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A new class of identities involving Cauchy numbers, harmonic numbers and zeta values

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Abstract Improving an old idea of Hermite, we associate to each natural number k a modified zeta function of order k. The evaluation of the values of these functions F_k at positive integers reveals a wide class of identities linking Cauchy numbers, harmonic numbers and zeta values.

Keywords Cauchy numbers · Bell polynomials · Harmonic numbers · Laplace–Borel transform · Mellin transform · Zeta values · Ramanujan summation · Hermite's formula

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1 Introduction

It has been well known since the second half of the nineteenth century that the Riemann zeta function may be represented by the (normalized) Mellin transform (cf. [14])

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\frac{e^{-t}}{1-e^{-t}}\,dt \quad \text{for } \Re(s)>1,$$

and from late works of Hermite (cf. [11]) that one also has

$$\zeta(s)-\frac{1}{s-1}=\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\frac{e^{-t}}{1-e^{-t}}\left(\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\left(1-e^{-t}\right)^n\right)dt \quad \text{for } \Re(s)\geq 1,$$

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where $\lambda_1 = \frac{1}{2}$ and $\lambda_{n+1} = \int_0^1 x(1-x)\cdots(n-x)\,dx$ are the (non-alternating) Cauchy numbers.[1]

Improving Hermite's idea, one may, more generally, consider Mellin transforms of type

$$F(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f\left(1-e^{-t}\right) dt \quad \text{with } f(z) = \sum_{n=1}^{\infty} \omega_n \frac{z^n}{n^k}$$

for suitable sequences $(\omega_n)_{n\geq 1}$ of rational numbers. The simplest interesting case $\omega_n = 1$ corresponds to the *Arakawa–Kaneko zeta function* and has been studied extensively in [8]. In this article, we investigate the case $\omega_n = \frac{\lambda_n}{n!}$, i.e., we study the function

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_k\left(1-e^{-t}\right) dt$$
$$\text{with } f_k(z) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{z^n}{n^k} \; (k = 0, 1, 2, \ldots),$$

which is a priori defined in the half-plane $\Re(s) \geq 1$ but analytically continues in the whole complex s-plane (Theorem 7). We call this function F_k the *modified zeta function of order k*. An evaluation by two different methods of the values of F_k at positive integers q leads to a new class of identities linking Cauchy numbers, harmonic numbers, and zeta values. In the case $k = 0$, *Hermite's formula* for ζ (cf. [7]) is regained, i.e.,

$$F_0(q) = \zeta(q) - \frac{1}{q-1} = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} P_{q-1}\left(H_n^{(1)}, H_n^{(2)}, \ldots, H_n^{(q-1)}\right),$$

where the polynomials P_m are the *modified Bell polynomials* defined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1, \ldots, x_m) z^m,$$

evaluated at harmonic numbers $H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m}$. In the simplest higher case $k = 1$, this extension of Hermite's formula leads to the following new relation (Theorem 10):

$$F_1(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}\left(H_n, H_n^{(2)}, \ldots, H_n^{(q-1)}\right)$$
$$= \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}},$$

[1] The sequence of numbers $\frac{\lambda_n}{n!}$ appeared for the first time in a letter of James Gregory dated back to 1670 (cf. *The correspondence of Isaac Newton*, vol. 1, p. 46). For this reason, they are sometimes called *Gregory coefficients*.

where $H_n = H_n^{(1)}$, and $\gamma = \lim_{n\to\infty}(H_n - \log n)$ is the Euler–Mascheroni constant.

For example, for $q = 2$, since $P_1(H_n) = H_n$ and $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$ (cf. [6, 7]), then the previous relation may be written

$$F_1(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n^2} = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1,$$

and this generalizes the known formula

$$F_0(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n} = \zeta(2) - 1.$$

The function F_k also has an interesting interpretation in terms of Ramanujan summation (cf. [3]) as underscored by Theorem 11. In particular, one shows the identity

$$F_k(1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n\geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n},$$

where, in the right member, $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the sum (in the sense of Ramanujan) of the divergent series. This raises a kind of reciprocity between $F_k(1)$ and $F_0(k+1)$.

2 Preliminaries

2.1 The non-alternating Cauchy numbers

Definition 1 The *non-alternating Cauchy numbers* (cf. [7, 12]) are the sequence of (positive) rational numbers $(\lambda_n)_{n\geq 1}$ defined by the exponential generating function

$$\frac{z}{\log(1-z)} + 1 = \sum_{n\geq 1} \frac{\lambda_n}{n!} z^n. \tag{1}$$

Dividing by z and setting $z = 1 - e^{-t}$ and $t > 0$, this relation may be rewritten

$$\frac{1}{1-e^{-t}} - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \left(1 - e^{-t}\right)^{n-1}. \tag{2}$$

From (1), one may easily deduce the following recursive relation:

$$\sum_{j=1}^{n} \frac{\lambda_j}{j!(n-j+1)} - \frac{1}{n+1} = 0 \quad \text{for } n \geq 1.$$

Example 1 The first non-alternating Cauchy numbers are

$$\lambda_1 = \frac{1}{2}, \qquad \lambda_2 = \frac{1}{6}, \qquad \lambda_3 = \frac{1}{4}, \qquad \lambda_4 = \frac{19}{30}, \qquad \lambda_5 = \frac{9}{4}.$$

2.2 The modified Bell polynomials evaluated at harmonic numbers

Definition 2 The *modified Bell polynomials* (cf. [5, 7, 10]) are the polynomials P_m defined for all natural numbers m by $P_0 = 1$ and the generating function

$$\exp\left(\sum_{k\geq 1} x_k \frac{z^k}{k}\right) = 1 + \sum_{m\geq 1} P_m(x_1,\ldots,x_m)\, z^m. \tag{3}$$

The general explicit expression for P_m is

$$P_m(x_1,\ldots,x_m) = \sum_{k_1+2k_2+3k_3+\cdots=m} \frac{1}{k_1!k_2!k_3!\cdots}\left(\frac{x_1}{1}\right)^{k_1}\left(\frac{x_2}{2}\right)^{k_2}\left(\frac{x_3}{3}\right)^{k_3}\cdots.$$

One may also compute recursively the polynomials P_m by means of the following relation:

$$m\,P_m(x_1,\ldots,x_m) = \sum_{k=1}^{m} x_k\, P_{m-k}(x_1,\ldots,x_{m-k}) \quad (m\geq 1).$$

Proposition 1 *For all natural numbers m, and each integer $n\geq 1$,*

$$\int_0^{+\infty} e^{-t}\left(1-e^{-t}\right)^{n-1}\frac{t^m}{m!}\,dt = \frac{P_m(H_n,\ldots,H_n^{(m)})}{n}, \tag{4}$$

with

$$H_n^{(m)} = \sum_{j=1}^{n}\frac{1}{j^m} \quad \textit{and} \quad H_n = H_n^{(1)}.$$

Proof One starts from the classical Euler relation (cf. [14])

$$\mathrm{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

and substitute $u = e^{-t}$, $a = 1-z$, and $b = n+1$; then one obtains

$$\int_0^{+\infty} e^{-t}\left(1-e^{-t}\right)^{n} e^{tz}\,dt = \frac{n!}{(1-z)(2-z)\cdots(n+1-z)}.$$

Moreover, one has

$$\begin{aligned}\frac{n!}{(1-z)(2-z)\cdots(n+1-z)} &= \frac{n!}{(n+1)!}\times\prod_{j=0}^{n}\left(1-\frac{z}{j+1}\right)^{-1}\\ &= \frac{1}{(n+1)}\times\exp\left(-\sum_{j=0}^{n}\log\left(1-\frac{z}{j+1}\right)\right)\end{aligned}$$

$$= \frac{1}{(n+1)} \times \exp\left(\sum_{j=0}^{n}\sum_{k=1}^{\infty} \frac{z^k}{k(j+1)^k}\right)$$

$$= \frac{1}{(n+1)} \exp\left(\sum_{k=1}^{\infty} H_{n+1}^{(k)} \frac{z^k}{k}\right)$$

$$= \sum_{m=0}^{\infty} \frac{P_m(H_{n+1}^{(1)}, \dots, H_{n+1}^{(m)})}{n+1} z^m \quad \text{(by (3))}.$$

Thus (4) results by identification of the term in z^m. □

Example 2 For small values of m, one has

$$P_1(H_n) = H_n,\ P_2\big(H_n, H_n^{(2)}\big) = \frac{(H_n)^2}{2} + \frac{H_n^{(2)}}{2},$$

$$P_3\big(H_n, H_n^{(2)}, H_n^{(3)}\big) = \frac{(H_n)^3}{6} + \frac{H_n H_n^{(2)}}{2} + \frac{H_n^{(3)}}{3}.$$

2.3 The Laplace–Borel transformation

We consider the vector space E of complex-valued functions $f \in \mathcal{C}^1(]0, +\infty[)$ such that

$$\text{for all } \varepsilon > 0, \text{ there exists } C_\varepsilon > 0 \quad \text{such that} \quad \big|f(t)\big| \le C_\varepsilon e^{\varepsilon t} \quad \text{for all } t \in]0, +\infty[.$$

In particular, a function $f \in E$ satisfies the following two properties:

(a) for all x with $\Re(x) > 0$, $t \mapsto e^{-xt} f(t)$ is integrable on $]0, +\infty[$,
(b) for all β with $0 < \beta < 1$, $t \mapsto |f(t)| \frac{1}{t^\beta}$ is integrable on $]0, 1[$.

We recall now some basic properties (cf. [13]) of the Laplace transformation in this frame which are appropriate for our purpose.

Definition 3 Let f be a function in E. The *Laplace transform* $\mathcal{L}(f)$ of f is defined by

$$\mathcal{L}(f)(x) = \int_0^{+\infty} e^{-xt} f(t)\, dt \quad \text{for } \Re(x) > 0.$$

Proposition 2 (cf. [13]) *Let $\mathcal{E} = \mathcal{L}(E)$ be the image of E under $\mathcal{L}$. If a is a function in $\mathcal{E}$, then*

(a) *a is an analytic function of x in the half-plane $\Re(x) > 0$,*
(b) *$a(x) \to 0$ when $\Re(x) \to +\infty$,*
(c) *$\mathcal{L} : E \to \mathcal{E}$ is an isomorphism.*

Definition 4 Let $a \in \mathcal{E}$. The *Borel transform* of a is the unique function $\widehat{a} \in E$ such that $a = \mathcal{L}(\widehat{a})$. One has the two reciprocal formulas

$$\widehat{a}(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{zt} a(z)\, dz \quad \text{for all } c > 0 \text{ and } t > 0,$$

and

$$a(x) = \int_0^{+\infty} e^{-xt} \widehat{a}(t)\, dt \quad \text{for } \Re(x) > 0.$$

Definition 5 Let f and g be two functions in E. The *convolution product* $f * g$ of f and g is the function defined for all $t > 0$ by

$$(f * g)(t) = \int_0^t f(u) g(t-u)\, du.$$

Proposition 3 (cf. [13]) *If $f \in E$ and $g \in E$, then $f * g \in E$ and*

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g). \tag{5}$$

Hence, if $a \in \mathcal{E}$ and $b \in \mathcal{E}$ then $ab \in \mathcal{E}$ since $ab = \mathcal{L}(\widehat{a} * \widehat{b})$.

Theorem 1 *Let a be a function in $\mathcal{E}$. Then the series*

$$\sum_{n \geq 1} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t} \left(1 - e^{-t}\right)^{n-1} \widehat{a}(t)\, dt$$

converges and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t} \left(1 - e^{-t}\right)^{n-1} \widehat{a}(t)\, dt = \int_0^{+\infty} \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} \widehat{a}(t)\, dt. \tag{6}$$

Proof By (2)

$$\int_0^{+\infty} \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} \widehat{a}(t)\, dt = \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \left(1 - e^{-t}\right)^{n-1} e^{-t} \widehat{a}(t)\, dt.$$

In the right member, the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged since

$$\begin{aligned}
\int_0^{+\infty} \sum_{n=1}^{\infty} \left| \frac{\lambda_n}{n!} \left(1 - e^{-t}\right)^{n-1} e^{-t} \widehat{a}(t) \right| dt &= \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \left(1 - e^{-t}\right)^{n-1} e^{-t} \left| \widehat{a}(t) \right| dt \\
&= \int_0^{+\infty} \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} \left| \widehat{a}(t) \right| dt,
\end{aligned}$$

and the convergence of this last integral follows from the assumption that $a \in \mathcal{E}$. $\square$

Example 3 Let $a(x) = \frac{1}{x^s}$ with $\Re(s) \geq 1$. Then $a \in \mathcal{E}$ and $\widehat{a}(t) = \frac{t^{s-1}}{\Gamma(s)}$. Hence

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}\big(1-e^{-t}\big)^{n-1} \frac{t^{s-1}}{\Gamma(s)}\,dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-t}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) t^{s-1}\,dt$$

$$= \begin{cases} \gamma & \text{if } s = 1, \\ \zeta(s) - \frac{1}{s-1} & \text{if } s \neq 1 \end{cases}$$

where γ refers to the Euler constant. In particular, since

$$\int_0^{+\infty} e^{-t}\big(1-e^{-t}\big)^{n-1}\,dt = \frac{1}{n} \quad \text{for each integer } n \geq 1,$$

then

$$\gamma = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!}\frac{1}{n}.$$

3 The operator D

Proposition 4 *If $a \in \mathcal{E}$, then the integral*

$$\int_0^{+\infty} e^{-t}\big(1-e^{-t}\big)^{x-1}\widehat{a}(t)\,dt$$

converges for all x with $\Re(x) > 0$.

Proof If $a \in \mathcal{E}$ and $\Re(x) > 0$, we may write for $t \in]0, +\infty[$,

$$\left|e^{-t}\big(1-e^{-t}\big)^{x-1}\widehat{a}(t)\right| \leq e^{-t} e^{(1-\Re(x))(-\log(1-e^{-t}))}\left|\widehat{a}(t)\right|.$$

The convergence when $t \to +\infty$ results from the inequality

$$e^{-t} e^{(1-\Re(x))(-\log(1-e^{-t}))}\left|\widehat{a}(t)\right| \leq \frac{e^{-t}}{1-e^{-t}}\left|\widehat{a}(t)\right| \leq 2e^{-t}\left|\widehat{a}(t)\right| \quad (\text{for } t \geq \log 2).$$

The convergence when $t \to 0$ results from the inequality

$$e^{(1-\Re(x))(-\log(1-e^{-t}))} \leq \begin{cases} 1 & \text{if } \Re(x) \geq 1, \\ \frac{1}{(1-e^{-t})^{(1-\Re(x))}} & \text{if } 0 < \Re(x) < 1 \end{cases}$$

since the function $t \mapsto e^{-t}|\widehat{a}(t)|\frac{1}{(1-e^{-t})^{\beta}}$ is integrable at 0 for $0 < \beta < 1$ by the definition of E (note that $(1-e^{-t})^{-\beta} \leq (kt)^{-1}$ for small enough t). $\square$

Definition 6 Let a be a function in $\mathcal{E}$. We call $D(a)$ the function defined for all x with $\Re(x) > 0$ by

$$D(a)(x) = \int_0^{+\infty} e^{-t}\big(1 - e^{-t}\big)^{x-1}\widehat{a}(t)\,dt. \tag{7}$$

Remark 1

(a) By Theorem 1, the series $\sum_{n\ge 1} \frac{\lambda_n}{n!} D(a)(n)$ converges and its sum is given by formula (6).
(b) The values of $D(a)$ at positive integers may be computed directly without recourse to $\widehat{a}$. The development of $(1 - e^{-t})^n$ by the binomial theorem gives

$$D(a)(n+1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a(k+1) \quad \text{for all integer } n \ge 0. \tag{8}$$

Definition 7 We call Λ the C^1-diffeomorphism of $\mathbb{R}_+$ defined by $\Lambda(u) = -\log(1 - e^{-u})$. In particular, it is important to note that Λ is involutive:

$$\Lambda^{-1} = \Lambda.$$

Theorem 2 *Let a be a function in $\mathcal{E}$. Then the function $D(a) \in \mathcal{E}$ and, moreover, verifies the relation*

$$\widehat{D(a)} = \widehat{a}(\Lambda), \tag{9}$$

where $\widehat{a}(\Lambda)$ denotes $\widehat{a} \circ \Lambda$.

Proof The change of variables $t = \Lambda(u)$ in (7) gives

$$D(a)(x) = \int_0^{+\infty} e^{-xu}\widehat{a}\big(\Lambda(u)\big)\,du \quad \text{for } \Re(x) > 0.$$

Thus, $D(a) = \mathcal{L}(\widehat{a}(\Lambda))$. It remains to prove that $D(a) \in \mathcal{E}$. One has only to check that the function $\widehat{a}(\Lambda)$ is in E. This function being in $\mathcal{C}^1(]0, +\infty[)$, it suffices to show that for all $\varepsilon > 0$, the function $u \mapsto e^{-\varepsilon u}|\widehat{a}(-\log(1 - e^{-u}))|$ is bounded on $]0, +\infty[$. This results from the existence of $C_\varepsilon > 0$ such that

$$\big|\widehat{a}\big(-\log\big(1 - e^{-u}\big)\big)\big| \le C_\varepsilon\big(1 - e^{-u}\big)^{\varepsilon} \quad \text{for all } u \in]0, +\infty[. \qquad \square$$

Example 4 Let $a(x) = \frac{1}{x^s}$ with $\Re(s) \ge 1$. Then $\widehat{a}(t) = \frac{t^{s-1}}{\Gamma(s)}$. Thus, by (9),

$$D\left(\frac{1}{x^s}\right) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right), \tag{10}$$

and if $s = m + 1$ with m a natural number and $n \ge 1$, then by (4),

$$D\left(\frac{1}{x^{m+1}}\right)(n) = \frac{P_m(H_n, \dots, H_n^{(m)})}{n}. \tag{11}$$

Remark 2 Theorem 2 may be summarized in the following diagram:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{D} & \mathcal{E} \\ \big\downarrow{\scriptstyle \mathcal{L}^{-1}} & & \big\uparrow{\scriptstyle \mathcal{L}} \\ E & \xrightarrow{\Lambda^\star} & E \end{array}$$

where $\Lambda^\star(\widehat{a}) = \widehat{a}(\Lambda)$. The algebraic properties of D are summed up in the following theorem.

Theorem 3 *The operator D is an automorphism of $\mathcal{E}$ which verifies $D = D^{-1}$ and lets the function $x \mapsto \frac{1}{x}$ invariant.*

Proof We can write $D = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}$ and $\Lambda^\star$ is an automorphism of E, which verifies $\Lambda^\star = (\Lambda^\star)^{-1}$ since $\Lambda = \Lambda^{-1}$. Furthermore,

$$D\left(\frac{1}{x}\right) = \mathcal{L}(1) = \frac{1}{x}.$$

□

4 The harmonic product

Our aim is to define the harmonic product of two functions a and b in $\mathcal{E}$ as being the unique function f of $\mathcal{E}$ such that

$$D(a)(x).D(b)(x) = D(f)(x).$$

Thus, we have to establish that such a function exists and is unique. In order to do this, we introduce first a Λ-convolution product of two functions in E.

4.1 The Λ-convolution product

Proposition 5 *If a and b are in $\mathcal{E}$, then $\widehat{a}(\Lambda) * \widehat{b}(\Lambda) \in E$.*

Proof From the definition of the convolution product, one may write

$$\big(\widehat{a}(\Lambda) * \widehat{b}(\Lambda)\big)(t) = \int_0^t \widehat{a}\big(\Lambda(u)\big)\widehat{b}\big(\Lambda(t-u)\big)\,du.$$

Now, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ and $D_\varepsilon > 0$ such that

$$\begin{aligned} &\left|\widehat{a}\big(-\log\big(1-e^{-u}\big)\big)\right| \le C_\varepsilon\big(1-e^{-u}\big)^\varepsilon \quad \text{and} \\ &\left|\widehat{b}\big(-\log\big(1-e^{-(t-u)}\big)\big)\right| \le D_\varepsilon\big(1-e^{-(t-u)}\big)^\varepsilon \quad \text{for all } u \in]0,+\infty[. \end{aligned}$$

It follows that

$$\left|\big(\widehat{a}(\Lambda) * \widehat{b}(\Lambda)\big)(t)\right| \le C_\varepsilon D_\varepsilon \int_0^t \big(1-e^{-u}\big)^\varepsilon \big(1-e^{-(t-u)}\big)^\varepsilon\,du.$$

One has also

$$\begin{aligned}
&\int_0^t \left(1-e^{-u}\right)^{\varepsilon}\left(1-e^{-(t-u)}\right)^{\varepsilon} du \\
&\quad = \left(1-e^{-t}\right)^{1+2\varepsilon} \int_0^1 u^{\varepsilon}(1-u)^{\varepsilon} \frac{1}{(1-(1-e^{-t})u)^{\varepsilon+1}} du \\
&\quad \le \left(1-e^{-t}\right)^{1+2\varepsilon} \int_0^1 \frac{1}{(1-(1-e^{-t})u)^{\varepsilon+1}} du \\
&\quad \le \left(1-e^{-t}\right)^{1+2\varepsilon} \frac{e^{t\varepsilon}-1}{(1-e^{-t})\varepsilon} \\
&\quad \le \left(1-e^{-t}\right)^{2\varepsilon} \frac{e^{t\varepsilon}-1}{\varepsilon} \le \frac{e^{t\varepsilon}}{\varepsilon}.
\end{aligned}$$

Hence, $|(\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t)| \le C_\varepsilon D_\varepsilon \frac{e^{t\varepsilon}}{\varepsilon}$, which proves that this function belongs to E as required. □

Definition 8 Let a and b be two functions in $\mathcal{E}$. The *Λ-convolution product* $\widehat{a} \circledast \widehat{b}$ of $\widehat{a}$ and $\widehat{b}$ is defined by

$$\widehat{a} \circledast \widehat{b} = \Lambda^\star\left(\Lambda^\star\left(\widehat{a}\right) * \Lambda^\star\left(\widehat{b}\right)\right),$$

or equivalently (since $\Lambda^\star = (\Lambda^\star)^{-1}$)

$$\left(\widehat{a} \circledast \widehat{b}\right)(\Lambda) = \widehat{a}(\Lambda) * \widehat{b}(\Lambda).$$

Remark 3 The Λ-convolution product inherits the algebraic properties of the ordinary convolution product, i.e., bilinearity, commutativity, and associativity.

4.2 The harmonic product

Definition 9 Let a and b be two functions in $\mathcal{E}$. The *harmonic product* $a \bowtie b$ of a and b is defined by

$$a \bowtie b = \mathcal{L}\left(\widehat{a} \circledast \widehat{b}\right) \in \mathcal{E}.$$

This construction may be summarized in the following diagram:

$$\begin{array}{ccccc}
(a,b) & \longrightarrow & (\widehat{a},\widehat{b}) & \longrightarrow & (\widehat{a}(\Lambda),\widehat{b}(\Lambda)) \\
\downarrow & & \downarrow & & \downarrow \\
a \bowtie b & \longleftarrow & \widehat{a} \circledast \widehat{b} & \longleftarrow & \widehat{a}(\Lambda) * \widehat{b}(\Lambda)
\end{array}$$

Remark 4 The harmonic product inherits the properties of the Λ-convolution product: it is bilinear, commutative, and associative.

Theorem 4 *Let a and b be in $\mathcal{E}$. Then,*

$$D(a \bowtie b) = D(a)D(b), \tag{12}$$

and

$$D(ab) = D(a) \bowtie D(b). \tag{13}$$

Proof One knows from Theorem 2 that

$$D = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}.$$

Hence

$$D(a \bowtie b) = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}(a \bowtie b) = \mathcal{L}\Lambda^\star\big(\widehat{a} \circledast \widehat{b}\big) = \mathcal{L}\big(\Lambda^\star\big(\widehat{a}\big) * \Lambda^\star\big(\widehat{b}\big)\big),$$

and it follows from (5) and (9) that

$$\mathcal{L}\big(\Lambda^\star\big(\widehat{a}\big) * \Lambda^\star\big(\widehat{b}\big)\big) = \mathcal{L}\big(\Lambda^\star\big(\widehat{a}\big)\big)\mathcal{L}\big(\Lambda^\star\big(\widehat{b}\big)\big) = D(a)D(b)$$

which proves (12). Moreover, (12) enables us to write

$$D\big(D(a) \bowtie D(b)\big) = D^2(a)D^2(b) = ab \quad \big(\text{since } D = D^{-1}\big),$$

and so

$$D(a\,b) = D^2\big(D(a) \bowtie D(b)\big) = D(a) \bowtie D(b)$$

which proves (13). □

Remark 5 The values of $(a \bowtie b)(n)$ may be computed without recourse to $\widehat{a}$ and $\widehat{b}$. By elementary transformations, it can be shown that

$$(a \bowtie b)(n+1) = \int_0^{+\infty}\int_0^{+\infty}\big(e^{-t-s}\big)\big(e^{-t} + e^{-s} - e^{-t}e^{-s}\big)^n\widehat{a}(t)\widehat{b}(s)\,dt\,ds.$$

Hence, if the numbers $C_n^{k,l}$ are defined by

$$(X + Y - XY)^n = \sum_{\substack{0\le k\le n,\\ 0\le l\le n}} C_n^{k,l}X^kY^l,$$

then one has the following explicit formula:

$$(a \bowtie b)(n+1) = \sum_{\substack{0\le k\le n,\\ 0\le l\le n}} C_n^{k,l}a(k+1)b(l+1).$$

For small values of n, this enables one to compute

$$(a \bowtie b)(1) = a(1)b(1),$$

$$(a \bowtie b)(2) = a(2)b(1) + a(1)b(2) - a(2)b(2),$$
$$(a \bowtie b)(3) = a(3)b(1) + a(1)b(3) + 2a(2)b(2) - 2a(3)b(2) - 2a(2)b(3) + a(3)b(3).$$

Theorem 5 *Let*

$$\left(\frac{1}{x}\right)^{\bowtie k} = \underbrace{\frac{1}{x} \bowtie \frac{1}{x} \bowtie \cdots \bowtie \frac{1}{x}}_{k} \quad (k = 1, 2, 3, \ldots),$$

where $\frac{1}{x}$ *denotes (improperly) the function* $x \mapsto \frac{1}{x}$. *Then, for all natural numbers* $m \geq 0$,

$$\left(\frac{1}{x}\right)^{\bowtie (m+1)} = D\left(\frac{1}{x^{m+1}}\right).$$

In particular, for all integers $n \geq 1$,

$$\left(\frac{1}{x}\right)^{\bowtie (m+1)}(n) = \frac{P_m(H_n, \ldots, H_n^{(m)})}{n}. \tag{14}$$

Proof By (13) we have

$$D\left(\frac{1}{x^{m+1}}\right) = D\left(\underbrace{\frac{1}{x} \cdots \frac{1}{x}}_{m+1}\right) = \left(D\left(\frac{1}{x}\right)\right)^{\bowtie (m+1)}$$
$$= \left(\frac{1}{x}\right)^{\bowtie (m+1)} \quad \text{since } D\left(\frac{1}{x}\right) = \frac{1}{x}.$$

Thus, (14) results from (11). □

4.3 The harmonic property

The following theorem explains the main reason why the harmonic product is called 'harmonic'.

Theorem 6 *Let* $a \in \mathcal{E}$. *Then*

$$\frac{1}{x} \bowtie a = \frac{A(x)}{x},$$

where A *denotes the function defined for* $\Re(x) > 0$ *by*

$$A(x) = \int_0^{+\infty} \frac{e^{-xt} - 1}{e^{-t} - 1} e^{-t} \widehat{a}(t)\, dt.$$

In particular, for each integer $n \geq 1$,

$$\left(\frac{1}{x} \bowtie a\right)(n) = \frac{A(n)}{n} = \frac{1}{n}\left(\sum_{k=1}^{n} a(k)\right). \tag{15}$$

Proof By the definition of the harmonic product, one has

$$\frac{1}{x} \bowtie a = \mathcal{L}\big(1 \circledast \widehat{a}\big).$$

Now

$$\big(1 \circledast \widehat{a}\big)\big(\Lambda(u)\big) = \big(1 * \widehat{a}(\Lambda)\big)(u) = \int_0^u \widehat{a}\big(\Lambda(v)\big)\, dv = -\int_{+\infty}^{\Lambda(u)} \widehat{a}(t) \frac{e^{-t}}{1-e^{-t}}\, dt$$

(by the change of variables $t = \Lambda(v)$). Hence,

$$\big(1 \circledast \widehat{a}\big)(u) = \int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1-e^{-t}}\, dt.$$

Thus, we have

$$\begin{aligned}
\frac{1}{x} \bowtie a &= \int_0^{+\infty} e^{-xu} \left(\int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1-e^{-t}}\, dt\right) du \\
&= \int_0^{+\infty} \left(\int_0^t e^{-xu}\, du\right) \widehat{a}(t) \frac{e^{-t}}{1-e^{-t}}\, dt \\
&= \frac{1}{x} \int_0^{+\infty} \big(1 - e^{-xt}\big) \widehat{a}(t) \frac{e^{-t}}{1-e^{-t}}\, dt \\
&= \frac{A(x)}{x}.
\end{aligned}$$

Furthermore, for each integer $n \geq 1$, we have

$$A(n) = \int_0^{+\infty} \frac{e^{-nt}-1}{e^{-t}-1} e^{-t} \widehat{a}(t)\, dt = \sum_{k=1}^{n} a(k).$$

□

Example 5

$$\frac{1}{x} \bowtie \frac{1}{x} = D\left(\frac{1}{x^2}\right) = \mathcal{L}(\Lambda) = \frac{H(x)}{x} \quad \text{with } H(x) = \psi(x+1) + \gamma,$$

ψ denoting the logarithmic derivative of Γ. In particular, for each integer $n \geq 1$,

$$\left(\frac{1}{x} \bowtie \frac{1}{x}\right)(n) = \frac{H(n)}{n} = \frac{H_n}{n}.$$

Example 6 For $\Re(s) \geq 1$,

$$\frac{1}{x} \bowtie \frac{1}{x^s} = \frac{H^{(s)}(x)}{x},$$

with

$$H^{(s)}(x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1 - e^{-xt}}{1 - e^{-t}} e^{-t} t^{s-1} \, dt.$$

For each integer $n \geq 1$,

$$\left(\frac{1}{x} \bowtie \frac{1}{x^s}\right)(n) = \frac{H^{(s)}(n)}{n} = \frac{H_n^{(s)}}{n} = \frac{1}{n}\left(\sum_{m=1}^{n} \frac{1}{m^s}\right).$$

From (15), by induction on k, we deduce the following important corollary.

Corollary 1 *For each integer $k \geq 2$,*

$$\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie a\right)(n) = \frac{1}{n}\left(\sum_{n \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{a(n_k)}{n_1 \cdots n_{k-1}}\right). \tag{16}$$

Example 7 Applying (16) with $a(x) = \frac{1}{x}$ (and $k = m$), we get

$$\left(\frac{1}{x}\right)^{\bowtie (m+1)}(n) = \frac{1}{n}\left(\sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1 \cdots n_m}\right). \tag{17}$$

Hence, it follows from (14) and (17) that

$$P_m\left(H_n, H_n^{(2)}, \ldots, H_n^{(m)}\right) = \sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1 \cdots n_m}, \tag{18}$$

which is a nice reformulation of Dilcher's formula (cf. [2, 9]).

5 The modified zeta function F_k

5.1 Integral representation

Definition 10 For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$ and each natural number k, the *modified zeta function of order* k is defined by

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1 - e^{-t}} f_k\left(1 - e^{-t}\right) dt \quad \text{with } f_k(z) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{z^n}{n^k}. \tag{19}$$

Remark 6 By (2) and Example 3, one has $F_0(s) = \zeta(s) - \frac{1}{s-1}$.

The fact that F_k may be represented by a Mellin transform enables us to analytically continue this function outside its half-plane of definition by a standard analytic method (cf. [14, Sect. 6.7]).

Theorem 7 *The function F_k analytically continues in the whole complex plane as an entire function.*

Proof The function $z \mapsto \frac{1}{\log(1-z)} + \frac{1}{z}$ being analytic in the disc $D(0,1)$ with a singularity at 1, we deduce from (1) that the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{\lambda_n z^n}{n!}$ is equal to 1. Thus 1 is also the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{\lambda_n z^n}{n! n^k}$ which defines an analytic function f_k in the disc $D(0,1)$. Hence, the function

$$g_k\colon \quad t \mapsto f_k\bigl(1-e^{-t}\bigr)$$

is analytic for all $t \in \mathbb{C}$ such that $1-e^{-t} \in D(0,1)$. Since $1-e^0 = 0$, it follows that g_k is analytic in a neighbourhood of 0. Since $g_k(0) = 0$, the function $t \mapsto g_k(t)\frac{e^{-t}}{1-e^{-t}}$ is itself analytic in a neighbourhood of 0. It follows that its Mellin transform analytically continues in the complex plane with simple poles at negative integers which are all cancelled by the poles of Γ. □

Theorem 8 *For all s with $\Re(s) > 1$ and each integer $k \geq 1$,*

$$F_k(s) = \vartheta(k)\zeta(s) + \sum_{j=1}^{k}(-1)^j \vartheta(k-j) Z_j(s) + (-1)^k \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} T^k\left(\frac{e^{-t}-1}{t}\right) dt \tag{20}$$

with

$$\vartheta(k) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^k}, \tag{21}$$

$$Z_j(s) = \sum_{n>n_1>n_2>\cdots>n_j>0} \frac{1}{n^s n_1 n_2 \cdots n_j}, \tag{22}$$

$$Tf(t) = \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} f(u)\, du. \tag{23}$$

Proof Formula (20) results from the integral representation (19) and the two following lemmas. □

Lemma 1 *For all $t > 0$,*

$$f_k\bigl(1-e^{-t}\bigr) = \sum_{j=0}^{k}(-1)^j \vartheta(k-j) \frac{\Lambda^j(t)}{j!} + (-1)^k T^k\left(\frac{e^{-t}-1}{t}\right),$$

where ϑ is defined by (21) *and T is the operator defined by* (23).

Proof Let $g_k(t) = f_k(1-e^{-t})$. The function g_k verifies the recursive relation

$$g_k'(t) = e^{-t} f_k'\big(1-e^{-t}\big) = \frac{e^{-t}}{1-e^{-t}} f_{k-1}\big(1-e^{-t}\big) = \frac{e^{-t}}{1-e^{-t}} g_{k-1}(t).$$

Thus

$$g_k(t) = \int_0^t \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)\, du = g_k(+\infty) - \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)\, du$$

with

$$g_k(+\infty) = f_k(1) = \vartheta(k).$$

Thus, one has

$$g_k(t) = \vartheta(k) - \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)\, du = \vartheta(k) - T(g_{k-1}),$$

and a repeated iteration k times of this relation gives

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j)(-1)^j T^j(1) + (-1)^k T^k(g_0).$$

Now, by (2),

$$g_0(t) = \sum_{n=1}^{\infty} \frac{\lambda_n (1-e^{-t})^n}{n!} = \frac{e^{-t}-1}{t} + 1,$$

and thus

$$T^k(g_0) = T^k\left(\frac{e^{-t}-1}{t}\right) + T^k(1).$$

Hence

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j)(-1)^j T^j(1) + (-1)^k T^k(1) + (-1)^k T^k\left(\frac{e^{-t}-1}{t}\right).$$

Since $\vartheta(0) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} = 1$ (by (1) and a Tauberian theorem), one deduces that

$$g_k(t) = \sum_{j=0}^{k} \vartheta(k-j)(-1)^j T^j(1) + (-1)^k T^k\left(\frac{e^{-t}-1}{t}\right),$$

and, now, it remains to prove that

$$\frac{\Lambda^j(t)}{j!} = T^j(1),$$

which follows from the recursive relation

$$\frac{\Lambda^j(t)}{j!} = -\int_{+\infty}^{t} \frac{e^{-u}}{1-e^{-u}} \frac{\Lambda^{j-1}(u)}{(j-1)!}\,du = T\left(\frac{\Lambda^{j-1}}{(j-1)!}\right). \qquad \square$$

Lemma 2 *Let $Z_j(s)$ be defined by (22). Then, for all $s \in \mathbb{C}$ with $\Re(s) > 1$,*

$$Z_j(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j(t)}{j!}\,dt.$$

Proof From the recursive relation

$$\partial \frac{\Lambda^j(t)}{j!} = \frac{\Lambda^{j-1}(t)}{(j-1)!} \partial \Lambda(t) = -\frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^{j-1}(t)}{(j-1)!} = -\sum_{m>0} e^{-mt} \frac{\Lambda^{j-1}(t)}{(j-1)!},$$

and $\Lambda(t) = \sum_{n>0} \frac{e^{-nt}}{n}$, one may check by induction on j that

$$\frac{\Lambda^j(t)}{j!} = \sum_{n_1>n_2>\cdots>n_j>0} \frac{e^{-n_1 t}}{n_1} \frac{1}{n_2} \cdots \frac{1}{n_j}.$$

Furthermore, one has

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-Nt} \frac{e^{-t}}{1-e^{-t}}\,dt = \sum_{n>N} \frac{1}{n^s} \quad \big(\text{for } \Re(s) > 1\big).$$

Hence

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j(t)}{j!}\,dt = \sum_{n>n_1>n_2>\cdots>n_j>0} \frac{1}{n^s} \frac{1}{n_1} \frac{1}{n_2} \cdots \frac{1}{n_j} = Z_j(s). \qquad \square$$

5.2 Values of F_k at integers

Theorem 9 *For all s in $\mathbb{C}$ with $\Re(s) \geq 1$ and each natural number k, then*

$$F_k(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^k} D\left(\frac{1}{x^s}\right)(n). \tag{24}$$

In particular, for all natural numbers m,

$$F_k(m+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)})}{n^{k+1}}. \tag{25}$$

Proof The change of variables $t = \Lambda(u)$ in (19) enables to write

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f_k\big(e^{-u}\big)\big(\Lambda(u)\big)^{s-1}\,du.$$

Since $D(\frac{1}{x^s}) = \mathcal{L}(\frac{\Lambda^{s-1}}{\Gamma(s)})$, we deduce (24) from this last expression of $F_k(s)$. Moreover, by (11), one also has $D(\frac{1}{x^{m+1}})(n) = \frac{P_m(H_n,\ldots,H_n^{(m)})}{n}$, which proves (25). □

Corollary 2 *Let $\vartheta(s)$ be the Dirichlet series defined for $\Re(s) > 0$ by*

$$\vartheta(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^s}.$$

Then for each natural number k,

$$\vartheta(k+1) = F_k(1). \tag{26}$$

Example 8

$$F_0(1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} = \gamma = \vartheta(1),$$

$$F_0(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n} = \zeta(2) - 1,$$

$$F_0(3) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n} = \zeta(3) - \frac{1}{2},$$

$$F_1(1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} = \vartheta(2),$$

$$F_1(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2},$$

$$F_1(3) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n^2}.$$

5.3 Identities linking Cauchy numbers, harmonic numbers and zeta values

Theorem 10 *For all integers $q \geq 2$,*

$$\begin{aligned} F_1(q) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}\big(H_n, H_n^{(2)}, \ldots, H_n^{(q-1)}\big) \\ &= \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}. \end{aligned} \tag{27}$$

Proof By (20) and (25), one may write

$$F_k(q)=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!n^{k+1}}P_{q-1}\big(H_n,H_n^{(2)},\dots,H_n^{(q-1)}\big)$$
$$=\vartheta(k)\zeta(q)+\sum_{j=1}^{k}(-1)^j\vartheta(k-j)Z_j(q)$$
$$+(-1)^k\frac{1}{\Gamma(q)}\int_0^{+\infty}t^{q-1}\frac{e^{-t}}{1-e^{-t}}T^k\left(\frac{e^{-t}-1}{t}\right)dt. \tag{28}$$

We apply now (28) with $k=1$. This gives

$$F_1(q)=\gamma\zeta(q)-\sum_{n\geq 1}\frac{H_{n-1}}{n^q}+\frac{1}{\Gamma(q)}\int_0^{+\infty}t^{q-1}\frac{e^{-t}}{1-e^{-t}}\mathrm{E}_1(t)\,dt,$$

with $\mathrm{E}_1(t)=-\mathrm{Ei}(-t)=\int_t^{+\infty}\frac{e^{-u}}{u}\,du$. Thus

$$F_1(q)=\gamma\zeta(q)-\sum_{n\geq 1}\frac{H_n}{n^q}+\zeta(q+1)+I(q),$$

where

$$I(q)=\frac{1}{\Gamma(q)}\int_0^{+\infty}t^{q-1}\frac{e^{-t}}{1-e^{-t}}\mathrm{E}_1(t)\,dt=\frac{1}{\Gamma(q)}\sum_{n=1}^{\infty}\int_0^{+\infty}e^{-nt}t^{q-1}\mathrm{E}_1(t)\,dt.$$

Since

$$\mathrm{E}_1(t)=-\gamma-\log t+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{t^n}{n!},$$

and $-\gamma-\log t=\widehat{\frac{\log x}{x}}$ (cf. [13]), then $\mathrm{E}_1=\widehat{\frac{\log(x+1)}{x}}$. Thus

$$\int_0^{+\infty}e^{-nt}t^{q-1}\mathrm{E}_1(t)\,dt=(-1)^{q-1}\left(\frac{\log(x+1)}{x}\right)^{(q-1)}(n).$$

Hence, by a calculation of the $(q-1)$th derivative, we get

$$I(q)=\frac{(-1)^{q-1}}{(q-1)!}\sum_{n=1}^{\infty}\left(\frac{\log(x+1)}{x}\right)^{(q-1)}(n)$$
$$=\sum_{n=1}^{\infty}\frac{\log(n+1)}{n^q}-\sum_{k=1}^{q-1}\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{(n+1)^k n^{q-k}}.$$

□

Remark 7

(1) We recall *Euler's formula* (cf. [6])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q} = \begin{cases} \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k) & \text{for } q>2, \\ 2\zeta(3) & \text{for } q=2. \end{cases}$$

(2) From $\sum_{n=1}^{\infty} \frac{1}{(n+1)n} = 1$, and the decomposition

$$\frac{1}{(n+1)^k n^{q-k}} = \frac{1}{(n+1)^{k-1} n^{q-k}} - \frac{1}{(n+1)^k n^{q-k-1}} \quad (0<k<q),$$

the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}$ may be expressed as a linear combination of zeta values and integers.

Example 9

$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1 = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2},$$

$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^3} + \gamma\zeta(3) - \frac{1}{10}\zeta(2)^2 - \frac{1}{2}\zeta(2) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n^2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n^2},$$

$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^4} + \gamma\zeta(4) - 2\zeta(5) + \zeta(2)\zeta(3) - \frac{2}{3}\zeta(3) + \frac{1}{3}\zeta(2) - \frac{1}{2}$$
$$= \frac{1}{6}\sum_{n=1}^{\infty} \frac{\lambda_n H_n^3}{n!n^2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\lambda_n H_n H_n^{(2)}}{n!n^2} + \frac{1}{3}\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(3)}}{n!n^2}.$$

5.4 Link with the Ramanujan summation

The function F_k has strong connections with the Ramanujan summation (cf. [3, 4]).

If $a \in \mathcal{E}$, then the series $\sum_{n\geq 1} a(n)$ may be written

$$\sum_{n\geq 1} a(n) = \sum_{n\geq 1} \int_0^{+\infty} e^{-nt}\widehat{a}(t)\,dt,$$

and a formal permutation of $\sum_{n\geq 1}$ and $\int_0^{+\infty}$ would lead us to write

$$\sum_{n\geq 1} a(n) = \int_0^{+\infty} \frac{1}{1-e^{-t}} e^{-t}\widehat{a}(t)\,dt.$$

However, this last integral may be divergent at 0. Nevertheless we can renormalize it by removing the singularity at zero. This may be done merely by subtracting the polar part $\frac{1}{t}$ of $\frac{1}{1-e^{-t}}$. From Theorem 1, we know that

$$\int_0^{+\infty}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right)e^{-t}\widehat{a}(t)\,dt=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\int_0^{+\infty}e^{-t}\left(1-e^{-t}\right)^{n-1}\widehat{a}(t)\,dt$$
$$=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D(a)(n).$$

This justifies the following definition.

Definition 11 Let a be a function in $\mathcal{E}=\mathcal{L}(E)$. The *Ramanujan sum* of the series $\sum_{n\geq 1}a(n)$ is defined by

$$\sum_{n\geq 1}^{\mathcal{R}}a(n)=\int_0^{+\infty}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right)e^{-t}\widehat{a}(t)\,dt=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D(a)(n). \tag{29}$$

Lemma 3 *Let a and b in $\mathcal{E}$. Then*

$$\sum_{n\geq 1}^{\mathcal{R}}(a\bowtie b)(n)=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D(a)(n)D(b)(n). \tag{30}$$

Proof This results directly from (12) and (29). □

Theorem 11 *For all $s\in\mathbb{C}$ with $\Re(s)\geq 1$, one has*

$$F_0(s)=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n^s}\quad \textit{and}\quad F_k(s)=\sum_{n\geq 1}^{\mathcal{R}}\left(\left(\frac{1}{x}\right)^{\bowtie k}\bowtie\frac{1}{x^s}\right)(n)\quad \textit{for } k\geq 1. \tag{31}$$

Proof By (24) and (30), taking into account the invariance of $\frac{1}{x}$ by D, one may write

$$\sum_{n\geq 1}^{\mathcal{R}}\left(\left(\frac{1}{x}\right)^{\bowtie k}\bowtie\frac{1}{x^s}\right)(n)=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D\left(\left(\frac{1}{x}\right)^{\bowtie k}\right)(n)D\left(\frac{1}{x^s}\right)(n)$$
$$=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\left(\frac{1}{x}\right)^{k}(n)D\left(\frac{1}{x^s}\right)(n)$$
$$=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!n^k}D\left(\frac{1}{x^s}\right)(n)=F_k(s).$$
□

In particular, by (14), one deduces from (31) the following identity.

Corollary 3 *For each natural number k,*

$$F_k(1) = \vartheta(k+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n\geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n}. \tag{32}$$

Example 10

$$\vartheta(1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma,$$

$$\vartheta(2) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n},$$

$$\vartheta(3) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^3} = \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^2}{n} + \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}.$$

Remark 8 Comparing (32) with

$$F_0(k+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} P_k\big(H_n, H_n^{(2)}, \ldots, H_n^{(k)}\big),$$

one may observe a kind of reciprocity between $F_k(1)$ and $F_0(k+1)$. This results from the fact that $D = D^{-1}$.

Remark 9 In the case $q = 1$, (27) is meaningless since both the series $\sum_{n\geq 1} \frac{\log(n+1)}{n}$ and $\sum_{n\geq 1} \frac{H_n}{n}$ diverge. However, since

$$\log(x+1) - \big(\psi(x+1) + \gamma\big) = \int_0^{+\infty} \big(e^{-xu} - 1\big)\left(\frac{1}{1-e^{-u}} - \frac{1}{u}\right) e^{-u}\, du,$$

it follows that

$$\left(\frac{\widehat{\log(x+1)}}{x} - \frac{\widehat{\psi(x+1)+\gamma}}{x}\right)(t) = \int_t^{+\infty} \left(\frac{1}{1-e^{-u}} - \frac{1}{u}\right) e^{-u}\, du,$$

and then one may easily deduce from (29) the relation

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\log(n+1)}{n} - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n} = -\frac{\gamma^2}{2},$$

which may be rewritten in the following form (cf. Example 10):

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\log(n+1)}{n} = \vartheta(2) - \frac{1}{2}\vartheta(1)^2.$$

5.5 Link with the Arakawa–Kaneko zeta function

For $\Re(s) \geq 1$ and $k \geq 1$, one can define in an algebraic fashion the function ξ_k by

$$\xi_k(s) = \sum_{n=1}^{\infty} D\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s}\right)(n) = \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{x^s}\right)(n). \tag{33}$$

In particular, for all natural numbers m, one has (cf. [8, Corollary 1])

$$\xi_k(m+1) = \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{x^{m+1}}\right)(n) = \sum_{n=1}^{\infty} \frac{P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)})}{n^{k+1}}.$$

Since $D(\frac{1}{x^s}) = \mathcal{L}(\frac{\Lambda^{s-1}}{\Gamma(s)})$, one may also rewrite (33) as

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Li}_k\left(e^{-u}\right)\left(\Lambda(u)\right)^{s-1} du,$$

and the change of variables $t = \Lambda(u)$ leads to the integral representation

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k\left(1-e^{-t}\right) dt,$$

which is the analogue of (19) (with Li_k in place of f_k) and also the original definition of the Arakawa–Kaneko zeta function (cf. [1, 8]).

Thus, taking into account the facts that $\xi_k(1) = \zeta(k+1)$ and $\mathrm{Li}_1(1-e^{-t}) = t$, and following the same process as in the proof of Theorem 8, one obtains the following analogue of (20):

$$\begin{aligned}\xi_{k+1}(s) = \zeta(k+1)\zeta(s) + \sum_{j=1}^{k-1} (-1)^j \zeta(k+1-j) Z_j(s) \\ + (-1)^k \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} T^k(t)\, dt.\end{aligned} \tag{34}$$

In particular, in the simplest case $k = 1$, since

$$T(t) = \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} u\, du = \sum_{m>0} \int_t^{+\infty} e^{-mu} u\, du = \sum_{m>0} \frac{e^{-tm}}{m} t + \sum_{m>0} \frac{e^{-tm}}{m^2},$$

(34) again gives the formula

$$\xi_2(s) = \zeta(2)\zeta(s) - s \sum_{n>m>0} \frac{1}{n^{s+1}} \frac{1}{m} - \sum_{n>m>0} \frac{1}{n^s} \frac{1}{m^2}$$

already obtained by Arakawa and Kaneko (cf. [1, Theorem 6(ii)]).

6 Conclusion

Most of the general results given for the modified zeta function F_k, especially Theorems 7, 8, and 9, also apply (with minor adaptations) to a wide class of functions including the Arakawa–Kaneko zeta function ξ_k, specifically to the class of functions represented by normalized Mellin transforms of type

$$F_{k,\omega}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_{k,\omega}\bigl(1-e^{-t}\bigr)\, dt$$

with $\omega = (\omega_n)_{n\geq 1}$ and $f_{k,\omega}(z) = \sum_{n=1}^{\infty} \frac{\omega_n}{n^k} z^n$. In particular, under the assumption that $\frac{|\omega_n|}{n^k} = O(\frac{1}{n})$, we have for positive integers m the nice formula

$$F_{k,\omega}(m+1) = \sum_{n=1}^{\infty} \frac{\omega_n}{n^k} D\biggl(\frac{1}{x^{m+1}}\biggr)(n) = \sum_{n=1}^{\infty} \omega_n \frac{P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)})}{n^{k+1}},$$

which extends (25). However, this formula is more theoretical than practical because of the fast increase in the size of polynomials P_m: the number of monomials in P_m is equal to the number $p(m)$ of partitions of m, as shown by the explicit expression of the mth modified Bell polynomial.

References

1. Arakawa, T., Kaneko, M.: Multiple zeta values, Poly–Bernoulli numbers and related zeta functions. Nagoya Math. J. **153**, 189–209 (1999)
2. Boyadzhiev, K.: Harmonic number identities via Euler's transform. J. Integer Seq. **12**, Article 09.6.1 (2009)
3. Candelpergher, B., Coppo, M.A., Delabaere, E.: La sommation de Ramanujan. Enseign. Math. **43**, 93–132 (1997)
4. Candelpergher, B., Gadiyar, H., Padma, R.: Ramanujan summation and the exponential generating function $\sum_{k=0}^{\infty} \frac{z^k}{k!} \zeta'(-k)$. Ramanujan J. **21**, 99–122 (2010)
5. Chen, X., Chu, W.: Dixon's $F_2(1)$-series and identities involving harmonic numbers and the Riemann zeta function. Discrete Math. **310**, 83–91 (2010)
6. Choi, J., Srivastava, H.M.: Explicit evaluation of Euler and related sums. Ramanujan J. **10**, 51–70 (2005)
7. Coppo, M.-A.: Nouvelles expressions des formules de Hasse et de Hermite pour la fonction zêta d'Hurwitz. Expo. Math. **27**, 79–86 (2009)
8. Coppo, M.-A., Candelpergher, B.: The Arakawa–Kaneko zeta function. Ramanujan J. **22**, 153–162 (2010)
9. Dilcher, K.: Some q-series identities related to divisors functions. Discrete Math. **145**, 83–93 (1995)
10. Flajolet, P., Sedgewick, R.: Mellin transforms and asymptotics: finite differences and Rice's integrals. Theor. Comput. Sci. **144**, 101–124 (1995)
11. Hermite, C.: Extrait de quelques lettres de M.Ch. Hermite à M.S. Pincherle. Ann. Mat. Pura Appl. **5**, 55–72 (1901)
12. Merlini, D., Sprugnoli, R., Verri, C.: The Cauchy numbers. Discrete Math. **306**, 1906–1920 (2006)
13. Schiff, J.: The Laplace Transform: Theory and Applications. Springer, New York (1999)
14. Zeidler, E.: Quantum Field Theory I: Basics in Mathematics and Physics. Springer, Berlin (2006)