular $g \in GL_2(\mathbb{Q}_p)$ on $\widehat{\Omega}_W$ for $p > 2$, following [1]. The computations are essentially the same. The point of view is based on fixed point locus, and we will use orders to determine the multiplicity in an easy way.

In section 4, we compute intersection numbers of $\mathrm{Fix}(g)$ with projective lines in special fiber of $\widehat{\Omega}$.

## 2. Construction

Fix a 2-dimensional $\mathbb{Q}_p$-vector space $V_0 \cong \mathbb{Q}_p^2$, and $\Omega = \mathbb{P}(V_0) - \mathbb{P}(V_0)(\mathbb{Q}_p) \cong \mathbb{P}^1_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$ with the action of $GL(V_0) \cong GL_2(\mathbb{Q}_p)$. For any lattice $L$ in $V_0$, the generic fiber of $\mathbb{P}(L)$ is canonically identified with $\mathbb{P}(V_0)$. Choose an basis $(e_1, e_2)$ of $L$, then we can identify $\mathbb{P}(L)$ with $\mathbb{P}^1_{\mathbb{Z}_p}$, and $\mathbb{P}(\mathbb{V}_0)(\mathbb{C}_p) \overset{L}{\cong} \mathbb{P}^1(O_{\mathbb{C}_p})$.

Consider the formal scheme $\widehat{\Omega}_L := (\mathbb{P}(L) - \mathbb{P}^1(\mathbb{F}_p))^\vee$ over $\mathrm{Spf}\mathbb{Z}_p$, where $(-)^\vee$ means completion along the ideal $(p)$. Its rigid generic fiber $\Omega_L$ over $\mathbb{Q}_p$ is an open rigid subvariety of $\Omega$, with $\mathbb{C}_p$-points

$$\Omega_L(\mathbb{C}_p) = \widehat{\Omega}_L(O_{\mathbb{C}_p}) = \lim_n \widehat{\Omega}_L(O_{\mathbb{C}_p}/p^n) = \mathbb{P}^1(\mathbb{C}_p) - red^{-1}(\mathbb{P}^1(\mathbb{F}_p)) \subseteq \Omega(\mathbb{C}_p).$$

Here red is the reduction map $\mathbb{P}^1(\mathbb{C}_p) \overset{L}{\cong} \mathbb{P}^1(O_{\mathbb{C}_p}) \to \mathbb{P}^1(\overline{\mathbb{F}_p})$. So $\Omega_L$ is a complement of finitely many open discs in $\mathbb{P}(V_0)$. More concretely, the basis $(e_1, e_2)$ provides a pair of coordinates $[X_1, X_2]$ on $\mathbb{P}(\mathbb{V}_0)$ i.e two sections of $O(1)$ that generates the line bundle $O(1)$. Let $T = X_1/X_2$ be the rational function on $\mathbb{P}(\mathbb{V}_0)$ and its restriction on $D(X_2) = \mathbb{P}(V_0) - [0,1]$ to be $z$, then we have

$$\widehat{\Omega}_L \cong \mathrm{Spf}\mathbb{Z}_p[T, (T^p - T)^{-1}]^\vee,$$

$$\Omega_L(\mathbb{C}_p) = \{z \in \mathbb{C}_p | |z| = 1\} - \{z \in \mathbb{C}_p | |z - a| < 1, \text{for some } a \in \mathbb{Z}_p\}.$$

It's easy to see for $z \in \Omega_L(\mathbb{C}_p)$, $|az + b| = \max\{|a|, |b|\}$ for any $a, b \in \mathbb{Q}_p$. So $\Omega_L(\mathbb{C}_p)$ is $GL(L) \overset{e_1,e_2}{\cong} GL_2(\mathbb{Z}_p)$-invariant. In general, we know by construction that $g\Omega_L = \Omega_{gL}$ for any $g \in GL(V_0)$. Note $GL(V_0)$ acts transitively on all lattices, so we fix one lattice $M_0 = \mathbb{Z}_p e_{01} \oplus \mathbb{Z}_p e_{02} \subseteq V_0$ as the standard lattice, and let $[M_0 M_1]$ be the standard edge where $M_1 = \mathbb{Z}_p e_{01} \oplus \mathbb{Z}_p p e_{02}$.

**Proposition 1.** If $[L_1] \neq [L_2] \in BT_0$, then $\Omega_{L_1}$ doesn't intersect with $\Omega_{L_2}$. Moreover, $\bigcup_{[L] \in BT_0} \Omega_L(\mathbb{C}_p) = \bigcup_{g \in GL_2(\mathbb{Q}_p)} g\Omega_{M_0}(\mathbb{C}_p)$ doesn't cover $\Omega(\mathbb{C}_p)$.

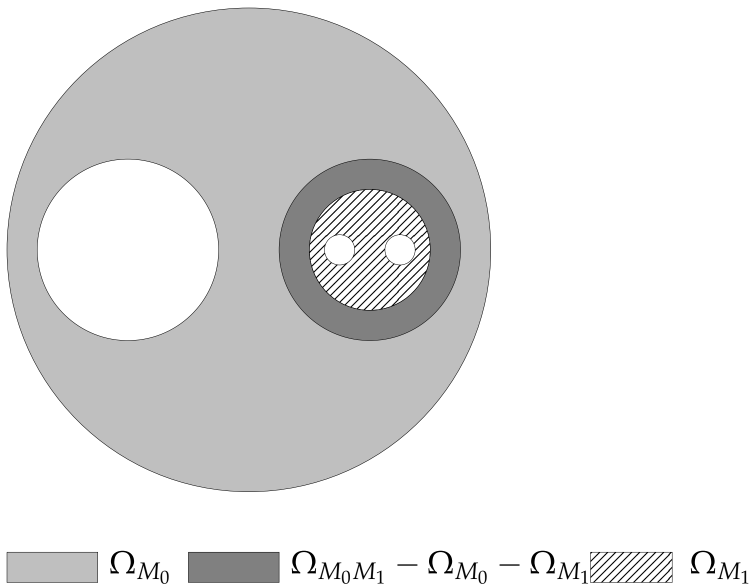
*Proof.* $\Omega(\mathbb{C}_p)$ can be identified with the collection of $\mathbb{C}^\times$-homothety classes of injective $\mathbb{Q}_p$-linear maps of $V_0$ into $\mathbb{C}_p$. $z \in \Omega_L(\mathbb{C}_p)$ corresponds to the map $f : V_0 \to \mathbb{C}_p$ such that $f(e_1) = z, f(e_2) = 1$. As $|az + b| = \max\{|a|, |b|\}$ for any $a, b \in \mathbb{Q}_p$, we see $f^{-1}(O_{\mathbb{C}_p}) =$

$L$. This shows two different $\Omega_L$ don't intersect. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p)$ and $z \in \Omega_{M_0}(\mathbb{C}_p)$, $gz = \frac{az+b}{cz+d}$, $|gz| = \frac{|az+b|}{|cz+d|} = \frac{\max\{|a|,|b|\}}{\max\{|c|,|d|\}} \in \mathbb{Q}$. So any $z \in \mathbb{C}_p$ with $|z| \notin \mathbb{Q}$ is not in $\bigcup_{[L]\in BT_0} \Omega_L(\mathbb{C}_p)$. □

What's missing in the generic fiber can be seen as follows. Under the standard basis $e_{01}, e_{02}$,

$$\Omega_{M_0}(\mathbb{C}_p) = \{z \in \mathbb{C}_p | |z| = 1\} - \cup_{a\in\mathbb{Z}_p}\{z \in \mathbb{C}_p | |z-a| < 1\},$$
$$\Omega_{M_1}(\mathbb{C}_p) = \{z \in \mathbb{C}_p | |z| = p^{-1}\} - \cup_{a\in p\mathbb{Z}_p}\{z \in \mathbb{C}_p | |z-a| < p^{-1}\}.$$

So if we define $\Omega_{M_0M_1}(\mathbb{C}_p) = \{z \in \mathbb{C}_p | p^{-1} \le |z| \le 1\} - \cup_{a\in\mathbb{Z}_p - p\mathbb{Z}_p}\{z \in \mathbb{C}_p | |z-a| < 1\} - \cup_{a\in p\mathbb{Z}_p}\{z \in \mathbb{C}_p | |z-a| < p^{-1}\}$, it will contain $\Omega_{M_0}(\mathbb{C}_p)$ and $\Omega_{M_1}(\mathbb{C}_p)$ and fill the "gap" between them naturally. The picture below explains their relations:




We need a formal model for $\Omega_{M_0M_1}$.

**Definition 1.** For any edge $[L_1L_2] \in BT_1$, let $\mathbb{P}_{L_1L_2}$ be the blow up of $\mathbb{P}(L_1)$ at the $\mathbb{F}_p$-point $pt_{L_1L_2} = L_2/pL_1 \in \mathbb{P}_{L_1}(\mathbb{F}_p)$, which is equal to the blow up o $\mathbb{P}(L_2)$ at the $\mathbb{F}_p$-point $pL_1/pL_2$. Its generic fiber is canonically identified with $\mathbb{P}(V)$, and it special fiber has an unique singular point (still denoted by $pt_{L_1L_2}$).

We define $\widehat{\Omega}_{L_1L_2}$ as the formal scheme $(\mathbb{P}_{L_1L_2} - (\mathbb{P}_{L_1L_2}(\mathbb{F}_p) - pt_{L_1L_2}))^\vee$, and its rigid generic fiber over $\mathbb{Q}_p$ by $\Omega_{L_1L_2}$.

**Proposition 2.** $\widehat{\Omega}_{M_0M_1} = \mathrm{Spf}(\mathbb{Z}_p[T_0, T_1, (T_0^{p-1}-1)^{-1}, (T_1^{p-1}-1)^{-1}]/(T_0T_1 - p))^\vee$ where $T_0 = X_2/X_1, T_1 = pX_1/X_2$. The embedding $\widehat{\Omega}_{M_0} \hookrightarrow \widehat{\Omega}_{M_0M_1}$ sends $T_0$ to $T^{-1}$, and $T_1$ to $pT_0$. In particular $T_1$ vanishes on $\widehat{\Omega}_{M_0,s}$. The embedding $\widehat{\Omega}_{M_1} \hookrightarrow \widehat{\Omega}_{M_0M_1}$ sends $T_0$ to $pT^{-1}$, and $T_1$ to $T$. In particular $T_0$ vanishes on $\widehat{\Omega}_{M_1,s}$.

It's not hard to show $\Omega_{M_0M_1}(\mathbb{C}_p)$ agrees with the first definition by hand, and all $\Omega_{L_1L_2}$ cover $\Omega$. Just as $\{z \in \mathbb{C}_\infty | |z| > 1, |Re(z)| \le 1/2\}$ gives a fundamental domain of $\mathbb{H}_\infty$ for the action of $GL_2(\mathbb{Z})$, one can think $\Omega_{M_0M_1}$ as a fundamental domain of the $p$-adic half plane for $GL_2(\mathbb{Z}_p) \times \mathrm{diag}\{\mathbb{Q}_p^\times, 1\}$.

For latter consideration of intersection theory, it's better to base change and assume the residue field is algebraically closed. We denote $W = W(\overline{\mathbb{F}_p})$ to be the ring of Witt vectors of $\overline{\mathbb{F}_p}$.

**Theorem 1.** There is a natural regular 2-dimensional formal model $\widehat{\Omega}$ over $\mathrm{Spf}Z_p$ of $\Omega$, inheriting the action of $GL(V_0) \cong GL_2(\mathbb{Q}_p)$. Moreover,

(1) Its special fiber $\Omega_s$ is reduced, and is a union of projective lines $\mathbb{P}_L$ indexed by vertices of $BT$. $\mathbb{P}_{L_1}$ and $\mathbb{P}_{L_2}$ intersects iff $L_1, L_2$ form an edge, in which case they intersect transversally at the $\mathbb{F}_p$-point $pt_{L_1L_2} = L_2/pL_1 \in \mathbb{P}_{L_1}(\mathbb{F}_p)$.

(2) $\Omega \to \mathrm{Spf}\mathbb{Z}_p$ is of strictly semi-stable reduction. In particular, for any point $x \in |\Omega_W| = |\Omega_{W,s}|$, if $x$ in the intersection of two $\mathbb{P}_L$, the completed local ring $O_x$ is isomorphic to $W[[T_0, T_1]]/(T_0T_1 - p)$; if $x$ is in $\mathbb{P}_L - \mathbb{P}_L(\mathbb{F}_p)$ for some $L$, then $O_x \cong W[[T]]$. $x$ is called a superspecial point/ ordinary point respectively.

(3) If $x = pt_{L_1L_2}$ is a superspecial point, then $\widehat{\Omega}_{L_1L_2}$ is an affinoid open neighborhood of $x$; If $x \in \mathbb{P}_L$ is an ordinary point, then $\widehat{\Omega}_L$ is an affinoid open neighborhood of $x$.

(4) The action of $GL_2(\mathbb{Q}_p)$ is compatible with the action of $GL_2(\mathbb{Q}_p)$ on $BT$. In particular, $g\mathbb{P}_L = \mathbb{P}_{gL}$.

*Proof.* Just glue $\widehat{\Omega}_{L_1L_2}$ along $\widehat{\Omega}_L$, and note the Bruhat-Tits tree is connected so they glue together to $\widehat{\Omega}$. □

## 3. FIXED POINT LOCUS

Special points over $\mathbb{H}_\infty$ can also be characterized as fixed points of regular $g \in GL_2(\mathbb{Q})$ on $\mathbb{H}_\infty$, where regularness means $g$ has two distinct eigenvalues. Note non-regular $g$ must have rational eigenvalues as $\lambda = \mathrm{Tr}(g)/2 \in \mathbb{Q}$.

For regular $g \in GL_2(\mathbb{Q}_p)$, we can also consider fixed points of $g$ on $\mathbb{H}_p = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$, and call them special points on $\mathbb{H}_p$ (note $\sqrt{p}$ is a special point on $\mathbb{H}_p$ but not on $\mathbb{H}_\infty$). Under the standard basis, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p)$ fixes $z \in \mathbb{C}_p - \mathbb{Q}_p$ iff $\frac{az+b}{cz+d} = z$ iff $cz^2 + (d-a)z - b = 0$.

Let $\mathrm{char}(g)$ be the characteristic polynomial of $g$, and $\mathbb{Q}_p[g] \subseteq M_2(\mathbb{Q}_p)$ be the 2-dimensional commutative algebra generated by $g$. By linear algebra, $\#\mathrm{Fix}(g)(\mathbb{C}_p) = 0, 2$. If $\mathrm{char}(g)$ has roots in $\mathbb{Q}_p$ i.e $\mathbb{Q}_p[g] \cong \mathbb{Q}_p \times \mathbb{Q}_p$ as algebras, then $\#\mathrm{Fix}(g)(\mathbb{C}_p) = 0$. If $\mathrm{char}(g)$ has no root in $\mathbb{Q}_p$ i.e $\mathbb{Q}_p[g]$ is a quadratic field extension of $\mathbb{Q}_p$, then $\#\mathrm{Fix}(g)(\mathbb{C}_p) = 2$.

It's better to consider the fixed point locus as a (formal) scheme. Let $K = W(\overline{\mathbb{F}_p})[1/p]$, the maximal unramified complete extension of $\mathbb{Q}_p$.

**Proposition 3.** The fixed point locus $(\Omega_K)^g = \begin{cases} \emptyset & \text{if } \mathbb{Q}_p[g] \cong \mathbb{Q}_p \times \mathbb{Q}_p \\ \mathrm{Spec}K \coprod \mathrm{Spec}K & \text{if } \mathbb{Q}_p[g] \text{unramfied field} \\ \mathrm{Spec}K[g] & \text{if } \mathbb{Q}_p[g] \text{ramified field} \end{cases}$

*Proof.* We just need to compute fixed point locus of $g$ in $\mathbb{P}^1_K$ assuming it's non-empty. For general $g$, there may not be an invariant open neighborhood of a fixed point $z$. But if $g$ is regular, we just consider the complement of $\mathbb{P}^1_K$ by the another fixed point. □

Now the question is: **what is the fixed point locus Fix($g$) on the formal model $\widehat{\Omega}_W$?** We have already computed its rigid generic fiber which is small, the next step is to compute its special fiber.

Note the center of $GL(V_0)$ acts trivially on $\widehat{\Omega}_W$, so by scaling we can assume $\mathrm{val}(\det(g)) = 0, 1$. From now on in this section, we assume $p > 2$, $\mathrm{val}(\det(g)) = 0, 1$, and work over $W$. So every $\widehat{\Omega}_L$ and $\widehat{\Omega}_{L_1L_2}$ means its base change to $W$.

**Remark 1.** The case $\mathrm{val}(\det(g)) = 1$ can be reduced to a special case of $\mathrm{val}(\det g) = 0$. We will prove later $\mathrm{Fix}(g) = \mathrm{Fix}(g + \lambda \mathrm{id})$ for any $\lambda \in \mathbb{Z}_p$ such that $g + \lambda\mathrm{id} \in GL_2(\mathbb{Q}_p)$.

**Remark 2.** It's natural to guess the structure of $\mathbb{Z}_p[g]$ will play a role. But to make $\mathbb{Z}_p[g]$ finite over $\mathbb{Z}_p$, we need $\mathrm{Tr}(g) \in \mathbb{Z}_p$. We will see shortly if $\mathrm{Tr}(g) \notin \mathbb{Z}_p$, then $\mathrm{Fix}(g)$ is only supported on generic fiber hence described completely as above.

Consider any $x \in \widehat{\Omega}_{W,s}$ fixed by $g$.

Firstly, we assume $x$ is ordinary, then there is an unique $[L] \in BT_0$ such $x \in \widehat{\Omega}_L$. So $\widehat{\Omega}_L \cap \widehat{\Omega}_{gL} \neq \emptyset$, which implies $[gL] = [L]$. As $\mathrm{val}(\det(g)) = 0, 1$, in fact $gL = L$ and $\mathrm{val}(\det(g)) = 0$. In particular, $\mathrm{char}(g) \in \mathbb{Z}_p[T]$.

**Remark 3.** This shows if $\mathrm{val}(\det(g))$ is odd, then $g$ can't fix any ordinary points, therefore the special fiber of $Fix(g)$ can only be a collection of discrete points. The case $\mathrm{val}(\det(g))$ is 0 is more interesting.

Secondly, if $x$ is superspecial, then there is an unique $[L_1L_2] \in BT_1$ such that $x \in \widehat{\Omega}_{L_1L_2} - \widehat{\Omega}_{L_1} - \widehat{\Omega}_{L_2}$. So $\widehat{\Omega}_{L_1L_2} - \widehat{\Omega}_{L_1} - \widehat{\Omega}_{L_2} \cap \widehat{\Omega}_{gL_1gL_2} - \widehat{\Omega}_{gL_1} - \widehat{\Omega}_{gL_2} \neq \emptyset$, which implies $[gL_1gL_2] = [L_1L_2]$. As $\mathrm{val}(\det(g)) = 0, 1$, we have $gL_1 = L_1, gL_2 = L_2$ if $\mathrm{val}(\det(g)) = 0$, and $gL_1 = L_2, gL_2 = pL_1$ if $\mathrm{val}(\det(g)) = 1$ (recall $pL_1 \subseteq L_2 \subseteq L_1$). In particular, $\mathrm{char}(g) \in \mathbb{Z}_p[T]$. We have proved

**Proposition 4.** For $g \in GL_2(\mathbb{Q}_p)$ such that $\mathrm{val}(\det g) = 0, 1$. If $\mathrm{Tr}(g) \notin \mathbb{Z}_p$, then $\mathrm{Fix}(g) = \mathrm{Fix}(g)|_\Omega$.

Therefore, we assume $\mathrm{char}(g) \in \mathbb{Z}_p[T]$ from now on.

**Definition 2.** For $g \in GL(V_0)$ such that $\mathrm{char}(g) \in \mathbb{Z}_p[T]$, we define its associated traceless endomorphism by $j_g := g - \frac{\mathrm{Tr} g}{2} id$. Then the finite $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[g] = \mathbb{Z}_p[j_g]$ is isomorphic to $\mathbb{Z}_p[X]/(X^2 - \det(j_g))$.

We return to determine the fixed locus of $g$.

**Proposition 5.** Let $g \in GL_2(\mathbb{Q}_p)$ such that $gL = L$, and choose a basis $(e_1, e_2)$ of $L$. Under this basis, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p)$ acts on $\widehat{\Omega}_L \cong \mathrm{Spf} W[T, (T^p - T)^{-1}]^\vee$ (recall $T = X_1/X_2$) by

$$g^* : W[T, (T^p - T)^{-1}]^\vee \to W[T, (T^p - T)^{-1}]^\vee,$$

$$T \mapsto \frac{aX_1 + bX_2}{cX_1 + dX_2} = \frac{aT+b}{cT+d}.$$

*Proof.* $g^*X_1 \in O(\mathbb{A}^2)$ satisfies $g^*X_1(e_1) = X_1(ge_1) = X_1(ae_1 + ce_2) = a$ and $g^*X_1(e_2) = X_1(ge_2) = X_1(be_1 + de_2) = b$, therefore $g^*X_1 = aX_1 + bX_2$. Similarly $g^*X_2 = cX_1 + dX_2$. We get $g^*T = \frac{aT+b}{cT+d}$. Note $cT + d | T^p - T \mod p$, so $cT + d$ is an unit mod $p$ on $W[T, (T^p - T)^{-1}]$. Hence $cT + d$ is an unit after $p$-adic completion, the ring map $g^*$ is well-defined. □

**Corollary 1.** Let $g \in GL_2(\mathbb{Q}_p)$ such that $gL = L$, then $Fix(g)|_{\widehat{\Omega}_L} = \widehat{\Omega}_L^g$ is defined by the equation $\frac{aT+b}{cT+d} = T$ i.e $cT^2 + (d-a)T - b = 0$ on $\widehat{\Omega}_L$ . It's an divisor on the 2-dimensional regular formal scheme $\widehat{\Omega}_L$. Moreover, for $\lambda \in \mathbb{Z}_p$ such that $(g + \lambda id)L = L$, we have $\mathrm{Fix}(g)|_{\widehat{\Omega}_L} = \mathrm{Fix}(g + \lambda \mathrm{id})|_{\widehat{\Omega}_L}$.

**Definition 3.** For regular $g \in GL(V_0)$ such that $gL = L$, the multiplicity $n(g, L)$ is defined as $\max\{n | d - a = c = d = 0 \mod p^n\} = \max\{n | g = \lambda \mathrm{id} \quad \text{on} \quad L/p^n L, \text{for some } \lambda \in \mathbb{Z}_p\ \}$. As $g$ is regular, $n(g, L)$ is finite.

**Proposition 6.** As cycles on $\widehat{\Omega}_L$, we have

$$\mathrm{Fix}(g)|_{\widehat{\Omega}_L} = n(g, L)(\mathbb{P}^1_L - \mathbb{P}^1_L(\mathbb{F}_p)) + \mathrm{Fix}(g)^h_L,$$

where $\mathrm{Fix}(g)^h_L$ is empty unless $\mathbb{F}_p[g_0]$ is a field, where $g_0 := p^{-n(g,L)} j_g = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ $(a_0 = -d_0, b_0, c_0 \in \mathbb{Z}_p)$. If $\mathbb{F}_p[g_0]$ is a field, then $\mathrm{Fix}(g)^h_L \cong \mathrm{Spf}W \coprod \mathrm{Spf}W$.

*Proof.* $cT^2 + (d-a)T - b = p^{n(g,L)}(c_0T^2 + (d_0 - a_0)T - b_0)$, and $\mathbb{P}^1_L - \mathbb{P}^1_L(\mathbb{F}_p)$ is defined by $p = 0$, hence we get the first part with $\mathrm{Fix}(g)^h_L = \mathrm{Spf}W[T, (T^p - T)^{-1}]^\vee/(c_0T^2 + (d_0 - a_0)T - b_0)$. As $g_0$ is not divided by $p$, $\mathrm{Fix}(g)^h_L$ will have at most 2 points, corresponding to finitely many non-rational lines invariant under $g_0$ on $L/pL$. By linear algebra, if it has a non-rational invariant line, then it has exactly two invariant non-rational lines, in which case $\mathbb{F}_p[g_0] \cong \mathbb{F}_{p^2}$. By Hensel lemma, we get $\mathrm{Fix}(g)^h_L \cong \mathrm{Spf}W \coprod \mathrm{Spf}W$. If it has no non-rational invariant line i.e $\mathrm{Fix}(g)^h_L =$, then it must have a rational invariant line, which implies $\mathbb{F}_p[g_0] \cong \mathbb{F}_p \times \mathbb{F}_p$ or $\mathbb{F}_p[x]/x^2$ □

Note $g_0$ only depends on the number $n(g, L)$. And $\mathbb{Z}_p[g_0] = \mathbb{Z}_p[p^{-n(g,L)} j_g] \cong \mathbb{Z}_p[X]/(X^2 - p^{-2n(g,L)} \det(j_g))$, therefore $n(g, L) \leq \mathrm{val}(\det(j_g))/2$. $\mathbb{F}_p[g_0]$ is a field iff $\mathbb{Q}_p[g]$ is an unramified field extension (so $\mathrm{val}(\det(j_g))$ is even), and $n(g, L) = \mathrm{val}(\det(j_g))/2$. Moreover, as $\mathrm{Fix}(g)|_\Omega$ has at most 2 points,there is only one possible $L$ such that $\mathrm{Fix}(g)^h_L \neq \emptyset$ even if $\mathbb{Q}_p[g]$ is unramified, .

In fact, we can determine $n(g, L)$ by the distance of $L$ to the "core" of $g$:

**Definition 4.** For any regular $g \in GL(V_0)$ with $\mathrm{char}(g) \in \mathbb{Z}_p[T]$, let $O_{\max}$ be the integral closure of $\mathbb{Z}_p[g]$ in $\mathbb{Q}_p[g]$. Then $O_{max} \cong \begin{cases} \mathbb{Z}_p \times \mathbb{Z}_p & \text{if } \mathbb{Q}_p[g] \cong \mathbb{Q}_p \times \mathbb{Q}_p \\ O_{\mathbb{Q}_p[g]} & \text{if } \mathbb{Q}_p[g] \text{unramfied field} \\ O_{\mathbb{Q}_p[g]} & \text{if } \mathbb{Q}_p[g] \text{ramified field} \end{cases}$

Let the core of $g$ be $B^g \subseteq BT_0$ be these lattices $[L]$ such that $O_{\max} L = L$.

**Proposition 7.** If $\mathbb{Q}_p[g]$ is split, then $B^g$ is an infinite chain given by $\{\mathbb{Z}_p p^n e_1 \oplus \mathbb{Z}_p e_2$, where $e_1, e_2$ is any basis of $V_0$ such that $e_i$ are eigenvectors of $g$. If $\mathbb{Q}_p[g]$ is unramified, then $B^g$ is a singleton. If $\mathbb{Q}_p[g]$ is ramified, then $B^g$ consists of two points, which form an edge.

*Proof.* We just need to classify $O_{\mathbb{Q}_p[g]}$ submodules in $\mathbb{Q}_p[g] \cong V_0$, which is easy as $\mathrm{val}(\det(g)) = 0, 1$ is a product of DVRs. If $\mathbb{Q}_p[g]$ is ramified, then $p$ is not an uniformizer for $\mathbb{Q}_p[g]$, that's why $B^g$ consists of two points: we're only consider equivalence under $\mathbb{Q}_p^\times$-scaling. □

**Theorem 2.** For any regular $g \in GL(V_0)$ such that $gL = L$, we have

$$n(g, L) = \log_p [O_{\max} : \mathbb{Z}_p[g]] - d(L, B^g).$$

where $d(L, B^g)$ is the distance of the vertex $[L]$ from $B^g$ on the Bruhat-Tits tree. If $gL \neq p^k L$ for any integer $k$, then $Fix(g) \cap \mathbb{P}_L^{ord} = \emptyset$.

*Proof.* $\mathbb{Z}_p[g] \cong \mathbb{Z}_p[x]/(x^2 - \det(j_g))$. If $[O_{\max} : \mathbb{Z}_p[g]] = p^n$, then $\mathbb{Z}_p[g] = \mathbb{Z}_p + p^n O_{max}$. □

Now we analyze the fixed point locus of $g$ near a superspecial point $x$. As before, there is an edge $[L_1 L_2]$ invariant under $g$. Without loss of generality, we assume $[L_1 L_2]$ is the standard edge $[M_0 M_1]$, where $M_0 = \mathbb{Z}_p e_{01} \oplus \mathbb{Z}_p e_{02} \subseteq V_0$, $M_1 = \mathbb{Z}_p e_{01} \oplus \mathbb{Z}_p p e_{02}$. Recall $gL_1 = L_1, gL_2 = L_2$ if $\mathrm{val}(\det(g)) = 0$, and $gL_1 = L_2, gL_2 = pL_1$ if $\mathrm{val}(\det(g)) = 1$. So under the standard basis, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \Gamma_0(p) \coprod \Gamma_0(p)$, where $\Gamma_0(p) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) | p | c\}$ is the Iwahori group. Recall we have shown $\widehat{\Omega}_{M_0 M_1} = \mathrm{Spf}(\mathbb{Z}_p[T_0, T_1, (T_0^{p-1} - 1)^{-1}, (T_1^{p-1} - 1)^{-1}]/(T_0 T_1 - p))^\vee$ where $T_0 = X_2/X_1$, $T_1 = pX_1/X_2$, and $g^* X_1 = aX_1 + bX_2, g^* X_2 = cX_1 + dX_2$. We get

**Proposition 8.** Let $g \in GL_2(\mathbb{Q}_p)$ such that $g[L_1 L_2] = [L_1 L_2]$, and choose a basis $(e_1, e_2)$ of $L_1$ such that $L_2 = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p p e_2$. Under this basis, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p)$ acts on $\widehat{\Omega}_{L_1 L_2}$ by

$$g^* : T_0 \mapsto \frac{dT_0 + c}{bT_0 + a}, T_1 \mapsto \frac{aT_1 + pb}{\frac{c}{p} T_1 + d}.$$

*Proof.* $T_0 = X_2/X_1, T_1 = pX_1/X_2$, $g^* X_1 = aX_1 + bX_2, g^* X_2 = cX_1 + dX_2$. Note $X_1$, $X_2$ globalize to sections of $O(1)$ on $\widehat{\Omega}$. □

**Corollary 2.** Keep assumptions as the previous proposition.

(1) If $g \in \Gamma_0(p)$, then $\widehat{\Omega}^g_{L_1 L_2}$ is given by

$$T_0(bT_0 + (a - d) + \frac{c}{p} T_1) = 0,$$
$$T_1(bT_0 + (a - d) + \frac{c}{p} T_1) = 0.$$

So $\widehat{\Omega}^g_{L_1 L_2}$ has an embedding superspecial point. The remaining part of $\widehat{\Omega}^g_{L_1 L_2}$ is an divisor on the 2-dimensional regular formal scheme $\widehat{\Omega}_{L_1 L_2}$, extending $\widehat{\Omega}^g_{L_i}$ on $\widehat{\Omega}_{L_i}$.

(2) If $g \in \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \Gamma_0(p)$, then $\widehat{\Omega}^g_{L_1L_2}$ is given by $bT_0 + a - d + \frac{c}{p}T_1 = 0$ which is a divisor.

*Proof.* Note we always have $p|c$. If $g \in \Gamma_0(p)$, then $bT_0 + a$ and $\frac{c}{p}T_1 + d$ are units, we just expand $\frac{dT_0+c}{bT_0+a} = T_0$, $\frac{aT_1+pb}{\frac{c}{p}T_1+d} = T_1$. If $g \in \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \Gamma_0(p)$, then $p|a, c, d$ and $b$ is an unit. So $bT_0 + a$ and $\frac{c}{p}T_1 + d$ are not units, but we can use $\frac{dT_0+c}{bT_0+a} = \frac{d+\frac{c}{p}T_1}{b+\frac{a}{p}T_1}$, $\frac{aT_1+pb}{\frac{c}{p}T_1+d} = \frac{a+bT_0}{\frac{c}{p}+\frac{d}{p}T_0}$ to proceed. □

Note if $\mathbb{Q}_p[g]$ ramified, then horizontal components of $\mathrm{Fix}(g)$ can only lie on superspecial points by previous analysis on ordinary locus.

From above discussion, we deduce one main theorem of Kudla-Rapoport [1]:

**Theorem 3.** For regular $g \in GL_2(\mathbb{Q}_p)$, by scaling we can assume $\mathrm{val}(\det(g)) = 0, 1$. If $\mathrm{Tr}(g) \notin \mathbb{Z}_p$, then $\mathrm{Fix}(g)$ is supported on the rigid generic fiber. If $\mathrm{Tr}(g) \in \mathbb{Z}_p$ and $\mathrm{val}(\det(g)) = 0$, then as cycles on $\widehat{\Omega}$, we have

$$\mathrm{Fix}(g) = \mathrm{Fix}(g)^h + \sum_L m(g, L)\mathbb{P}^1_L + (\text{embedding superspecial points}),$$

where

(1) $\mathrm{Fix}(g)^h = \begin{cases} \varnothing & \text{if } \mathbb{Q}_p[g] \cong \mathbb{Q}_p \times \mathbb{Q}_p \\ \mathrm{Spf}W \coprod \mathrm{Spf}W & \text{if } \mathbb{Q}_p[g] \text{unramfied field} \\ \mathrm{Spf}W[g] & \text{if } \mathbb{Q}_p[g] \text{ramified field} \end{cases}$

If $\mathrm{Fix}(g)^h$ is non-empty, then it points lie in $\mathbb{P}^1_L$ for any $L \in B^g$, the core of $g$.

(2) $m(g, L) = \max\{n(g, L), 0\} = \max\{\log_p[O_{\max} : \mathbb{Z}_p[g]] - d(L, B^g), 0\}$.

(3) $pt_{L_1L_2}$ is an embedding point of $\mathrm{Fix}(g)$ iff $gL_1 = L_1$ and $gL_2 = L_2$.

The case $\mathrm{Tr}(g) \in \mathbb{Z}_p$ and $\mathrm{val}(\det(g)) = 1$ is simpler. In this case, $m(g, L) = 0$ for all $L$ and $\mathbb{Z}_p[g] = O_{\max}$ is split or ramified, see Remark 3. The above formulas still apply, and $\mathrm{Fix}(g) = \mathrm{Fix}(g)^h$.

**Remark 4.** det on $\mathbb{Q}_p[g]$ can be identified with the norm map $\mathbb{Q}_p[g] \to \mathbb{Q}_p$.

**Example 1.** $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has no fixed points on $\Omega$, but preserves $L_n = \mathbb{Z}_p p^n e_{01} \oplus \mathbb{Z}_p e_{02}$. Its fixed point locus is union of infinitely many superspecial poiints $pt_{L_nL_{n+1}} = \mathbb{P}_{L_n} \cap \mathbb{P}_{L_{n+1}}$. This is compatible with that $\mathbb{Q}_p[g]$ is split.

**Example 2.** $g = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ switches $T_0$ and $T_1$ on $\widehat{\Omega}_{M_0M_1}$, with $\mathrm{val}(\det g) = 1$. Its fixed point locus is isomorphic to $\mathrm{Spf}W[T]/(T^2 - p)$. This is compatible with that $\mathbb{Q}_p[g]$ is ramified and $\Omega^g = \{\pm\frac{1}{\sqrt{p}}\}$.

**Example 3.** $g = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$ with $c \in \mathbb{Z}_p^\times - \mathbb{Z}_p^{\times 2}$ acts by $T \to cT^{-1}$ on $\widehat{\Omega}_{M_0}$. Its fixed point locus isomorphic to $\mathrm{Spf}W[T]/(T^2 - c) \cong \mathrm{Spf}W \coprod \mathrm{Spf}W$. This is compatible with that $\mathbb{Q}_p[g]$ is unramified and $\Omega^g = \{\pm \frac{1}{\sqrt{c}}\}$.

**Example 4.** $g = \begin{pmatrix} 1+p^n & 0 \\ 0 & 1 \end{pmatrix}$ has no fixed points on $\Omega$, but $\mathrm{Fix}(g)$ contains infinitely many $\mathbb{P}^1$.

Two important observations:

**Corollary 3.** For any regular $g \in GL_2(\mathbb{Q}_p)$, $\mathrm{Fix}(g)$ only depends on on the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[g]$. In other words, if $\mathbb{Z}_p[g_1] = \mathbb{Z}_p[g_2]$, then $\mathrm{Fix}(g_1) = \mathrm{Fix}(g_2)$.

**Corollary 4.** $\mathrm{Fix}(g) - \mathrm{Fix}(g)^h$ is locally constant for regular $g \in GL_2(\mathbb{Q}_p)$ (even for the embedding points part). If $\mathrm{char}(g) \in \mathbb{Z}_p[T]$, then $\mathrm{Fix}(g)^h$ agrees with $\mathrm{Fix}(g_0)^h$ where $g_0 \in \mathbb{Q}_p[g]$ such that $\mathbb{Z}_p[g_0] = O_{\mathbb{Q}_p[g]}$.

**Remark 5.** If $g$ is not regular and not a scalar matrix, then $\mathbb{Q}_p[g] \cong \mathbb{Q}_p[x]/x^2$, and $\mathrm{Fix}(g)$ has empty generic fiber. It's an interesting question to study limit behaviors of $\mathrm{Fix}(g)$ when $g$ tends to a non-regular element.

## 4. INTERSECTION

In this section we assume $g \in GL_2(\mathbb{Q}_p)$ is regular, $\mathrm{val}(\det g) = 0$ and $\mathrm{char}(g) \in \mathbb{Z}_p[x]$. $(-,-) := (-,-)_{\widehat{\Omega}}$ means the intersection pairing on $\widehat{\Omega}$. Let $\alpha = \alpha_g = \log_p[O_{\max} : \mathbb{Z}_p[g]]$, and $B^g$ be the core of $g$.

Then $BT(g) = \{L | n(g, L) \geq 0\} = \{L | d(L, B^g) \leq \alpha\}$ is a ball if $\mathbb{Q}_p[g]$ is unramified; it is a ball center at the middle point of the core of $g$ if $\mathbb{Q}_p[g]$ is ramified; it is a tube around the line $B^g$ if $\mathbb{Q}_p[g]$ is split.

**Proposition 9.** $(\mathbb{P}^1_{L'}, \mathbb{P}^1_L)_{\widehat{\Omega}} = 0, 1, -1-p$, if $d([L],[L']) \geq 1, = 1, = 0$ respectively.

Then we compute the intersection number $(\mathrm{Fix}(g), \mathbb{P}^1_L)$, the result is [2, Lemma 6.2]:

**Proposition 10.** For regular $g$ as above and any lattice $L$, set $d = d(L, B^g)$, $\alpha = \alpha_g = \log_p[O_{\max} : \mathbb{Z}_p[g]]$. $(\mathrm{Fix}(g), \mathbb{P}^1_L) = 0$ unless $gL = L$ i.e $d \leq \alpha$. Now assume $gL = L$.

(1) If $1 \leq d \leq \alpha - 1$, then $(\mathrm{Fix}(g), \mathbb{P}^1_L) = 1 - p$.
(2) If $\alpha \geq 1, d = 0$, then $(\mathrm{Fix}(g), \mathbb{P}^1_L) = 1 - p$.
(3) If $\alpha = 0, d = 0$, then $(\mathrm{Fix}(g), \mathbb{P}^1_L) = 2, 1, 0$, if $\mathbb{Q}_p[g]$ is unramified, ramified, split respectively.
(4) If $\alpha \geq 1, d = \alpha$, then $(\mathrm{Fix}(g), \mathbb{P}^1_L) = 1$.

*Proof.* It's zero unless $\mathrm{Fix}(g) \cap \mathbb{P}^1_L$ is non-empty, which implies $\mathbb{P}^1_L \cap \mathbb{P}^1_{gL}$ is non-empty hence $gL = L$.

Assume $1 \leq d = d(L, B^g) \leq \alpha - 1$, then all neighbors of $L$ have possible multiplicities: 1 neighbor has distance $d - 1$, $p$ neighbors have distances $d + 1$. $\mathrm{Fix}(g)^h$ doesn't intersect with $\mathbb{P}^1_L$. Therefore

$$(\mathrm{Fix}(g), \mathbb{P}^1_L) = \sum_{L'} (m(g, L')\mathbb{P}^1_{L'}, \mathbb{P}^1_L) = (\alpha - d)(-1-p) + (\alpha - d - 1)p + (\alpha - d + 1) = 1 - p.$$

Assume $d = 0, \alpha \geq 1$. Note we includes the horizontal part of $\mathrm{Fix}(g)$. In the unramified case, $(\mathrm{Fix}(g), \mathbb{P}^1_L)$ is $2 + \alpha(-1-p) + (\alpha - 1)(p+1) = 1 - p$. In the split case, it's $0 + \alpha(-1-p) + 2\alpha + (\alpha - 1)(p - 1) = 1 - p$. In the ramified case, it's $1 + \alpha(-1-p) + \alpha + (\alpha - 1)p = 1 - p$.

The case $\alpha = 0, d = 0$ follows from the number of horizontal components. If $\alpha \geq 1, d = \alpha$, then there is only one neighbor of $L$ that is closer to $B^g$ than $L$, which contributes 1.

□

**Remark 6.** Let $\omega$ be the relative dualizing sheaf of $\widehat{\Omega}$. The dualizing sheaf on a nodal curve $C$ over a field is equal to $\Omega^1_C \otimes O(D)$ where $D$ are nodes of $C$. From this we get

$$\deg(\omega|_{\mathbb{P}^1_L}) = \deg(\Omega_{\mathbb{P}^1}) + \#\mathbb{P}^1(\mathbb{F}_p) = -2 + p + 1 = p - 1.$$

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## References


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