Notes on operator algebras

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Chapter 1

Spectral theory

If A is a complex unital algebra then we denote by $G(A)$ the set of elements which have a two sided inverse. If $x \in A$, the **spectrum** of x is

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \notin G(A)\}.$$

The complement of the spectrum is called the **resolvent** and denoted $\rho_A(x)$.

Proposition 1.0.1. *Let A be a unital algebra over $\mathbb{C}$, and consider $x, y \in A$. Then $\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}$.*

Proof. If $1 - xy \in G(A)$ then we have

$$\begin{aligned}(1 - yx)(1 + y(1 - xy)^{-1}x) &= 1 - yx + y(1 - xy)^{-1}x - yxy(1 - xy)^{-1}x \\ &= 1 - yx + y(1 - xy)(1 - xy)^{-1}x = 1.\end{aligned}$$

Similarly, we have

$$(1 + y(1 - xy)^{-1}x)(1 - yx) = 1,$$

and hence $1 - yx \in G(A)$. ∎

Knowing the formula for the inverse beforehand of course made the proof of the previous proposition quite a bit easier. But this formula is quite natural to consider. Indeed, if we just consider formal power series then we have

$$(1 - yx)^{-1} = \sum_{k=0}^{\infty} (yx)^k = 1 + y(\sum_{k=0}^{\infty} (xy)^k)x = 1 + y(1 - xy)^{-1}x.$$

1.1 Banach and C^*-algebras

A **Banach algebra** is a Banach space A, which is also an algebra such that

$$\|xy\| \leq \|x\|\|y\|.$$

An **involution** $*$ on a Banach algebra is a conjugate linear period two anti-isomorphism such that $\|x^*\| = \|x\|$, for all $x \in A$. An **involutive** Banach algebra is a Banach algebra, together with a fixed involution.

If an involutive Banach algebra A additionally satisfies

$$\|x^*x\| = \|x\|^2,$$

for all $x \in A$, then we say that A is a C^***-algebra**. If a Banach or C^*-algebra is unital, then we further require $\|1\| = 1$.

Note that if A is a unital involutive Banach algebra, and $x \in G(A)$ then $(x^{-1})^* = (x^*)^{-1}$, and hence $\sigma_A(x^*) = \overline{\sigma_A(x)}$.

Lemma 1.1.1. *Let A be a unital Banach algebra and suppose $x \in A$ such that $\|1 - x\| < 1$, then $x \in G(A)$.*

Proof. Since $\|1-x\| < 1$, the element $y = \sum_{k=0}^{\infty}(1-x)^k$ is well defined, and it is easy to see that $xy = yx = 1$. ∎

Proposition 1.1.2. *Let A be a unital Banach algebra, then $G(A)$ is open, and the map $x \mapsto x^{-1}$ is a continuous map on $G(A)$.*

Proof. If $y \in G(A)$ and $\|x - y\| < \|y^{-1}\|$ then $\|1 - xy^{-1}\| < 1$ and hence by the previous lemma $xy^{-1} \in G(A)$ (hence also $x = xy^{-1}y \in G(A)$) and

$$\begin{aligned}\|xy^{-1}\| &\leq \sum_{n=0}^{\infty} \|(1 - xy^{-1})\|^n \\ &\leq \sum_{n=0}^{\infty} \|y^{-1}\|^n\|y - x\|^n = \frac{1}{1 - \|y\|^{-1}\|y - x\|}.\end{aligned}$$

Hence,

$$\begin{aligned}\|x^{-1} - y^{-1}\| &= \|x^{-1}(y - x)y^{-1}\| \\ &\leq \|y^{-1}(xy^{-1})^{-1}\|\|y^{-1}\|\|y - x\| \leq \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\|\|y - x\|}\|y - x\|.\end{aligned}$$

Thus continuity follows from continuity of the map $t \mapsto \frac{\|y^{-1}\|^2}{1-\|y^{-1}\|t}t$, at $t = 0$. ∎

Proposition 1.1.3. *Let A be a unital Banach algebra, and suppose $x \in A$, then $\sigma_A(x)$ is a non-empty compact set.*

Proof. If $\|x\| < |\lambda|$ then $\frac{x}{\lambda} - 1 \in G(A)$ by Lemma 1.1.1, also $\sigma_A(x)$ is closed by Proposition 1.1.2, thus $\sigma_A(x)$ is compact.

To see that $\sigma_A(x)$ is non-empty note that for any linear functional $\varphi \in A^*$, we have that $f(z) = \varphi((x-z)^{-1})$ is analytic on $\rho_A(x)$. Indeed, if $z, z_0 \in \rho_A(x)$ then we have

$$(x-z)^{-1} - (x-z_0)^{-1} = (x-z)^{-1}(z-z_0)(x-z_0)^{-1}.$$

Since inversion is continuous it then follows that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \varphi((x-z_0)^{-2}).$$

We also have $\lim_{z \to \infty} f(z) = 0$, and hence if $\sigma_A(x)$ were empty then f would be a bounded entire function and we would then have $f = 0$. Since $\varphi \in A^*$ were arbitrary this would then contradict the Hahn-Banach theorem. ∎

Theorem 1.1.4 (Gelfand-Mazur)**.** *Suppose A is a unital Banach algebra such that every non-zero element is invertible, then $A \cong \mathbb{C}$.*

Proof. Fix $x \in A$, and take $\lambda \in \sigma(x)$. Since $x - \lambda$ is not invertible we have that $x - \lambda = 0$, and the result then follows. ∎

If $f(z) = \sum_{k=0}^n a_k z^k$ is a polynomial, and $x \in A$, a unital Banach algebra, then we define $f(x) = \sum_{k=0}^n a_k x^k \in A$.

Proposition 1.1.5 (The spectral mapping formula for polynomials)**.** *Let A be a unital Banach algebra, $x \in A$ and f a polynomial. then $\sigma_A(f(x)) = f(\sigma_A(x))$.*

Proof. If $\lambda \in \sigma_A(x)$, and $f(z) = \sum_{k=0}^n a_k z^k$ then

$$\begin{aligned} f(x) - f(\lambda) &= \sum_{k=1}^n a_k (x^k - \lambda^k) \\ &= (x - \lambda) \sum_{k=1} a_k \sum_{j=0}^{k-1} x^j \lambda^{k-j-1}, \end{aligned}$$

hence $f(\lambda) \in \sigma_A(x)$. conversely if $\mu \notin f(\sigma_A(x))$ and we factor $f - \mu$ as

$$f - \mu = \alpha_n(x - \lambda_1) \cdots (x - \lambda_n),$$

then since $f(\lambda) - \mu \neq 0$, for all $\lambda \in \sigma_A(x)$ it follows that $\lambda_i \notin \sigma_A(x)$, for $1 \leq i \leq n$, hence $f(x) - \mu \in G(A)$. ■

If A is a unital Banach algebra and $x \in A$, the **spectral radius** of x is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

Note that by Proposition 1.1.3 the spectral radius is finite, and the supremum is attained. Also note that by Proposition 1.0.1 we have the very useful equality $r(xy) = r(yx)$ for all x and y in a unital Banach algebra A. A priori the spectral radius depends on the Banach algebra in which x lives, but we will show now that this is not the case.

Proposition 1.1.6 (The spectral radius formula)**.** *Let A be a unital Banach algebra, and suppose $x \in A$. Then $\lim_{n\to\infty} \|x^n\|^{1/n}$ exists and we have*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Proof. By Proposition 1.1.5 we have $r(x^n) = r(x)^n$, and hence

$$r(x)^n \leq \|x^n\|,$$

showing that $r(x) \leq \liminf_{n\to\infty} \|x^n\|^{1/n}$.

To show that $r(x) \geq \limsup_{n\to\infty} \|x^n\|^{1/n}$, consider the domain $\Omega = \{z \in \mathbb{C} \mid |z| > r(x)\}$, and fix a linear functional $\varphi \in A^*$. We showed in Proposition 1.1.3 that $z \mapsto \varphi((x - z)^{-1})$ is analytic in Ω and as such we have a Laurent expansion

$$\varphi((z - x)^{-1}) = \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

for $|z| > r(x)$. However, we also know that for $|z| > \|x\|$ we have

$$\varphi((z - x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n}.$$

By uniqueness of the Laurent expansion we then have that

$$\varphi((z-x)^{-1}) = \sum_{n=1}^{\infty} \frac{\varphi(x^{n-1})}{z^n},$$

for $|z| > r(x)$.

Hence for $|z| > r(x)$ we have that $\lim_{n\to\infty} \frac{\varphi(x^{n-1})}{|z|^n} = 0$, for all linear functionals $\varphi \in A^*$. By the uniform boundedness principle we then have $\lim_{n\to\infty} \frac{\|x^{n-1}\|}{|z|^n} = 0$, hence $|z| > \limsup_{n\to\infty} \|x^n\|^{1/n}$, and thus

$$r(x) \geq \limsup_{n\to\infty} \|x^n\|^{1/n}. \qquad \blacksquare$$

An element x of a involutive algebra A is **normal** if $x^*x = xx^*$. An element x is **self-adjoint** (resp. **skew-adjoint**) if $x^* = x$ (resp. $x^* = -x$). Note that self-adjoint and skew-adjoint elements are normal.

Corollary 1.1.7. *Let A be a unital involutive Banach algebra and $x \in A$ normal, then $r(x^*x) \leq r(x)^2$. Moreover, if A is a C^*-algebra, then we have equality $r(x^*x) = r(x)^2$.*

Proof. By the previous proposition we have

$$r(x^*x) = \lim_{n\to\infty} \|(x^*x)^n\|^{1/n} = \lim_{n\to\infty} \|(x^*)^n x^n\|^{1/n} \leq \lim_{n\to\infty} \|x^n\|^{2/n} = r(x)^2.$$

By the C^*-identity, the inequality above becomes equality in a C^*-algebra. $\blacksquare$

Proposition 1.1.8. *Let A be a C^*-algebra and $x \in A$ normal, then $\|x\| = r(x)$.*

Proof. We first show this if x is self-adjoint, in which case we have $\|x^2\| = \|x\|^2$, and by induction we have $\|x^{2^n}\| = \|x\|^{2^n}$ for all $n \in \mathbb{N}$. Therefore, $\|x\| = \lim_{n\to\infty} \|x^{2^n}\|^{2^n} = r(x)$.

If x is normal then by Corollary 1.1.7 we have

$$\|x\|^2 = \|x^*x\| = r(x^*x) = r(x)^2. \qquad \blacksquare$$

Corollary 1.1.9. *Let A and B be two unital C^*-algebras and $\Phi : A \to B$ a unital $*$-homomorphism, then Φ is contractive. If Φ is a $*$-isomorphism, then Φ is isometric.*

Proof. Since Φ is a unital $*$-homomorphism we clearly have $\Phi(G(A)) \subset G(B)$, from which it follows that $\sigma_B(\Phi(x)) \subset \sigma_A(x)$, and hence $r(\Phi(x)) \leq r(x)$, for all $x \in A$. By Proposition 1.1.8 we then have

$$\|\Phi(x)\|^2 = \|\Phi(x^*x)\| = r(\Phi(x^*x)) \leq r(x^*x) = \|x^*x\| = \|x\|^2.$$

If Φ is a $*$-isomorphism then so is Φ^{-1} which then shows that Φ is isometric. ■

Corollary 1.1.10. *Let A be a unital complex involutive algebra, then there is at most one norm on A which makes A into a C^*-algebra.*

Proof. If there were two norms which gave a C^*-algebra structure to A then by the previous corollary the identity map would be an isometry. ■

Lemma 1.1.11. *Let A be a unital C^*-algebra, if $x \in A$ is self-adjoint then $\sigma_A(x) \subset \mathbb{R}$.*

Proof. Suppose $\lambda = \alpha + i\beta \in \sigma_A(x)$ where $\alpha, \beta \in \mathbb{R}$. If we consider $y = x - \alpha + it$ where $t \in \mathbb{R}$, then we have $i(\beta + t) \in \sigma_A(y)$ and y is normal. Hence,

$$\begin{aligned}(\beta + t)^2 \leq r(y)^2 &= \|y\|^2 = \|y^*y\| \\ &= \|(x - \alpha)^2 + t^2\| \leq \|x - \alpha\|^2 + t^2,\end{aligned}$$

and since $t \in \mathbb{R}$ was arbitrary it then follows that $\beta = 0$. ■

Lemma 1.1.12. *Let A be a unital Banach algebra and suppose $x \notin G(A)$. If $x_n \in G(A)$ such that $\|x_n - x\| \to 0$, then $\|x_n^{-1}\| \to \infty$.*

Proof. If $\|x_n^{-1}\|$ were bounded then we would have that $\|1 - xx_n^{-1}\| < 1$ for some n. Thus, we would have that $xx_n^{-1} \in G(A)$ and hence also $x \in G(A)$. ■

Proposition 1.1.13. *Let B be a unital C^*-algebra and $A \subset B$ a unital C^*-subalgebra. If $x \in A$ then $\sigma_A(x) = \sigma_B(x)$.*

Proof. Note that we always have $G(A) \subset G(B)$. If $x \in A$ is self-adjoint such that $x \notin G(A)$, then by Lemma 1.1.11 we have $it \in \rho_A(x)$ for $t > 0$. By the previous lemma we then have

$$\lim_{t \to 0} \|(x - it)^{-1}\| = \infty,$$

and thus $x \notin G(B)$ since inversion is continuous in $G(B)$.

For general $x \in A$ we then have

$$x \in G(A) \Leftrightarrow x^*x \in G(A) \Leftrightarrow x^*x \in G(B) \Leftrightarrow x \in G(B).$$

In particular, we have $\sigma_A(x) = \sigma_B(x)$ for all $x \in A$. ∎

Because of the previous result we will henceforth write simply $\sigma(x)$ for the spectrum of an element in a C^*-algebra.

Exercise 1.1.14. Suppose A is a unital Banach algebra, and $I \subset A$ is a closed two sided ideal, then A/I is again a unital Banach algebra, when given the norm $\|a + I\| = \inf_{y \in I} \|a + y\|$, and $(a + I)(b + I) = (ab + I)$.

1.1.1 Examples

The most basic example of a C^*-algebra is found by considering a locally compact Hausdorff space K. Then the space $C_0(K)$ of complex valued continuous functions which vanish at infinity is a C^*-algebra when given the supremum norm $\|f\|_\infty = \sup_{x \in K} |f(x)|$. This is unital if and only if K is compact. Another commutative example is given by considering (X, μ) a measure space. Then we let $L^\infty(X, \mu)$ be the space of complex valued essentially bounded functions with two functions identified if they agree almost everywhere. This is a C^*-algebra with the norm being given by the essential supremum.

For a noncommutative example consider a Hilbert space $\mathcal{H}$. Then the space of all bounded operators $\mathcal{B}(\mathcal{H})$ is a C^*-algebra when endowed with the operator norm $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$.

Proposition 1.1.15. *Let $\mathcal{H}$ be a Hilbert space and suppose $x \in \mathcal{B}(\mathcal{H})$, then $\|x^*x\| = \|x\|^2$.*

Proof. We clearly have $\|x^*x\| \leq \|x^*\|\|x\|$. Also, $\|x^*\| = \|x\|$ since

$$\|x\| = \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x\xi, \eta \rangle| = \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle \xi, x^*\eta \rangle| = \|x^*\|.$$

To see the reverse inequality just note that

$$\begin{aligned}\|x\|^2 &= \sup_{\xi\in\mathcal{H},\|\xi\|\leq 1} \langle x\xi, x\xi\rangle \\ &\leq \sup_{\xi,\eta\in\mathcal{H},\|\xi\|,\|\eta\|\leq 1} |\langle x\xi, x\eta\rangle| \\ &= \sup_{\xi,\eta\in\mathcal{H},\|\xi\|,\|\eta\|\leq 1} |\langle x^*x\xi, \eta\rangle| = \|x^*x\|.\end{aligned}$$

■

1.2 The Gelfand transform

Let A be a abelian Banach algebra, the **spectrum** of A, denoted by $\sigma(A)$, is the set of non-zero continuous homomorphsims $\varphi : A \to \mathbb{C}$, which we endow with the weak*-topology as a subset of A^*.

Note that if A is unital, and $\varphi : A \to \mathbb{C}$ is a homomorphism, then it follows easily that $\ker(\varphi) \cap G(A) = \emptyset$. In particular, this shows that $\varphi(x) \in \sigma(x)$, since $x - \varphi(x) \in \ker(\varphi)$. Hence, for all $x \in A$ we have $|\varphi(x)| \leq r(x) \leq \|x\|$. Since, $\varphi(1) = 1$ this shows that $\|\varphi\| = 1$. We also have $\|\varphi\| = 1$ in the non-unital case, this will follow from Theorem 2.1.1.

It is easy to see that when A is unital $\sigma(A)$ is closed and bounded, by the Banach-Alaoglu theorem it is then a compact Hausdorff space.

Proposition 1.2.1. *Let A be a unital abelian Banach algebra. Then the association $\varphi \mapsto \ker(\varphi)$ gives a bijection between the spectrum of A and the space of maximal ideals.*

Proof. If $\varphi \in \sigma(A)$ then $\ker(\varphi)$ is clearly an ideal, and if we have a larger ideal I, then there exists $x \in I$ such that $\varphi(x) \neq 0$, hence $1 - x/\varphi(x) \in \ker(\varphi) \subset I$ and so $1 = (1 - x/\varphi(x)) + x/\varphi(x) \in I$ which implies $I = A$.

Conversely, if $I \subset A$ is a maximal ideal, then $I \cap G(A) = \emptyset$ and hence $\|1 - y\| \geq 1$ for all $y \in I$. Thus, $\overline{I}$ is also an ideal and $1 \notin I$ which shows that $I = \overline{I}$ by maximality. We then have that A/I is a unital Banach algebra, and since I is maximal we have that all non-zero elements of A/I are invertible. Thus, by the Gelfand-Mazur theorem we have $A/I \cong \mathbb{C}$ and hence the projection map $\pi : A \to A/I \cong \mathbb{C}$ gives a continuous homomorphism with kernel I. ■

Corollary 1.2.2. *Let A be a unital abelian Banach algebra, and $x \in A \setminus G(A)$. Then there exists $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. In particular, $\sigma(A)$ is a non-empty compact set.*

Proof. If $x \notin G(A)$ then the ideal generated by x is proper, hence by Zorn's lemma we see that x is contained in a maximal ideal $I \subset A$, and from Proposition 1.2.1 there exists $\varphi \in \sigma(A)$ such that $\Gamma(x)(\varphi) = \varphi(x) = 0$.

Considering $x = 0$ shows that $\sigma(A) \neq \emptyset$. We leave it as an exercise to see that $\sigma(A)$ is a weak*-closed subset of A^*, which is then compact by the Banach-Alaoglu theorem. ∎

Suppose A is a unital C^*-algebra which is generated (as a unital C^*-algebra) by a single element x, if $\lambda \in \sigma(x)$ then we can consider the closed ideal generated by $x - \lambda$ which is maximal since x generates A. This therefore induces a map from $\sigma(x)$ to $\sigma(A)$. We leave it to the reader to check that this map is actually a homeomorphism.

Theorem 1.2.3. *Let K be a compact Hausdorff space. For each $k \in K$ denote by $\varphi_k : C(K) \to \mathbb{C}$ the homomorphism given by $\varphi_k(f) = f(k)$, then $K \ni k \mapsto \varphi_k \in \sigma(C(K))$ is a homeomorphism.*

Proof. Since $C(K)$ separates points it follows that $k \mapsto \varphi_k$ is injective. If $\{k_i\} \subset K$ is a net such that $k_i \to k$, then for any $f \in C(K)$ we have $\varphi_{k_i}(f) = f(k_i) \to f(k) = \varphi_k(f)$, hence $\varphi_{k_i} \to \varphi_k$ in the weak*-topology. Thus, $k \mapsto \varphi_k$ is continuous and then to see that it is a homeomorphism it is enough to show that it is surjective. Which, by Proposition 1.2.1 is equivalent to showing that for every maximal ideal I in $C(K)$ there exists $k \in K$, such that $I = \{f \in C(K) \mid f(k) = 0\}$.

Suppose therefore that $I \subset C(K)$ is an ideal such that $I \not\subset \{f \in C(K) \mid f(k) = 0\}$, for any $k \in K$. Thus, for every $k \in K$ there exists $f_k \in I \setminus \{f \in C(K) \mid f(k) = 0\}$. If we let $O_k = \{x \in K \mid f_k(x) \neq 0\}$, then O_k is open and $k \in O_k$. As K is compact there then exists $k_1, \ldots, k_n \in K$ such that $K = \cup_{i=1}^n O_{k_i}$. If $\tilde{f} = \sum_{i=1}^n |f_{k_i}|^2$, then $\tilde{f} \in I$, and $\tilde{f}(x) > 0$ for all $x \in K$. Hence, $\tilde{f}$ is invertible, showing that $I = C(K)$ is not proper. ∎

Let A be a unital abelian Banach algebra, the **Gelfand transform** is the map $\Gamma : A \to C(\sigma(A))$ defined by

$$\Gamma(x)(\varphi) = \varphi(x).$$

Theorem 1.2.4. *Let A be a unital abelian Banach algebra, then the Gelfand transform is a contractive homomorphism, and $\Gamma(x)$ is invertible in $C(\sigma(A))$ if and only if x is invertible in A.*

Proof. It is easy to see that the Gelfand transform is a contractive homomorphism. Also, if $x \in G(A)$, then $\Gamma(a)\Gamma(a^{-1}) = \Gamma(aa^{-1}) = \Gamma(1) = 1$, hence $\Gamma(x)$ is invertible. Conversely, if $x \notin G(A)$ then by Corollary 1.2.2 there exists $\varphi \in \sigma(A)$ such that $\Gamma(x)(\varphi) = \varphi(x) = 0$. Hence, in this case $\Gamma(x)$ is not invertible. ∎

Corollary 1.2.5. *Let A be a unital abelian Banach algebra, then $\sigma(\Gamma(x)) = \sigma(x)$, and in particular $\|\Gamma(x)\| = r(\Gamma(x)) = r(x)$, for all $x \in A$.*

Theorem 1.2.6. *Let A be a unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C(\sigma(A))$ gives an isometric $*$-isomorphism between A and $C(\sigma(A))$.*

Proof. If x is self-adjoint then from Lemma 1.1.11 we have $\sigma(\Gamma(x)) = \sigma(x) \subset \mathbb{R}$, and hence $\overline{\Gamma(x)} = \Gamma(x^*)$. In general, if $x \in A$ we can write x as $x = a + ib$ where $a = \frac{x+x^*}{2}$ and $b = \frac{i(x^*-x)}{2}$ are self-adjoint. Hence, $\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}$ and so Γ is a $*$-homomorphism.

By Proposition 1.1.8 and the previous corollary, if $x \in A$ we have

$$\|x\| = r(x) = r(\Gamma(x)) = \|\Gamma(x)\|,$$

and so Γ is isometric, and in particular injective.

To show that Γ is surjective note that $\Gamma(A)$ is self-adjoint, and closed since Γ is isometric. Moreover, $\Gamma(A)$ contains the constants and clearly separates points, hence $\Gamma(A) = C(\sigma(A))$ by the Stone-Weierstrauss theorem. ∎

1.3 Continuous functional calculus

Let A be a C^*-algebra. An element $x \in A$ is:

- **positive** if $x = y^*y$ for some $y \in A$.
- a **projection** if $x^* = x^2 = x$.
- **unitary** if A is unital, and $x^*x = xx^* = 1$.
- **isometric** if A is unital, and $x^*x = 1$.
- **partially isometric** if x^*x is a projection.

We denote by A_+ the set of positive elements, and $a, b \in A$ are two self-adjoint elements then we write $a \leq b$ if $b - a \in A_+$. Note that if $x \in A$ then $x^* A_+ x \subset A_+$, in particular, if $a, b \in A$ are self-adjoint such that $a \leq b$, then $x^* a x \leq x^* b x$.

Since we have seen above that if A is generated as a unital C^*-algebra by a single normal element $x \in A$, then we have a natural homeomorphism $\sigma(x) \cong \sigma(A)$. Thus by considering the inverse Gelfand transform we obtain an isomorphism between $C(\sigma(x))$ and A which we denote by $f \mapsto f(x)$.

Theorem 1.3.1 (Continuous functional calculus)**.** *Let A and B be a unital C^*-algebras, with $x \in A$ normal, then this functional calculus satisfies the following properties:*

(i) *The map $f \mapsto f(x)$ is a homomorphism from $C(\sigma(x))$ to A, and if $f(z, \overline{z}) = \sum_{j,k=0}^n a_{j,k} z^j \overline{z}^k$ is a polynomial, then $f(x) = \sum_{j,k=0}^n a_{j,k} x^j (x^*)^k$.*

(ii) *For $f \in C(\sigma(x))$ we have $\sigma(f(x)) = f(\sigma(x))$.*

(iii) *If $\Phi : A \to B$ is a C^*-homomorphism then $\Phi(f(x)) = f(\Phi(x))$.*

(iv) *If $x_n \in A$ is a sequence of normal elements such that $\|x_n - x\| \to 0$, Ω is a compact neighborhood of $\sigma(x)$, and $f \in C(\Omega)$, then for large enough n we have $\sigma(x_n) \subset \Omega$ and $\|f(x_n) - f(x)\| \to 0$.*

Proof. Parts (i), and (ii) follow easily from Theorem 1.2.6. Part (iii) is obvious for polynomials and then follows for all continuous functions by approximation.

For part (iv), the fact that $\sigma(x_n) \subset \Omega$ for large n follows from continuity of inversion. If we write $C = \sup_n \|x_n\|$ and we have $\varepsilon > 0$ arbitrary, then we may take a polynomial $g : \Omega \to \mathbb{C}$ such that $\|f - g\|_\infty < \varepsilon$ and we have

$$\limsup_{n\to\infty} \|f(x_n) - f(x)\| \leq 2\|f - g\|_\infty C + \limsup_{n\to\infty} \|g(x_n) + g(x)\| < 2C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have $\lim_{n\to\infty} \|f(x_n) - f(x)\| = 0$. ■

1.3.1 The non-unital case

If A is not a unital Banach-algebra then we may consider the space $\tilde{A} = A \oplus \mathbb{C}$ which is a algebra with multiplication

$$(x \oplus \alpha) \cdot (y \oplus \beta) = (xy + \alpha y + \beta x) \oplus \alpha\beta.$$

If, moreover, A has an involution $*$, then we may endow $\tilde{A}$ with an involution given by $(x \oplus \alpha)^* = x^* \oplus \overline{\alpha}$. We may place a norm on $\tilde{A}$ by setting $\|x \oplus \alpha\| = \|x\| + |\alpha|$, and in this way $\tilde{A}$ is a Banach algebra, the **unitization** of A, and the natural inclusion $A \subset \tilde{A}$ is an isometric inclusion.

If A is a C^*-algebra, then the norm defined above is not a C^*-norm. Instead, we may consider the norm given by

$$\|x \oplus \alpha\| = \sup_{y \in A, \|y\| \leq 1} \|xy + \alpha y\|.$$

In the setting of C^*-algebras we call $\tilde{A}$, with this norm, the **unitization** of A.

Proposition 1.3.2. *Let A be a non-unital C^*-algebra, then the unitization $\tilde{A}$ is again a C^*-algebra, and the map $x \mapsto x \oplus 0$ is an isometric $*$-isomorphism of A onto a maximal ideal in $\tilde{A}$.*

Proof. The map $x \mapsto x \oplus 0$ is indeed isometric since on one hand we have $\|x \oplus 0\| = \sup_{y \in A, \|y\| \leq 1} \|xy\| \leq \|x\|$, while on the other hand if $x \neq 0$, and we set $y = x^*/\|x^*\|$ then we have $\|x\| = \|xx^*\|/\|x^*\| = \|xy\| \leq \|x \oplus 0\|$.

The norm on $\tilde{A}$ is nothing but the operator norm when we view $\tilde{A}$ as acting on A by left multiplication and hence we have that this is at least a semi-norm such that $\|xy\| \leq \|x\|\|y\|$, for all $x, y \in \tilde{A}$. To see that this is actually a norm note that if $\alpha \neq 0$, but $\|x \oplus \alpha\| = 0$ then for all $y \in A$ we have $\|xy + \alpha y\| \leq \|x \oplus \alpha\|\|y\| = 0$, and hence $e = -x/\alpha$ is a left identity for A. Taking adjoints we see that e^* is a right identity for A, and then $e = ee^* = e^*$ is an identity for A which contradicts that A is non-unital. Thus, $\| \cdot \|$ is indeed a norm.

It is easy to see then that $\tilde{A}$ is then complete, and hence all that remains to be seen is the C^*-identity. Since, each for each $y \in A$, $\|y\| \leq 1$ we have $(y \oplus 0)^*(x \oplus \alpha) \in A \oplus 0 \cong A$ it follows that the C^*-identity holds here, and so

$$\begin{aligned}
\|(x \oplus \alpha)^*(x \oplus \alpha)\| &\geq \|(y \oplus 0)^*(x \oplus \alpha)^*(x \oplus \alpha)(y \oplus 0)\| \\
&= \|(x \oplus \alpha)(y \oplus 0)\|^2.
\end{aligned}$$

Taking the supremum over all $y \in A$, $\|y\| \leq 1$ we then have

$$\|(x \oplus \alpha)^*(x \oplus \alpha)\| \geq \|x \oplus \alpha\|^2 \geq \|(x \oplus \alpha)^*(x \oplus \alpha)\|.$$

Note that the C^*-identity also entails that the adjoint is isometric. Indeed, for $x \oplus \alpha \in \tilde{A}$ we have $\|x \oplus \alpha\|^2 = \|(x \oplus \alpha)^*(x \oplus \alpha)\| \leq \|(x \oplus \alpha)^*\|\|x \oplus \alpha\|$, and hence $\|x \oplus \alpha\| \leq \|(x \oplus \alpha)^*\|$, and the reverse inequality then follows from symmetry. ∎

Lemma 1.3.3. *If A is a non-unital abelian C^*-algebra, then any multiplicative linear functional $\varphi \in \sigma(A)$ has a unique extension $\tilde{\varphi} \in \sigma(\tilde{A})$.*

Proof. If we consider $\tilde{\varphi}(x \oplus \alpha) = \varphi(x) + \alpha$ then the result follows easily. ∎

In particular, this shows that $\sigma(A)$ is homeomorphic to $\sigma(\tilde{A}) \setminus \{\varphi_0\}$ where φ_0 is defined by $\varphi(x, \alpha) = \alpha$. Thus, $\sigma(A)$ is locally compact.

If $x \in A$ then the **spectrum** $\sigma(x)$ of x is defined to be the spectrum of $x \oplus 0 \in \tilde{A}$. Note that for a non-unital C^*-algebra A, since $A \subset \tilde{A}$ is an ideal it follows that $0 \in \sigma(x)$ whenever $x \in A$.

By considering the embedding $A \subset \tilde{A}$ we are able to extend the spectral theorem and continuous functional calculus to the non-unital setting. We leave the details to the reader.

Theorem 1.3.4. *Let A be a non-unital abelian C^*-algebra, then the Gelfand transform $\Gamma : A \to C_0(\sigma(A))$ gives an isometric isomorphism between A and $C_0(\sigma(A))$.*

Theorem 1.3.5. *Let A be a C^*-algebra, and $x \in A$ a normal element, then if $f \in C(\sigma(x))$ such that $f(0) = 0$, then $f(x) \in A \subset \tilde{A}$.*

Exercise 1.3.6. Suppose K is a non-compact, locally compact Hausdorff space, and $K \cup \{\infty\}$ is the one point compactification. Show that we have a natural isomorphism $C(K \cup \{\infty\}) \cong \widetilde{C_0(K)}$.

1.4 Applications of functional calculus

Given any element x in a C^*-algebra A, we can decompose x uniquely as a sum of a self-adoint and skew-adjoint elements $\frac{x+x^*}{2}$ and $\frac{x-x^*}{2}$. We refer to the self-adjoint elements $\frac{x+x^*}{2}$ and $i\frac{x^*-x}{2}$ the **real** and **imaginary** parts of x, note that the real and imaginary parts of x have norms no grater than that of x.

Proposition 1.4.1. *Let A be a unital C^*-algebra, then every element is a linear combination of four unitaries.*

Proof. If $x \in A$ is self-adjoint and $\|x\| \leq 1$, then $u = x + i(1 - x^2)^{1/2}$ is a unitary and we have $x = \frac{1}{2}(u + u^*)$. In general, we can decompose x into its real and imaginary parts and then write each as a linear combination of two unitaries. ∎

Also, if $x \in A$ is self-adjoint then from above we know that $\sigma(x) \subset \mathbb{R}$, hence by considering $x_+ = (0 \vee t)(x)$ and $x_- = -(0 \wedge t)(x)$ it follows easily from functional calculus that $\sigma(x_+), \sigma(x_-) \subset [0, \infty)$, $x_+x_- = x_-x_+ = 0$, and $x = x_+ - x_-$. We call x_+ and x_- the **positive** and **negative** parts of x.

1.4.1 The positive cone

Lemma 1.4.2. *Suppose we have self-adjoint elements $x, y \in A$ such that $\sigma(x), \sigma(y) \subset [0, \infty)$ then $\sigma(x + y) \subset [0, \infty)$.*

Proof. Let $a = \|x\|$, and $b = \|y\|$. Since x is self-adjoint and $\sigma(x) \subset [0, \infty)$ we have $\|a - x\| = r(a - x) = a$. Similarly we have $\|b - y\| = b$ and since $\|x + y\| \leq a + b$ we have

$$\begin{aligned} \sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} &= r((a + b) - (x + y)) = \|(a + b) - (x + y)\| \\ &\leq \|x - a\| + \|y - b\| = a + b. \end{aligned}$$

Therefore, $\sigma(x + y) \subset [0, \infty)$. ∎

Proposition 1.4.3. *Let A be a C^*-algebra. A normal element $x \in A$ is*

(i) *self-adjoint if and only if $\sigma(x) \subset \mathbb{R}$.*

(ii) *positive if and only if $\sigma(x) \subset [0, \infty)$.*

(iii) *unitary if and only if $\sigma(x) \subset \mathbb{T}$.*

(iv) *a projection if and only if $\sigma(x) \subset \{0, 1\}$.*

Proof. Parts (i), (iii), and (iv) all follow easily by applying continuous functional calculus. For part (ii) if x is normal and $\sigma(x) \subset [0, \infty)$ then $x = (\sqrt{x})^2 = (\sqrt{x})^* \sqrt{x}$ is positive. It also follows easily that if $x = y^*y$ where y is normal then $\sigma(x) \subset [0, \infty)$. Thus, the difficulty arises only when $x = y^*y$ where y is perhaps not normal.

Suppose $x = y^*y$ for some $y \in A$, to show that $\sigma(x) \subset [0, \infty)$, decompose x into its positive and negative parts $x = x_+ - x_-$ as described above. Set

$z = yx_-$ and note that $z^*z = x_-(y^*y)x_- = -x_-^3$, and hence $\sigma(zz^*) \subset \sigma(z^*z) \subset (-\infty, 0]$.

If $z = a+ib$ where a and b are self-adjoint, then we have $zz^*+z^*z = 2a^2+2b^2$, hence we also have $\sigma(zz^*+z^*z) \subset [0,\infty)$ and so by Lemma 1.4.2 we have $\sigma(z^*z) = \sigma((2a^2+2b^2) - zz^*) \subset [0,\infty)$. Therefore $\sigma(-x_-^3) = \sigma(z^*z) \subset \{0\}$ and since x_- is normal this shows that $x_-^3 = 0$, and consequently $x_- = 0$. ■

Corollary 1.4.4. *Let A be a C^*-algebra. An element $x \in A$ is a partial isometry if and only if x^* is a partial isometry.*

Proof. Since x^*x is normal, it follows from the previous proposition that x is a partial isometry if and only if $\sigma(x^*x) \subset \{0,1\}$. Since $\sigma(x^*x) \cup \{0\} = \sigma(xx^*) \cup \{0\}$ this gives the result. ■

Corollary 1.4.5. *Let A be a C^*-algebra, then the set of positive elements forms a closed cone. Moreover, if $a \in A$ is self-adjoint, and A is unital, then we have $a \leq \|a\|$.*

Proposition 1.4.6. *Let A be a C^*-algebra, and suppose $x, y \in A_+$ such that $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$. Moreover, if A is unital and $x, y \in A$ are invertible, then $y^{-1} \leq x^{-1}$.*

Proof. First consider the case that A is unital and x and y are invertible, then we have

$$y^{-1/2}xy^{-1/2} \leq 1,$$

hence

$$\begin{aligned} x^{1/2}y^{-1}x^{1/2} &\leq \|x^{1/2}y^{-1}x^{1/2}\| = r(x^{1/2}y^{-1}x^{1/2}) \\ &= r(y^{-1/2}xy^{-1/2}) \leq 1. \end{aligned}$$

Conjugating by $x^{-1/2}$ gives $y^{-1} \leq x^{-1}$.

We also have

$$\|y^{-1/2}x^{1/2}\|^2 = \|y^{-1/2}xy^{-1/2}\| \leq 1,$$

therefore

$$\begin{aligned} y^{-1/4}x^{1/2}y^{-1/4} &\leq \|y^{-1/4}x^{1/2}y^{-1/4}\| = r(y^{-1/4}x^{1/2}y^{-1/4}) \\ &= r(y^{-1/2}x^{1/2}) \leq \|y^{-1/2}x^{1/2}\| \leq 1. \end{aligned}$$

Conjugating by $y^{1/4}$ gives $x^{1/2} \leq y^{1/2}$.

In the general case we may consider the unitization of A, and note that if $\varepsilon > 0$, then we have $0 \leq x + \varepsilon \leq y + \varepsilon$, where $x + \varepsilon$, and $y + \varepsilon$ are invertible, hence from above we have

$$(x + \varepsilon)^{1/2} \leq (y + \varepsilon)^{1/2}.$$

Taking the limit as $\varepsilon \to 0$ we obtain the result. ■

In general, a continuous real valued function f defined on an interval I is said to be **operator monotone** if $f(a) \leq f(b)$ whenever $\sigma(a), \sigma(b) \subset I$, and $a \leq b$. The previous proposition shows that the functions $f(t) = \sqrt{t}$, and $f(t) = -1/t$, $t > 0$ are operator monotone.

Note that if $x \in A$ is an arbitrary element of a C^*-algebra A, then x^*x is positive and hence we may define the **absolute value** of x as the unique element $|x| \in A_+$ such that $|x|^2 = x^*x$.

Corollary 1.4.7. *Let A be a C^*-algebra, then for $x, y \in A$ we have $|xy| \leq \|x\||y|$.*

Proof. Since $|xy|^2 = y^*x^*xy \leq \|x\|^2 y^*y$, this follows from the previous proposition. ■

1.4.2 Extreme points

Given a involutive normed algebra A, we denote by $(A)_1$ the unit ball of A, and $A_{\text{s.a.}}$ the subspace of self-adjoint elements.

Proposition 1.4.8. *Let A be a C^*-algebra.*

(i) The extreme points of $(A_+)_1$ are the projections of A.

(ii) The extreme points of $(A_{\text{s.a.}})_1$ are the self-adjoint unitaries in A.

(iii) Every extreme point of $(A)_1$ is a partial isometry in A.

Proof. (i) If $x \in (A_+)_1$, then we have $x^2 \leq 2x$, and $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. Hence if x is an extreme point then we have $x = x^2$ and so x is a projection. For the converse we first consider the case when A is abelian, and so we may assume $A = C_0(K)$ for some locally compact Hausdorff space K. If x is a projection then $x = 1_E$ is the characteristic function on some open and closed set $E \subset K$, hence the result follows easily from the fact that 0 and 1 are extreme points of $[0, 1]$.

For the general case, suppose $p \in A$ is a projection, if $p = \frac{1}{2}(a + b)$ then $\frac{1}{2}a = p - b \leq p$, and hence $0 \leq (1 - p)a(1 - p) \leq 0$, thus $a = ap = pa$. We therefore have that a, b, and p commute and hence the result follows from the abelian case.

(ii) First note that if A is unital then 1 is an extreme point in the unit ball. Indeed, if $1 = \frac{1}{2}(a + b)$ where $a, b \in (A)_1$, then we have the same equation when replacing a and b by their real parts. Thus, assuming a and b are self-adjoint we have $\frac{1}{2}a = 1 - \frac{1}{2}b$ and hence a and b commute. By considering the unital C^*-subalgebra generated by a and b we may assume $A = C(K)$ for some compact Hausdorff space K, and then it is an easy exercise to conclude that $a = b = 1$.

If u is a unitary in A, then the map $x \mapsto ux$ is a linear isometry of A, thus since 1 is an extreme point of $(A)_1$ it follows that u is also an extreme point. In particular, if u is self-adjoint then it is an extreme point of $(A_{\text{s.a.}})_1$.

Conversely, if $x \in (A_{\text{s.a.}})_1$ is an extreme point then if $x_+ = \frac{1}{2}(a + b)$ for $a, b \in (A_+)_1$, then $0 = x_-x_+x_- = \frac{1}{2}(x_-ax_- + x_-bx_-) \geq 0$, hence we have $(a^{1/2}x_-)^*(a^{1/2}x_-) = x_-ax_- = 0$. We conclude that $ax_- = x_-a = 0$, and similarly $bx_- = x_-b = 0$. Thus, $a - x_-$ and $b - x_-$ are in $(A_{\text{s.a.}})_1$ and $x = \frac{1}{2}((a - x_-) + (b - x_-))$. Since x is an extreme point we conclude that $x = a - x_- = b - x_-$ and hence $a = b = x_+$.

We have shown now that x_+ is an extreme point in $(A_+)_1$ and thus by part (i) we conclude that x_+ is a projection. The same argument shows that x_- is also a projection, and thus x is a self-adjoint unitary.

(iii) If $x \in (A)_1$ such that x^*x is not a projection then by applying functional calculus to x^*x we can find an element $y \in A_+$ such that $x^*xy = yx^*x \neq 0$, and $\|x(1 \pm y)\|^2 = \|x^*x(1 \pm y)^2\| \leq 1$. Since $xy \neq 0$ we conclude that $x = \frac{1}{2}((x + xy) + (x - xy))$ is not an extreme point of $(A)_1$. ■

Chapter 2

Representations and states

2.1 Approximate identities

If A is a Banach algebra, then a **left (resp. right) approximate identity** consists of a uniformly bounded net $\{a_\lambda\}_\lambda$ such that $\|a_\lambda x - x\| \to 0$, for all $x \in A$. An **approximate identity** is a net which is both a left and right approximate identity. If A is a Banach algebra, then the **opposite algebra** $A^{\rm op}$ is the Banach algebra which has the same Banach space structure as A, but with a new multiplication given by $x \cdot y = yx$. Then A has a left approximate identity if and only if $A^{\rm op}$ has a right approximate identity.

Theorem 2.1.1. *Let A be a C^*-algebra, and let $I \subset A$ be a left ideal, then there exists an increasing net $\{a_\lambda\}_\lambda \subset I$ of positive elements such that for all $x \in I$ we have*

$$\|xa_\lambda - x\| \to 0.$$

In particular, I has a right approximate identity. Moreover, if A is separable then the net can be taken to be a sequence.

Proof. Consider Λ to be the set of all finite subsets of $I \subset A \subset \tilde{A}$, ordered by inclusion. If $\lambda \in \Lambda$ we consider

$$h_\lambda = \sum_{x \in \lambda} x^* x, \qquad a_\lambda = |\lambda| h_\lambda (1 + |\lambda| h_\lambda)^{-1}.$$

Then we have $a_\lambda \in I$ and $0 \leq a_\lambda \leq 1$. If $\lambda \leq \lambda'$ then we clearly have $h_\lambda \leq h_{\lambda'}$ and hence by Proposition 1.4.6 we have that

$$\frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_{\lambda'}\right)^{-1} \leq \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_\lambda\right)^{-1}.$$

Therefore

$$a_\lambda = 1 - \frac{1}{|\lambda|}\left(\frac{1}{|\lambda|} + h_\lambda\right)^{-1} \leq 1 - \frac{1}{|\lambda'|}\left(\frac{1}{|\lambda'|} + h_{\lambda'}\right)^{-1} = a_{\lambda'}.$$

If $y \in \lambda$ then we have

$$(y(1-a_\lambda))^*(y(1-a_\lambda)) \leq \sum_{x\in\lambda}(x(1-a_\lambda))^*(x(1-a_\lambda)) = (1-a_\lambda)h_\lambda(1-a_\lambda).$$

But $\|(1-a_\lambda)h_\lambda(1-a_\lambda)\| = \|h_\lambda(1+|\lambda|h_\lambda)^{-2}\| \leq \frac{1}{4|\lambda|}$, from which it follows easily that $\|y - ya_\lambda\| \to 0$, for all $y \in I$.

If A is separable then so is $\overline{I}$, hence there exists a countable subset $\{x_n\}_{n\in\mathbb{N}} \subset I$ which is dense in I. If we take $\lambda_n = \{x_1, \ldots, x_n\}$, then clearly $a_n = a_{\lambda_n}$ also satisfies

$$\|y - ya_n\| \to 0. \qquad ■$$

If I is self-adjoint then we also have $\|a_\lambda x - x\| = \|x^* a_\lambda - x^*\| \to 0$ and in this case $\{a_\lambda\}$ is an approximate identity. Taking $I = A$ we obtain the following corollary.

Corollary 2.1.2. *Every C^*-algebra has an approximate identity consisting of an increasing net of positive elements.*

Using the fact that the adjoint is an isometry we also obtain the following corollary.

Corollary 2.1.3. *Let A be a C^*-algebra, and $I \subset A$ a closed two sided ideal. Then I is self-adjoint. In particular, I is a C^*-algebra.*

Exercise 2.1.4. Show that if A is a C^*-algebra such that $x \leq y \implies x^2 \leq y^2$, for all $x, y \in A_+$, then A is abelian.

Exercise 2.1.5. Let A be a C^*-algebra and $I \subset A$ a non-trivial closed two sided ideal. Show that A/I is again a C^*-algebra.

2.2 The Cohen-Hewitt factorization theorem

Let A be a Banach algebra with a left approximate identity. If X is a Banach space and $\pi : A \to \mathcal{B}(X)$ is a continuous representation, then a point $x \in X$ is a point of continuity if $\lim_{\lambda\to\infty} \|\pi(e_\lambda)x - x\| \to 0$, for some left approximate identity $\{e_\lambda\}$. Note that if $\{\tilde{e}_\alpha\}$ is another left approximate identity and $x \in X$ is a point of continuity, then we have $\lim_{\alpha\to\infty} \|\tilde{e}_\alpha e_\lambda x - e_\lambda x\| = 0$, for each λ, and hence it follows that x is a point of continuity with respect to any left approximate identity. We denote by X_c the set of points of continuity.

Theorem 2.2.1 (The Cohen-Hewitt factorization theorem)**.** *Let A be a Banach algebra with a left approximate identity, X a Banach space, and $\pi : A \to \mathcal{B}(X)$ a continuous representation. Then X_c is a closed invariant subspace, and we have $X_c = \pi(A)X$.*

Proof. It's easy to see that X_c is a closed invariant subspace, and it is also easy to see that $\pi(A)X \subset X_c$. Thus, it suffices to show $X_c \subset \pi(A)X$. To show this, we consider the Banach algebra unitization $\tilde{A}$, and extend π to a representation $\tilde{\pi} : \tilde{A} \to \mathcal{B}(X)$ by $\tilde{\pi}(x, \alpha) = \pi(x) + \alpha$.

Let $\{e_i\}_{i\in I}$ denote a left approximate unit, and set $M = \sup_i \|e_i\|$, so that $1 \leq M < \infty$. Set $\gamma = 1/4M$. We claim that $\gamma e_i + (1-\gamma)$ is invertible and $\lim_{i\to\infty} \tilde{\pi}((\gamma e_i + (1-\gamma))^{-1})x = x$, for all $x \in X_c$. Indeed, we have

$$\|\gamma e_i - \gamma\| = \gamma(\|e_i\| + 1) \leq (M+1)/4M \leq 1/2. \tag{2.1}$$

Thus, $\gamma e_i + (1-\gamma)$ is invertible, and we have $(\gamma e_i + (1-\gamma))^{-1} = \sum_{k=0}^{\infty}(\gamma - \gamma e_i)^k$, hence

$$\|(\gamma e_i + (1-\gamma))^{-1}\| \leq \sum_{k=0}^{\infty} \gamma^k (1+M)^k \leq 2. \tag{2.2}$$

We then have

$$\lim_{i\to\infty} \tilde{\pi}((\gamma e_i + (1-\gamma))^{-1})x = \lim_{i\to\infty} \tilde{\pi}((\gamma e_i + (1-\gamma))^{-1}(\gamma e_i + (1-\gamma)))x = x.$$

Fix $x \in X_c$. We set $a_0 = 1$, and inductively define a sequence of invertible elements $\{a_n\} \subset \tilde{A}$, satisfying the following properties:

- $a_n - (1-\gamma)^n \in A$.
- $\|a_n - a_{n-1}\| < 2^{-n} + (1-\gamma)^{n-1}$.
- $\|a_n^{-1}\| \leq 2^n$.
- $\|\pi(a_n^{-1})x - \pi(a_{n-1}^{-1})x\| < 2^{-n}$.

Indeed, suppose that a_{n-1} has been constructed satisfying the above properties. Since $\{e_i\}_{i\in I}$ is a left approximate unit for A, and since $a_{n-1} - (1-\gamma)^{n-1} \in A$, there exists $i \in I$ such that $\|(e_i - 1)(a_{n-1} - (1-\gamma)^{n-1})\| < 2^{-n}$. Moreover from above we may choose i so that it also satisfies $\|\pi((\gamma e_i + (1-\gamma))^{-1})x - x\| < 2^{-2n}$.

If we set $a_n = (\gamma e_i + (1-\gamma))a_{n-1}$ then

$$\begin{aligned} a_n - (1-\gamma)^n &= (\gamma e_i + (1-\gamma))a_{n-1} - (1-\gamma)^n \\ &= \gamma e_i a_{n-1} + (1-\gamma)(a_{n-1} - (1-\gamma)^{n-1}) \in A, \end{aligned}$$

and from (2.1) we then have

$$\begin{aligned} \|a_n - a_{n-1}\| &= \|(\gamma e_i - \gamma)a_{n-1}\| \\ &\leq \|(\gamma e_i - \gamma)(a_{n-1} - (1-\gamma)^{n-1})\| + (1-\gamma)^{n-1}\|\gamma e_i - \gamma\| \\ &< 2^{-n} + (1-\gamma)^{n-1}. \end{aligned}$$

Moreover, from (2.2) we have

$$\|a_n^{-1}\| \leq \|a_{n-1}^{-1}\|\|(\gamma e_i + (1-\gamma))^{-1}\| \leq 2^n,$$

and hence

$$\|\pi(a_n^{-1})x - \pi(a_{n-1}^{-1})x\| \leq \|a_{n-1}^{-1}\|\|\pi((\gamma e_i + (1-\gamma))^{-1})x - x\| < 2^{-n}.$$

Thus, we have that $\{a_n\}$ and $\{\tilde{\pi}(a_n^{-1})x\}$ are Cauchy and hence converge to elements a and y respectively. Note that since $a_n - (1-\gamma)^n \in A$ it follows that $a \in A$. We then have $x = \lim_{n\to\infty} \tilde{\pi}(a_n)(\tilde{\pi}(a_n^{-1})x) = \pi(a)y \in \pi(A)X$. ■

Corollary 2.2.2. *Let A be a Banach algebra with a left or right approximate identity. Then $A^2 = A$.*

Proof. If A has a left approximate identity then by considering left multiplication we obtain a representation of A into $\mathcal{B}(A)$ such that every point is a point of continuity. Hence the Cohen-Hewitt factorization theorem gives the result. If A has a right approximate identity then A^{op} has a left approximate identity the result again follows. ■

2.3 States

If A is a C^*-algebra then A^* is a Banach space which is also an A-bimodule given by $(a \cdot \psi \cdot b)(x) = \psi(bxa)$. Moreover, the bimodule structure is continuous since

$$\|a \cdot \psi \cdot b\| = \sup_{x \in (A)_1} |\psi(bxa)| \leq \sup_{x \in (A)_1} \|\psi\| \|bxa\| \leq \|\psi\| \|b\| \|a\|.$$

A linear functional $\varphi : A \to \mathbb{C}$ on a C^*-algebra A is **positive** if $\varphi(x) \geq 0$, whenever $x \in A_+$. Note that if $\varphi : A \to \mathbb{C}$ is positive then so is $a^* \cdot \varphi \cdot a$ for all $a \in A$. A positive linear functional is **faithful** if $\varphi(x) \neq 0$ for every non-zero $x \in A_+$, and a **state** if φ is positive, and $\|\varphi\| = 1$. The state space $S(A)$ is a convex closed subspace of the unit ball of A^*, and as such it is a compact Hausdorff space when endowed with the weak*-topology.

Note that if $\varphi \in S(A)$ then for all $x \in A$, $x = x^*$, then $\varphi(x) = \varphi(x_+ - x_-) \in \mathbb{R}$. Hence, if $y \in A$ then writing $y = y_1 + iy_2$ where y_j are self-adjoint for $j = 1, 2$, we have $\varphi(y^*) = \varphi(y_1) - i\varphi(y_2) = \overline{\varphi(y)}$. In general, we say a functional is **Hermitian** if $\varphi(y^*) = \overline{\varphi(y)}$, for all $y \in A$. Note that by defining $\varphi^*(y) = \overline{\varphi(y^*)}$ then we have that $\varphi + \varphi^*$, and $i(\varphi - \varphi^*)$ are each Hermitian.

Also note that a positive linear functional $\varphi : A \to \mathbb{C}$ is bounded. Indeed, if $\{x_n\}_n$ is any sequence of positive elements in $(A)_1$ then for any $(a_n)_n \in \ell^1\mathbb{N}$ we have $\sum_n a_n \varphi(x_n) = \varphi(\sum_n a_n x_n) < \infty$. This shows that $(\varphi(x_n))_n \in \ell^\infty\mathbb{N}$ and since the sequence was arbitrary we have that φ is bounded on the set of positive elements in $(A)_1$. Writing an element x in the usual way as $x = x_1 - x_2 + ix_3 - ix_4$ then shows that φ is bounded on the whole unit ball.

Lemma 2.3.1. *Let $\varphi : A \to \mathbb{C}$ be a positive linear functional on a C^*-algebra A, then for all $x, y \in A$ we have $|\varphi(y^*x)|^2 \leq \varphi(y^*y)\varphi(x^*x)$.*

Proof. Since φ is positive, the sesquilinear form defined by $\langle x, y\rangle = \varphi(y^*x)$ is non-negative definite. Thus, the result follows from the Cauchy-Schwarz inequality. ∎

Lemma 2.3.2. *Suppose A is a unital C^*-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is positive if and only $\|\varphi\| = \varphi(1)$.*

Proof. First suppose φ is a positive linear functional, then for all $x \in A$ we have $\varphi(\|x + x^*\| \pm (x + x^*)) \geq 0$. Since φ is Hermitian we then have

$$|\varphi(x)| = |\varphi(\frac{x + x^*}{2})| \leq \|\frac{x + x^*}{2}\|\varphi(1) \leq \|x\|\varphi(1),$$

showing $\|\varphi\| \leq \varphi(1) \leq \|\varphi\|$.

Now suppose $\|\varphi\| = \varphi(1)$, and $x \in A$ is a positive element such that $\varphi(x) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. For all $t \in \mathbb{R}$ we have

$$\begin{aligned}\alpha^2 + (\beta + t\|\varphi\|)^2 &= |\varphi(x + it)|^2 \\ &\leq \|x + it\|^2 \|\varphi\|^2 = (\|x\|^2 + t^2)\|\varphi\|^2.\end{aligned}$$

Subtracting $t^2\|\varphi\|^2$ from both sides of this inequality shows $2\beta t\|\varphi\| \leq \|x\|^2\|\varphi\|$, thus $\beta = 0$.

Also, we have

$$\|x\|\|\varphi\| - \varphi(x) = \varphi(\|x\| - x) \leq \|\|x\| - x\|\|\varphi\| \leq \|x\|\|\varphi\|,$$

hence $\alpha > 0$. ■

Proposition 2.3.3. *If $\varphi : A \to \mathbb{C}$ is a positive linear functional on a C^*-algebra A, then φ has a unique extension to a positive linear functional $\tilde{\varphi}$ on the unitization $\tilde{A}$, such that $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. Suppose $\varphi : A \to \mathbb{C}$ is a positive linear functional. If $\{a_\lambda\}_\lambda$ is an approximate identity consisting of positive contractions as given by Theorem 2.1.1, then we have that $\varphi(a_\lambda^2)$ is a bounded net and hence has a cluster point $\beta > 0$. If $x \in A$, $\|x\| \leq 1$, then $|\varphi(x)| = \lim_{\lambda\to\infty} |\varphi(a_\lambda x)| \leq \liminf_{\lambda\to\infty} \varphi(a_\lambda^2)^{1/2}\varphi(x^*x)^{1/2} \leq \beta^{1/2}\|x\|\|\varphi\|^{1/2}$. Thus, we have $\|\varphi\| \leq \beta$ and hence $\beta = \|\varphi\|$, since we also have $\varphi(a_\lambda^2) \leq \|\varphi\|$, for all λ. Since β was an arbitrary cluster point we then have $\|\varphi\| = \lim_{\lambda\to\infty} \varphi(a_\lambda)$.

If we define $\tilde{\varphi}$ on $\tilde{A}$ by $\tilde{\varphi}(x, \alpha) = \varphi(x) + \alpha\|\varphi\|$, then for all $x \in A$, and $\alpha \in \mathbb{C}$ we then have $\tilde{\varphi}(x, \alpha) = \lim_{\lambda\to\infty} \varphi(a_\lambda x a_\lambda + \alpha a_\lambda^2)$. Thus, we have

$$\tilde{\varphi}((x, \alpha)^*(x, \alpha)) = \lim_{\lambda\to\infty} \varphi((xa_\lambda + a_\lambda)^*(xa_\lambda + a_\lambda)) \geq 0.$$

Uniqueness of such an extension follows from the previous lemma. ■

Proposition 2.3.4. *Let A be a C^*-algebra and $x \in A$. For each $\lambda \in \sigma(x)$ there exists a state $\varphi \in S(A)$ such that $\varphi(x) = \lambda$.*

Proof. By considering the unitization, we may assume that A is unital. Consider the subspace $\mathbb{C}x + \mathbb{C}1 \subset A$, with the linear functional φ_0 on this space defined by $\varphi_0(\alpha x + \beta) = \alpha\lambda + \beta$, for $\alpha, \beta \in \mathbb{C}$. Since $\varphi_0(\alpha x + \beta) \in \sigma(\alpha x + \beta)$ we have that $\|\varphi_0\| = 1$.

By the Hahn-Banach theorem there exists an extension $\varphi : A \to \mathbb{C}$ such that $\|\varphi\| = 1 = \varphi(1)$. By Lemma 2.3.2 $\varphi \in S(A)$, and we have $\varphi(x) = \lambda$. ■

Proposition 2.3.5. *Let A be a C^*-algebra, and $x \in A$.*

(i) $x = 0$ if and only if $\varphi(x) = 0$ for all $\varphi \in S(A)$.

(ii) x is self-adjoint if and only if $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$.

(iii) x is positive if and only if $\varphi(x) \geq 0$ for all $\varphi \in S(A)$.

Proof. (i) If $\varphi(x) = 0$ for all $\varphi \in S(A)$ then writing $x = x_1 + ix_2$ where $x_j = x_j^*$, for $j = 1, 2$, we have $\varphi(x_j) = 0$ for all $\varphi \in S(A)$, $j = 1, 2$. Thus, $x_1 = x_2 = 0$ by Proposition 2.3.4

(ii) If $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$ then $\varphi(x - x^*) = \varphi(x) - \overline{\varphi(x)} = 0$, for all $\varphi \in S(A)$. Hence $x - x^* = 0$.

(iii) If $\varphi(x) \geq 0$ for all $\varphi \in S(A)$ then $x = x^*$ and by Proposition 2.3.4 we have $\sigma(x) \subset [0, \infty)$. ∎

2.3.1 The Gelfand-Naimark-Segal construction

A **representation** of a C^*-algebra A is a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. If $\mathcal{K} \subset \mathcal{H}$ is a closed subspace such that $\pi(x)\mathcal{K} \subset \mathcal{K}$ for all $x \in A$ then the restriction to this subspace determines a **sub-representation**. If the only sub-representations are the restrictions to $\{0\}$ or $\mathcal{H}$ then π is **irreducible** , which by the double commutant theorem is equivalent to the von Neumann algebra generated by $\pi(A)$ being $\mathcal{B}(\mathcal{H})$. Two representations $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\rho : A \to \mathcal{B}(\mathcal{K})$ are **equivalent** if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that $U\pi(x) = \rho(x)U$, for all $x \in A$.

If $\pi : A \to \mathcal{B}(\mathcal{H})$ is a representation, and $\xi \in \mathcal{H}$, $\|\xi\| = 1$ then we obtain a state on A by the formula $\varphi_\xi(x) = \langle \pi(x)\xi, \xi \rangle$. Indeed, if $x \in A$ then $\langle \pi(x^*x)\xi, \xi \rangle = \|\pi(x)\xi\|^2 \geq 0$. We now show that every state arises in this way.

Theorem 2.3.6 (The GNS construction)**.** *Let A be a unital C^*-algebra, and consider $\varphi \in S(A)$, then there exists a Hilbert space $L^2(A, \varphi)$, and a unique (up to equivalence) representation $\pi : A \to \mathcal{B}(L^2(A, \varphi))$, with a unit cyclic vector $1_\varphi \in L^2(A, \varphi)$ such that $\varphi(x) = \langle \pi(x)1_\varphi, 1_\varphi \rangle$, for all $x \in A$.*

Proof. By Corollary 2.3.3 we may assume that A is unital. Consider $A_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$. By Lemma 2.3.1 we have that $A_\varphi = \{x \in A \mid \varphi(yx) = 0, y \in A\}$, and from this we see that N_φ is a closed linear subspace.

We also see that N_φ is a left ideal since for $x \in N_\varphi$ Lemma 2.3.1 gives $\varphi((ax)^*(ax)) \leq \varphi(x^*x)^{1/2}\varphi(x^*(a^*a)^2x)^{1/2} = 0$.

We consider $\mathcal{H}_0 = A/N_\varphi$ which we endow with the inner product $\langle [x], [y] \rangle = \varphi(y^*x)$, where $[x]$ denotes the equivalence class of x in A/N_φ, (this is well defined since N_φ is a left ideal). This inner product is then positive definite, and hence we denote by $L^2(A, \varphi)$ the Hilbert space completion.

For $a \in A$ we consider the map $\pi_0(a) : \mathcal{H}_0 \to \mathcal{H}_0$ given by $\pi_0(a)[x] = [ax]$. Since N_φ is a left ideal this is well defined, and since $\|\pi_0(a)[x]\|^2 = \varphi((ax)^*(ax)) \leq \|a\|^2\varphi(x^*x)$ we have that this extends to a bounded operator $\pi(a) \in \mathcal{B}(L^2(A, \varphi))$ such that $\|\pi(a)\| \leq \|a\|$. The map $a \mapsto \pi(a)$ is clearly a homomorphism, and for $x, y \in A$ we have $\langle [x], \pi(a^*)[y] \rangle = \varphi(y^*a^*x) = \langle \pi(a)[x], [y] \rangle$, thus $\pi(a^*) = \pi(a)^*$. Also, if we consider $1_\varphi = [1] \in \mathcal{H}_0 \subset L^2(A, \varphi)$ then we have $\langle \pi(a)1_\varphi, 1_\varphi \rangle = \varphi(a)$.

If $\rho : A \to \mathcal{B}(\mathcal{K})$ and $\eta \in \mathcal{K}$ is a cyclic vector such that $\varphi(a) = \langle \rho(a)\eta, \eta \rangle$, then we can consider the map $U_0 : \mathcal{H}_0 \to \mathcal{K}$ given by $U_0([x]) = \rho(x)\eta$. We then have

$$\langle U_0([x]), U_0([y]) \rangle = \langle \rho(x)\eta, \rho(y)\eta = \langle \rho(y^*x)\eta, \eta = \varphi(y^*x) = \langle [x], [y] \rangle$$

which shows that U_0 is well defined and isometric. Also, for $a, x \in A$ we have

$$U_0(\pi(a)[x]) = U_0([ax]) = \rho(ax)\eta = \rho(a)U_0([x]).$$

Hence, U_0 extends to an isometry $U : L^2(A, \varphi) \to \mathcal{K}$ such that $U\pi(a) = \rho(a)U$ for all $a \in A$. Since η is cyclic, and $\rho(A)\eta \subset U(L^2(A, \varphi))$ we have that U is unitary. ∎

Corollary 2.3.7. *Let A be a C^*-algebra, then there exists a faithful representation.*

Proof. If we let π be the direct sum over all GNS representations corresponding to states, then this follows easily from Proposition 2.3.5. Note also that if A is separable, then so is $S(A)$ and by considering a countable dense subset of $S(A)$ we can construct a faithful representation onto a separable Hilbert space. ∎

If φ and ψ are two Hermitian linear functionals, we write $\varphi \leq \psi$ if $\varphi(a) \leq \psi(a)$ for all $a \in A_+$, alternatively, this is if and only if $\psi - \varphi$ is a positive linear functional. The following is a Radon-Nikodym type theorem for positive linear functionals.

Proposition 2.3.8. *Suppose φ and ψ are positive linear functionals on a C^*-algebra A such that ψ is a state. Then $\varphi \leq \psi$, if and only if there exists a unique $y \in \pi_\psi(A)'$ such that $0 \leq y \leq 1$ and $\varphi(a) = \langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle$ for all $a \in A$.*

Proof. First suppose that $y \in \pi_\psi(A)'$, with $0 \leq y \leq 1$. Then for all $a \in A$, $a \geq 0$ we have $\pi_\psi(a) y = \pi_\psi(a)^{1/2} y \pi_\psi(a)^{1/2} \leq \pi_\psi(a)$, hence $\langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle \leq \langle \pi_\psi(a) 1_\psi, 1_\psi \rangle = \psi(a)$.

Conversely, if $\varphi \leq \psi$, the Cauchy-Schwarz inequality implies

$$|\varphi(b^* a)|^2 \leq \varphi(a^* a)\varphi(b^* b) \leq \psi(a^* a)\psi(b^* b) = \|\pi_\psi(a) 1_\psi\|^2 \|\pi_\psi(b) 1_\psi\|^2.$$

Thus $\langle \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle_\varphi = \varphi(b^* a)$ is a well defined non-negative definite sesquilinear form on $\pi_\psi(A) 1_\psi$ which is bounded by 1, and hence extends to the closure $L^2(A, \psi)$.

Therefore there is an operator $y \in \mathcal{B}(L^2(A, \psi))$, $0 \leq y \leq 1$, such that $\varphi(b^* a) = \langle y \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle$, for all $a, b \in A$.

If $a, b, c \in A$ then

$$\begin{aligned} \langle y \pi_\psi(a) \pi_\psi(b) 1_\psi, \pi_\psi(c) 1_\psi \rangle &= \langle y \pi_\psi(ab) 1_\psi, \pi_\psi(c) 1_\psi \rangle = \varphi(c^* ab) \\ &= \langle y \pi_\psi(b) 1_\psi, \pi_\psi(a^*) \pi_\psi(c) 1_\psi \rangle \\ &= \langle \pi_\psi(a) y \pi_\psi(b) 1_\psi, \pi_\psi(c) 1_\psi \rangle. \end{aligned}$$

Thus, $y \pi_\psi(a) = \pi_\psi(a) y$, for all $a \in A$.

To see that y is unique, suppose that $0 \leq z \leq 1$, $z \in \pi_\psi(A)'$ such that $\langle \pi_\psi(a) z 1_\psi, 1_\psi \rangle = \langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle$ for all $a \in A$. Then $\langle (z - y) 1_\psi, \pi_\psi(a^*) 1_\psi \rangle = 0$ for all $a \in A$ and hence $z - y = 0$ since 1_ψ is a cyclic vector for $\pi_\psi(A)$. ■

2.3.2 Pure states

A state φ on a C^*-algebra A is said to be **pure** if it is an extreme point in $S(A)$.

Proposition 2.3.9. *A state φ on a C^*-algebra A is a pure state if and only if the corresponding GNS representation $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$ with corresponding cyclic vector 1_φ is irreducible.*

Proof. Suppose first that φ is pure. If $\mathcal{K} \subset L^2(A, \varphi)$ is a closed invariant subspace, then so is $\mathcal{K}^\perp$ and we may consider $\xi_1 = [\mathcal{K}](1_\varphi) \in \mathcal{K}$ and $\xi_2 = 1_\varphi - \xi_1 \in \mathcal{K}^\perp$. For $x \in A$ we have

$$\langle x \xi_1, \xi_1 \rangle + \langle x \xi_2, \xi_2 \rangle = \langle x 1_\varphi, 1_\varphi \rangle = \varphi(x).$$

Thus, either $\xi_1 = 0$, or $\xi_2 = 0$, since φ is pure. Since 1_φ is cyclic, we have that ξ_1 is cyclic for $\mathcal{K}$ and ξ_2 is cyclic for $\mathcal{K}^\perp$ showing that either $\mathcal{K} = \{0\}$ or else $\mathcal{K}^\perp = \{0\}$.

Conversely, suppose that π_φ is irreducible and $\varphi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$ where $\varphi_j \in S(A)$ for $j = 1, 2$, then we may consider the map $U : L^2(A, \varphi) \to L^2(A, \varphi_1) \oplus L^2(A, \varphi_2)$ such that $U(x1_\varphi) = (x\frac{1}{\sqrt{2}}1_{\varphi_1}) \oplus (x\frac{1}{\sqrt{2}}1_{\varphi_2})$, for all $x \in A$. It is not hard to see that U is a well defined isometry and $U\pi_\varphi(x) = (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))U$ for all $x \in A$. If we denote by $p_1 \in \mathcal{B}(L^2(A, \varphi_1) \oplus L^2(A, \varphi_2)$ the orthogonal projection onto $L^2(A, \varphi_1)$ then the operator $U^*p_1U \in \mathcal{B}(L^2(A, \varphi))$ commutes with $\pi_\varphi(A)$, and since π_φ is irreducible we then have that $U^*p_1U = \alpha \in \mathbb{C}$, and in fact $U^*p_1U = \langle U^*p_1U1_\varphi, 1_\varphi\rangle = 1/2$. Thus, $u_1 = \sqrt{2}p_1U$ implements an isometry from $L^2(A, \varphi)$ to $L^2(A, \varphi_1)$ such that $u_11_\varphi = 1_{\varphi_1}$, and $u_1\pi_\varphi(x) = \pi_{\varphi_1(x)}u_1$ for all $x \in A$. It then follows, in particular, that $\varphi_1(x) = \varphi(x)$, for all $x \in A$, hence $\varphi = \varphi_1 = \varphi_2$ showing that φ is pure. ∎

Note that the previous proposition, together with Proposition 2.3.8 shows also that a state φ is pure if and only if for any positive linear functional ψ such that $\psi \leq \varphi$ there exists a constant $\alpha \geq 0$ such that $\psi = \alpha\varphi$.

Since irreducible representations of an abelian C^*-algebra must be one dimensional, the following corollary follows from the above Proposition.

Corollary 2.3.10. *Let A be an abelian C^*-algebra, then the pure states on A agree with the spectrum $\sigma(A)$.*

Theorem 2.3.11. *Let A be a C^*-algebra, then the convex hull of the pure states on A are weak*-dense in $S(A)$.*

Proof. If A is unital, then the state space $S(A)$ is a weak* compact convex subset of A^*, and hence the convex hull of extreme states are dense in $S(A)$ by the Krein-Milman theorem.

If A is not unital, then consider the unitization $\tilde{A}$. Any irreducible representation of A extends to an irreducible representation of $\tilde{A}$, and conversely, for any irreducible representation of $\tilde{A}$ we must have that its restriction to A is irreducible, or else contains A in its kernel and hence is the representation given by $\pi_0(x, \alpha) = \alpha$.

Thus, any pure state on A extends uniquely to a pure state on $\tilde{A}$, and the only pure state on $\tilde{A}$ which does not arise in this way is $\varphi_0(x, \alpha) = \alpha$. Since every state on A extends to a state on $\tilde{A}$ we may then use the Krein-Milman theorem on the state space of $\tilde{A}$ to conclude that any state on A is a weak* limit of convex combinations of pure states on A and 0. However, since states satisfy $\|\varphi\| = 1$, we see that there must be no contribution from 0. ■

Corollary 2.3.12. *Let A be a C^*-algebra and $x \in A$, $x \neq 0$, then there exists an irreducible representation $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $\pi(x) \neq 0$.*

Proof. By Proposition 2.3.5 there exists a state φ on A such that $\varphi(x) \neq 0$, and hence by the previous theorem there exists a pure state φ_0 on A such that $\varphi_0(x) \neq 0$. Proposition 2.3.9 then shows that the corresponding GNS-representation gives an irreducible representation π such that $\pi(x) \neq 0$. ■

2.3.3 Jordan Decomposition

Theorem 2.3.13 (Jordan Decomposition)**.** *Let A be a C^*-algebra and $\varphi \in A^*$, a Hermitian linear functional, then there exist unique positive linear functionals $\varphi_+, \varphi_- \in A^*$ such that $\varphi = \varphi_+ - \varphi_-$, and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. Suppose $\varphi \in A^*$ is Hermitian. Let Σ denote the set of positive linear functionals on A, then Σ is a compact Hausdorff space when given the weak$*$ topology. Consider the map $\gamma : A \to C(\Sigma)$ given by $\gamma(a)(\psi) = \psi(a)$, then by Proposition 2.3.4 γ is isometric, and we also have $\gamma(A_+) \subset C(\Sigma)_+$.

By the Hahn-Banach theorem there exists a linear functional $\tilde{\varphi} \in C(\Sigma)^*$ such that $\|\tilde{\varphi}\| = \|\varphi\|$, and $\tilde{\varphi}(\gamma(a)) = \varphi(a)$, for all $a \in A$. By replading $\tilde{\varphi}$ with $\frac{1}{2}(\tilde{\varphi} + \tilde{\varphi}^*)$ we may assume that $\tilde{\varphi}$ is also Hermitian. By the Reisz representation theorem there then exists a signed Radon measure ν on Σ such that $\tilde{\varphi}(f) = \int f \, \mathrm{d}\nu$ for all $f \in C(\Sigma)$. By the Jordan decomposition of measures there exist positive measures ν_+, and ν_- such that $\nu = \nu_+ - \nu_-$, and $\|\nu\| = \|\nu_+\| + \|\nu_-\|$.

Define the linear functionals φ_+, and φ_- by setting $\varphi_+(a) = \int \gamma(a) \, \mathrm{d}\nu_+$, and $\varphi_-(a) = \int \gamma(a) \, \mathrm{d}\nu_-$, for all $a \in A$. Then since $\gamma(A_+) \subset C(\Sigma)_+$ it follows that φ_+, and φ_- are positive. Moreover, we have $\varphi = \varphi_+ - \varphi_-$, and we have $\|\varphi\| \leq \|\varphi_+\| + \|\varphi_-\| \leq \|\nu_+\| + \|\nu_-\| = \|\nu\| = \|\varphi\|$. ■

Corollary 2.3.14. *Let A be a C^*-algebra, then A^* is the span of positive linear functionals.*

Corollary 2.3.15. *Let A be a C^*-algebra and $\varphi \in A^*$, then there exists a representation $\pi : A \to \mathcal{B}(\mathcal{H})$, and vectors $\xi, \eta \in \mathcal{H}$, such that $\varphi(a) = \langle \pi(a)\xi, \eta \rangle$, for all $a \in A$.*

Proof. Let $\varphi \in A^*$ be given. By the previous corollary we have that $\varphi = \sum_{i=1}^n \alpha_i \psi_i$ for some $\alpha_i \in \mathbb{C}$, and ψ_i states. If we consider the GNS-representations $\pi_i : A \to \mathcal{B}(L^2(A, \psi_i))$, then setting $\pi = \oplus_{i=1}^n$, $\xi = \oplus_{i=1}^n \alpha_i 1_{\psi_i}$, and $\eta = \oplus_{i=1}^n 1_{\psi_i}$, we have $\varphi(a) = \langle \pi(a)\xi, \eta \rangle$, for all $a \in A$. ■

Note that $\|\varphi\| \leq \|\xi\| \|\eta\|$, however we may not have equality. We'll show in Theorem 3.11.7 below that we may also choose a representation π and vectors $\xi, \eta \in \mathcal{H}$ which additionally satisfy $\|\varphi\| = \|\xi\| \|\eta\|$.

Chapter 3

Bounded linear operators

Recall that if $\mathcal{H}$ is a Hilbert space then $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators is a C^*-algebra with norm

$$\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|,$$

and involution given by the adjoint, i.e., x^* is the unique bounded linear operator such that

$$\langle \xi, x^*\eta \rangle = \langle x\xi, \eta \rangle,$$

for all $\xi, \eta \in \mathcal{H}$.

Lemma 3.0.16. *Let $\mathcal{H}$ be a Hilbert space and consider $x \in \mathcal{B}(\mathcal{H})$, then $\ker(x) = R(x^*)^\perp$.*

Proof. If $\xi \in \ker(x)$, and $\eta \in \mathcal{H}$, then $\langle \xi, x^*\eta \rangle = \langle x\xi, \eta \rangle = 0$, hence $\ker(x) \subset R(x^*)^\perp$. If $\xi \in R(x^*)^\perp$ then for any $\eta \in \mathcal{H}$ we have $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle = 0$, hence $\xi \in \ker(x)$. ■

The **point spectrum** $\sigma_p(x)$ of an operator $x \in \mathcal{B}(\mathcal{H})$ consists of all points $\lambda \in \mathbb{C}$ such that $x - \lambda$ has a non-trivial kernel. The operator x has an **approximate kernel** if there exists a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that $\|x\xi_n\| \to 0$. The **approximate point spectrum** $\sigma_{ap}(x)$ consists of all points $\lambda \in \mathbb{C}$ such that $x - \lambda$ has an approximate kernel. Note that we have $\sigma_p(x) \subset \sigma_{ap}(x) \subset \sigma(x)$.

Proposition 3.0.17. *Let $x \in \mathcal{B}(\mathcal{H})$ be a normal operator, then $\sigma_p(x^*) = \overline{\sigma_p(x)}$. Moreover, eigenspaces for x corresponding to distinct eigenvalues are orthogonal.*

Proof. As x is normal, so is $x - \lambda$, and hence for each $\xi \in \mathcal{H}$ we have $\|(x - \lambda)\xi\| = \|(x^* - \overline{\lambda})\xi\|$. The first implication then follows.

If $\xi, \eta \in \mathcal{H}$ are eigenvectors for x with respective eigenvalues λ, μ, such that $\lambda \neq \mu$, Then we have $x^*\eta = \overline{\mu}\eta$, and so $\lambda\langle \xi, \eta\rangle = \langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle = \mu\langle \xi, \eta\rangle$. As $\lambda \neq \mu$, it then follows $\langle \xi, \eta\rangle = 0$. ■

Proposition 3.0.18. *Let $x \in \mathcal{B}(\mathcal{H})$, then $\partial\sigma(x) \subset \sigma_{ap}(x)$.*

Proof. Suppose $\lambda \in \partial\sigma(x)$. Then there exists a sequence $\lambda_n \in \rho(x)$, such that $\lambda_n \to \lambda$. By Lemma 1.1.12 we then have that $\|(x-\lambda_n)^{-1}\| \to \infty$, hence there exists a sequence of unit vectors $\xi_n \in \mathcal{H}$, such that $\|\xi_n\| \to 0$, and $\|(x - \lambda_n)^{-1}\xi_n\| = 1$. We then have $\|(x - \lambda)(x - \lambda_n)^{-1}\xi_n\| \leq |\lambda - \lambda_n|\|(x - \lambda_n)^{-1}\xi_n\| + \|\xi_n\| \to 0$. Hence, $\lambda \in \sigma_{ap}(x)$. ■

Lemma 3.0.19. *Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$, then x is invertible in $\mathcal{B}(\mathcal{H})$ if and only if neither x nor x^* has an approximate kernel. Consequently, $\sigma(x) = \sigma_{ap}(x) \cup \overline{\sigma_{ap}(x^*)}$, for all $x \in \mathcal{B}(\mathcal{H})$.*

Proof. If x is invertible, then for all $\xi \in \mathcal{H}$ we have $\|x^{-1}\|\|x\xi\| \geq \|x^{-1}x\xi\| = \|\xi\|$, and hence x cannot have an approximate kernel. Neither can x^* since it is then also invertible.

Conversely, if neither x nor x^* has an approximate kernel then x is injective, and the previous lemma applied to x^* shows that x has dense range. If $\{x\xi_n\} \subset R(x)$ is Cauchy then we have $\lim_{n,m\to\infty} \|x(\xi_n - \xi_m)\| \to 0$. Since x does not have an approximate kernel it then follows that $\{\xi_n\}$ is also Cauchy (Otherwise an approximate kernel of the form $(\xi_n - \xi_m)/\|\xi_n - \xi_m\|$ may be found) and hence converges to a vector ξ. We then have $\lim_{n\to\infty} x\xi_n = x\xi \in R(x)$, thus $R(x)$ is closed and hence x is surjective. The open mapping theorem then implies that x has a bounded inverse. ■

The **numerical range** $W(x)$ of an operator $x \in \mathcal{B}(\mathcal{H})$ is the closure of the set $\{\langle x\xi, \xi\rangle \mid \xi \in \mathcal{H}, \|\xi\| = 1\}$.

Lemma 3.0.20. *Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$, then $\sigma(x) \subset W(x)$.*

Proof. Suppose $\lambda \in \sigma(x)$. Then from the previous lemma either $x - \lambda$ or $(x-\lambda)^*$ has an approximate kernel. In either case there then exists a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that $\langle (x - \lambda)\xi_n, \xi_n\rangle \to 0$. Hence, $\lambda \in W(x)$. ■

Proposition 3.0.21. *Let $\mathcal{H}$ be a Hilbert space, then an operator $x \in \mathcal{B}(\mathcal{H})$ is* (*i*) *normal if and only if* $\|x\xi\| = \|x^*\xi\|$, *for all* $\xi \in \mathcal{H}$.

(*ii*) *self-adjoint if and only if* $\langle x\xi, \xi\rangle \in \mathbb{R}$, *for all* $\xi \in \mathcal{H}$.

(*iii*) *positive if and only if* $\langle x\xi, \xi\rangle \geq 0$, *for all* $\xi \in \mathcal{H}$.

(*iv*) *an isometry if and only if* $\|x\xi\| = \|\xi\|$, *for all* $\xi \in \mathcal{H}$.

(*v*) *a projection if and only if* x *is the orthogonal projection onto some closed subspace of* $\mathcal{H}$.

(*vi*) *a partial isometry if and only if there is a closed subspace* $\mathcal{K} \subset \mathcal{H}$ *such that* $x_{|\mathcal{K}}$ *is an isometry while* $x_{|\mathcal{K}^\perp} = 0$.

Proof.

(i) If x is normal than for all $\xi \in \mathcal{H}$ we have $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \langle xx^*\xi, \xi\rangle = \|x^*\xi\|^2$. Conversely, is $\langle (x^*x - xx^*)\xi, \xi\rangle = 0$, for all $\xi \in \mathcal{H}$, then for all $\xi, \eta \in \mathcal{H}$, by polarization we have

$$\langle (x^*x - xx^*)\xi, \eta\rangle = \sum_{k=0}^{3} i^k \langle (x^*x - xx^*)(\xi + i^k\eta), (\xi + i^k\eta)\rangle = 0.$$

Hence $x^*x = xx^*$.

(ii) If $x = x^*$ then $\overline{\langle x\xi, \xi\rangle} = \langle \xi, x\xi\rangle = \langle x\xi, \xi\rangle$. The converse follows again by a polarization argument.

(iii) If $x = y^*y$, then $\langle x\xi, \xi\rangle = \|y\xi\|^2 \geq 0$. Conversely, if $\langle x\xi, \xi\rangle \geq 0$, for all $\xi \in \mathcal{H}$ then we know from part (ii) that x is normal. From Lemma 3.0.20 we have that $\sigma(x) \subset [0, \infty)$ and hence x is positive by functional calculus.

(iv) If x is an isometry then $x^*x = 1$ and hence $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \|\xi\|^2$ for all $\xi \in \mathcal{H}$. The converse again follows from the polarization identity.

(v) If x is a projection then let $\mathcal{K} = \overline{R(x)} = \ker(x)^\perp$, and note that for all $\xi \in \mathcal{K}, \eta \in \ker(x), x\zeta \in R(x)$ we have $\langle x\xi, \eta + x\zeta\rangle = \langle \xi, x\zeta\rangle$, hence $x\xi \in \mathcal{K}$, and $x\xi = \xi$. This shows that x is the orthogonal projection onto the subspace $\mathcal{K}$.

(vi) This follows directly from (iv) and (v). ∎

Proposition 3.0.22 (Polar decomposition)**.** *Let $\mathcal{H}$ be a Hilbert space, and $x \in \mathcal{B}(\mathcal{H})$, then there exists a partial isometry v such that $x = v|x|$, and* $\ker(v) = \ker(|x|) = \ker(x)$*. Moreover, this decomposition is unique, in that if $x = wy$ where $y \geq 0$, and w is a partial isometry with* $\ker(w) = \ker(y)$ *then $y = |x|$, and $w = v$.*

Proof. We define a linear operator $v_0 : R(|x|) \to R(x)$ by $v_0(|x|\xi) = x\xi$, for $\xi \in \mathcal{H}$. Since $\| |x|\xi \| = \|x\xi\|$, for all $\xi \in \mathcal{H}$ it follows that v_0 is well defined and extends to a partial isometry v from $\overline{R(|x|)}$ to $\overline{R(x)}$, and we have $v|x| = x$. We also have $\ker(v) = R(|x|)^\perp = \ker(|x|) = \ker(x)$.

To see the uniqueness of this decomposition suppose $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$. Then $|x|^2 = x^*x = yw^*wy = y^2$, and hence $|x| = (|x|^2)^{1/2} = (y^2)^{1/2} = y$. We then have $\ker(w) = \overline{R(|x|)}^\perp$, and $\|w|x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$, hence $w = v$. ∎

3.1 Trace class operators

Given a Hilbert space $\mathcal{H}$, an operator $x \in \mathcal{B}(\mathcal{H})$ has finite rank if $\overline{R(x)} = \ker(x^*)^\perp$ is finite dimensional, the **rank** of x is $\dim(\overline{R(x)})$. We denote the space of finite rank operators by $\mathcal{FR}(\mathcal{H})$. If x is finite rank than $R(x^*) = R(x^*_{|\ker(x^*)^\perp})$ is also finite dimensional being the image of a finite dimensional space, hence we see that x^* also has finite rank. If $\xi, \eta \in \mathcal{H}$ are vectors we denote by $\xi \otimes \overline{\eta}$ the operator given by

$$(\xi \otimes \overline{\eta})(\zeta) = \langle \zeta, \eta \rangle \xi.$$

Note that $(\xi \otimes \overline{\eta})^* = \eta \otimes \overline{\xi}$, and if $\|\xi\| = \|\eta\| = 1$ then $\xi \otimes \overline{\eta}$ is a rank one partial isometry from $C\eta$ to $\mathbb{C}\xi$. Also note that if $x, y \in \mathcal{B}(\mathcal{H})$, then we have $x(\xi \otimes \overline{\eta})y = (x\xi) \otimes (\overline{y^*\eta})$.

From above we see that any finite rank operator is of the form pxq where $p, q \in \mathcal{B}(\mathcal{H})$ are projections onto finite dimensional subspaces. In particular this shows that $\mathcal{FR}(\mathcal{H}) = \mathrm{sp}\{\xi \otimes \overline{\eta} \mid \xi, \eta \in \mathcal{H}\}$

Lemma 3.1.1. *Suppose $x \in \mathcal{B}(\mathcal{H})$ has polar decomposition $x = v|x|$. Then for all $\xi \in \mathcal{H}$ we have*

$$2|\langle x\xi, \xi \rangle| \leq \langle |x|\xi, \xi \rangle + \langle |x|v^*\xi, v^*\xi \rangle.$$

Proof. If $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, then we have

$$\begin{aligned} 0 &\leq \|(|x|^{1/2} - \lambda |x|^{1/2} v^*)\xi\|^2 \\ &= \||x|^{1/2}\xi\|^2 - 2\mathrm{Re}(\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi\rangle) + \||x|^{1/2}v^*\xi\|^2. \end{aligned}$$

Taking λ such that $\overline{\lambda}\langle |x|^{1/2}\xi, |x|^{1/2}v^*\xi\rangle \geq 0$, the inequality follows directly. ∎

If $\{\xi_i\}$ is an orthonormal basis for $\mathcal{H}$, and $x \in \mathcal{B}(\mathcal{H})$ is positive, then we define the trace of x to be

$$\mathrm{Tr}(x) = \sum_i \langle x\xi_i, \xi_i\rangle.$$

Lemma 3.1.2. *If $x \in \mathcal{B}(\mathcal{H})$ then $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$.*

Proof. By Parseval's identity and Fubini's theorem we have

$$\begin{aligned} \sum_i \langle x^*x\xi_i, \xi_i\rangle &= \sum_i \sum_j \langle x\xi_i, \xi_j\rangle \overline{\langle \xi_j, x\xi_i\rangle} \\ &= \sum_j \sum_i \langle \xi_i, x^*\xi_j\rangle \overline{\langle \xi_i, x^*\xi_j\rangle} = \sum_j \langle xx^*\xi_j, \xi_j\rangle. \end{aligned}$$ ∎

Corollary 3.1.3. *If $x \in \mathcal{B}(\mathcal{H})$ is positive and u is a unitary, then $\mathrm{Tr}(u^*xu) = \mathrm{Tr}(x)$. In particular, the trace is independent of the chosen orthonormal basis.*

Proof. If we write $x = y^*y$, then from the previous lemma we have

$$\mathrm{Tr}(y^*y) = \mathrm{Tr}(yy^*) = \mathrm{Tr}((yu)(u^*y^*)) = \mathrm{Tr}(u^*(y^*y)u).$$ ∎

An operator $x \in \mathcal{B}(\mathcal{H})$ is said to be of **trace class** if $\|x\|_1 := \mathrm{Tr}(|x|) < \infty$. We denote the set of trace class operators by $L^1(\mathcal{B}(\mathcal{H}))$ or $L^1(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$.

Given an orthonormal basis $\{\xi_i\}$, and $x \in L^1(\mathcal{B}(\mathcal{H}))$ we define the **trace** of x by

$$\mathrm{Tr}(x) = \Sigma_i \langle x\xi_i, \xi_i\rangle.$$

By Lemma 3.1.1 this is absolutely summable, and

$$2|\mathrm{Tr}(x)| \leq \mathrm{Tr}(|x|) + \mathrm{Tr}(v|x|v^*) \leq 2\|x\|_1.$$

Lemma 3.1.4. *$L^1(\mathcal{B}(\mathcal{H}))$ is a two sided self-adjoint ideal in $\mathcal{B}(\mathcal{H})$ which coincides with the span of the positive operators with finite trace. The trace is independent of the chosen basis, and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.*

Proof. If $x, y \in L^1(\mathcal{B}(\mathcal{H}))$ and we let $x + y = w|x + y|$ be the polar decomposition, then we have $w^*x, w^*y \in L^1(\mathcal{B}(\mathcal{H}))$, therefore $\sum_i \langle |x + y|\xi_i, \xi_i\rangle = \sum_i \langle w^*x\xi_i, \xi_i\rangle + \langle w^*y\xi_i, \xi_i\rangle$ is absolutely summable. Thus $x + y \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\|x + y\|_1 \leq \|w^*x\|_1 + \|w^*y\|_1 \leq \|x\|_1 + \|y\|_1.$$

Thus, it follows that $L^1(\mathcal{B}(\mathcal{H}))$ is a linear space which contains the span of the positive operators with finite trace, and $\| \cdot \|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.

If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ then

$$4a|x| = \sum_{k=0}^{3} i^k(a + i^k)|x|(a + i^k)^*,$$

and for each k we have

$$\mathrm{Tr}((a + i^k)|x|(a + i^k)^*) = \mathrm{Tr}(|x|^{1/2}|a + i^k|^2|x|^{1/2}) \leq \|a + i^k\|^2 \,\mathrm{Tr}(|x|).$$

Thus if we take a to be the partial isometry in the polar decomposition of x we see that x is a linear combination of positive operators with finite trace, (in particular, the trace is independent of the basis). This also shows that $L^1(\mathcal{B}(\mathcal{H}))$ is a self-adjoint left ideal, and hence is also a right ideal. ∎

Theorem 3.1.5. *If $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a, b \in \mathcal{B}(\mathcal{H})$ then*

$$\|x\| \leq \|x\|_1$$

$$\|axb\|_1 \leq \|a\| \; \|b\| \; \|x\|_1,$$

and

$$\mathrm{Tr}(ax) = \mathrm{Tr}(xa).$$

Proof. Since the trace is independent of the basis, and $\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\|$ it follows easily that $\|x\| \leq \|x\|_1$.

Since for $x \in L^1(\mathcal{B}(\mathcal{H}))$, and $a \in \mathcal{B}(\mathcal{H})$ we have $|ax| \leq \|a\||x|$ it follows that $\|ax\|_1 \leq \|a\|\|x\|_1$. Since $\|x\|_1 = \|x^*\|_1$ we also have $\|xb\|_1 \leq \|b\|\|x\|_1$.

Since the definition of the trace is independent of the chosen basis, if $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $u \in \mathcal{U}(\mathcal{H})$ we have

$$\mathrm{Tr}(xu) = \sum_i \langle xu\xi_i, \xi_i\rangle = \sum_i \langle uxu\xi_i, u\xi_i\rangle = \mathrm{Tr}(ux).$$

Since every operator $a \in \mathcal{B}(\mathcal{H})$ is a linear combination of four unitaries this also gives

$$\mathrm{Tr}(xa) = \mathrm{Tr}(ax).$$ ∎

We also remark that for all $\xi, \eta \in \mathcal{H}$, the operators $\xi \otimes \overline{\eta}$ satisfy $\mathrm{Tr}(\xi \otimes \overline{\eta}) = \langle \xi, \eta \rangle$. Also, it's easy to check that $\mathcal{FR}(\mathcal{H})$ is a dense subspace of $L^1(\mathcal{B}(\mathcal{H}))$, endowed with the norm $\| \cdot \|_1$.

Proposition 3.1.6. *The space of trace class operators $L^1(\mathcal{B}(\mathcal{H}))$, with the norm $\| \cdot \|_1$ is a Banach space.*

Proof. From Lemma 3.1.4 we know that $\| \cdot \|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$ and hence we need only show that $L^1(\mathcal{B}(\mathcal{H}))$ is complete. Suppose x_n is Cauchy in $L^1(\mathcal{B}(\mathcal{H}))$. Since $\|x_n - x_m\| \leq \|x_n - x_m\|_1$ it follows that x_n is also Cauchy in $\mathcal{B}(\mathcal{H})$, therefore we have $\|x - x_n\| \to 0$, for some $x \in \mathcal{B}(\mathcal{H})$, and by continuity of functional calculus we also have $\||x| - |x_n|\| \to 0$. Thus for any finite orthonormal set $\eta_1, \ldots, \eta_k$ we have

$$\begin{aligned}\sum_{i=1}^{k} \langle |x| \eta_i, \eta_i \rangle &= \lim_{n \to \infty} \sum_{i=1}^{k} \langle |x_n| \eta_i, \eta_i \rangle \\ &\leq \lim_{n \to \infty} \|x_n\|_1 < \infty.\end{aligned}$$

Hence $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $\|x\|_1 \leq \lim_{n \to \infty} \|x_n\|_1$.

If we let $\varepsilon > 0$ be given and consider $N \in \mathbb{N}$ such that for all $n > N$ we have $\|x_n - x_N\|_1 < \varepsilon/3$, and then take $\mathcal{H}_0 \subset \mathcal{H}$ a finite dimensional subspace such that $\|x_N P_{\mathcal{H}_0^\perp}\|_1, \|x P_{\mathcal{H}_0^\perp}\|_1 < \varepsilon/3$. Then for all $n > N$ we have

$$\begin{aligned}&\|x - x_n\|_1 \\ &\quad \leq \|(x - x_n) P_{\mathcal{H}_0}\|_1 + \|x P_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|(x_N - x_n) P_{\mathcal{H}_0^\perp}\|_1 \\ &\quad \leq \|(x - x_n) P_{\mathcal{H}_0}\|_1 + \varepsilon.\end{aligned}$$

Since $\|x - x_n\| \to 0$ it follows that $\|(x - x_n) P_{\mathcal{H}_0}\|_1 \to 0$, and since $\varepsilon > 0$ was arbitrary we then have $\|x - x_n\|_1 \to 0$. ■

Theorem 3.1.7. *The map $\psi : \mathcal{B}(\mathcal{H}) \to L^1(\mathcal{B}(\mathcal{H}))^*$ given by $\psi_a(x) = \mathrm{Tr}(ax)$, for $a \in \mathcal{B}(\mathcal{H})$, $x \in L^1(\mathcal{B}(\mathcal{H}))$, is a Banach space isomorphism.*

Proof. From Theorem 3.1.5 we have that ψ is a linear contraction.

Suppose $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$, then $(\xi, \eta) \mapsto \varphi(\xi \otimes \overline{\eta})$ defines a bounded sesquilinear form on $\mathcal{H}$ and hence there exists a bounded operator $a \in \mathcal{B}(\mathcal{H})$ such that $\langle a\xi, \eta \rangle = \varphi(\xi \otimes \overline{\eta})$, for all $\xi, \eta \in \mathcal{H}$. Since the finite rank operators is dense in $L^1(\mathcal{B}(\mathcal{H}))$, and since operators of the form $\xi \otimes \overline{\eta}$ span the finite rank operators we have $\varphi = \psi_a$, thus we see that ψ is bijective.

We also have

$$\begin{aligned}\|a\| &= \sup_{\substack{\xi,\eta\in\mathcal{H},\\ \|\xi\|,\|\eta\|\leq 1}} |\langle a\xi,\eta\rangle| \\ &= \sup_{\substack{\xi,\eta\in\mathcal{H},\\ \|\xi\|,\|\eta\|\leq 1}} |\operatorname{Tr}(a(\xi\otimes\overline{\eta}))| \leq \|\psi_a\|.\end{aligned}$$

Hence ψ is isometric. ∎

3.2 Hilbert-Schmidt operators

Given a Hilbert space $\mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$, we say that x is a Hilbert-Schmidt operator on $\mathcal{H}$ if $|x|^2 \in L^1(\mathcal{B}(\mathcal{H}))$. We define the set of Hilbert-Schmidt operators by $L^2(\mathcal{B}(\mathcal{H}))$, or $L^2(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

Lemma 3.2.1. *$L^2(\mathcal{B}(\mathcal{H}))$ is a self-adjoint ideal in $\mathcal{B}(\mathcal{H})$, and if $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ then $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$, and*

$$\operatorname{Tr}(xy) = \operatorname{Tr}(yx).$$

Proof. Since $|x+y|^2 \leq |x+y|^2+|x-y|^2 = 2(|x|^2+|y|^2)$ we see that $L^2(\mathcal{B}(\mathcal{H}))$ is a linear space, also since $|ax|^2 \leq \|a\|^2|x|^2$ we have that $L^2(\mathcal{B}(\mathcal{H}))$ is a left ideal. Moreover, since we have $\operatorname{Tr}(xx^*) = \operatorname{Tr}(x^*x)$ we see that $L^2(\mathcal{B}(\mathcal{H}))$ is self-adjoint. In particular, $L^2(\mathcal{B}(\mathcal{H}))$ is also a right ideal.

By the polarization identity

$$4y^*x = \sum_{k=0}^{3} i^k |x + i^k y|^2,$$

we have that $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ for $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, and

$$\begin{aligned}4\operatorname{Tr}(y^*x) &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x+i^k y)^*(x+i^k y)) \\ &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x+i^k y)(x+i^k y)^*) = 4\operatorname{Tr}(xy^*).\end{aligned}$$

∎

From the previous lemma we see that the sesquilinear form on $L^2(\mathcal{B}(\mathcal{H}))$ give by

$$\langle x, y \rangle_2 = \mathrm{Tr}(y^* x)$$

is well defined and positive definite. We again have $\|axb\|_2 \leq \|a\| \, \|b\| \, \|x\|_2$, and any $x \in L^2(\mathcal{B}(\mathcal{H}))$ can be approximated in $\| \cdot \|_2$ by operators px where p is a finite rank projection. Thus, the same argument as for the trace class operators shows that the Hilbert-Schmidt operators is complete in the Hilbert-Schmidt norm.

Also, note that if $x \in L^2(\mathcal{B}(\mathcal{H}))$ then since $\|y\| \leq \|y\|_2$ for all $y \in L^2(\mathcal{B}(\mathcal{H}))$ it follows that

$$\begin{aligned} \|x\|_2 &= \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})), \\ \|y\|_2 \leq 1}} |\mathrm{Tr}(y^* x)| \\ &\leq \sup_{\substack{y \in L^2(\mathcal{B}(\mathcal{H})), \\ \|y\|_2 \leq 1}} \|y\| \|x\|_1 \leq \|x\|_1. \end{aligned}$$

Proposition 3.2.2. *Let $\mathcal{H}$ be a Hilbert space and suppose $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then*

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2.$$

Proof. If we consider the polar decomposition $xy = v|xy|$, then by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \|xy\|_1 &= |\mathrm{Tr}(v^* xy)| = |\langle y, x^* v \rangle_2| \\ &\leq \|x^* v\|_2 \|y\|_2 \leq \|x\|_2 \|y\|_2. \end{aligned}$$

■

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then we may extend a bounded operator $x : \mathcal{H} \to \mathcal{K}$ to a bounded operator $\tilde{x} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ by $\tilde{x}(\xi \oplus \eta) = 0 \oplus x\xi$. We define $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ as the bounded operators $x : \mathcal{H} \to \mathcal{K}$ such that $\tilde{x} \in L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$. In this way $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ forms a closed subspace of $L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))$.

Note that $\mathrm{HS}(\mathcal{H}, \mathbb{C})$ is the dual Banach space of $\mathcal{H}$, and is naturally anti-isomorphic to $\mathcal{H}$, we denote this isomorphism by $\xi \mapsto \overline{\xi}$. We call this the **conjugate Hilbert space** of $\mathcal{H}$, and denote it by $\overline{\mathcal{H}}$. Note that we have the natural identification $\overline{\overline{\mathcal{H}}} = \mathcal{H}$. Also, we have a natural anti-linear map $x \mapsto \overline{x}$ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\overline{\mathcal{H}})$ given by $\overline{x}\overline{\xi} = \overline{x\xi}$.

If we wish to emphasize that we are considering only the Hilbert space aspects of the Hilbert-Schmidt operators, we often use the notation $\mathcal{H} \overline{\otimes} \mathcal{K}$ for the Hilbert-Schmidt operators $\mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$. In this setting we call $\mathcal{H}\overline{\otimes}\mathcal{K}$ the **Hilbert space tensor product** of $\mathcal{H}$ with $\mathcal{K}$. Note that if $\{\xi_i\}_i$ and $\{\eta_j\}_j$ form orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$ respectively, then $\{\xi_i \otimes \eta_j\}_{i,j}$ forms an orthonormal basis for $\mathcal{H}\overline{\otimes}\mathcal{K}$. We see that the algebraic tensor product $\mathcal{H}\otimes\mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ can be realized as the subspace of finite rank operators, i.e., we have $\mathcal{H} \otimes \mathcal{K} = \mathrm{sp}\{\xi \otimes \eta \mid \xi \in \mathcal{H}, \eta \in \mathcal{K}\}$.

If $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{K})$ then we obtain an operator $x \otimes y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ which is given by $(x \otimes y)\Xi = x\Xi\overline{y}^*$. We then have that $\|x \otimes y\| \leq \|x\|\|y\|$, and $(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$ for all $\xi \in \mathcal{H}$, and $\eta \in \mathcal{K}$. We also have $(x \otimes y)^* = x^* \otimes y^*$, and the map $(x, y) \mapsto x \otimes y$ is separately linear in each variable. If $A \subset \mathcal{B}(\mathcal{H})$ and $B \subset \mathcal{B}(\mathcal{K})$ are algebras then the tensor product $A \otimes B$ is the algebra generated by operators of the form $a \otimes b$ for $a \in A$ and $b \in B$.

If (X, μ) is a measure space then we have a particularly nice description of the Hilbert-Schmidt operators on $L^2(X, \mu)$.

Theorem 3.2.3. *For each $k \in L^2(X \times X, \mu \times \mu)$ the integral operator T_k defined by*

$$T_k\xi(x) = \int k(x, y)\xi(y)d\mu(y), \qquad \xi \in L^2(X, \mu),$$

is a Hilbert-Schmidt operator on $L^2(X, \mu)$. Moreover, the map $k \mapsto T_k$ is a unitary operator from $L^2(X \times X, \mu \times \mu)$ to $L^2(\mathcal{B}(L^2(X, \mu)))$. Moreover, if we define $k^*(x, y) = \overline{k(x, y)}$ then we have $T_k^* = T_{k^*}$.

Proof. For all $\eta \in L^2(X, \mu)$, the Cauchy-Schwarz inequality gives

$$\|k(x, y)\xi(y)\eta(x)\|_1 \leq \|k\|_2\|\|\xi\|_{L^2(X,\mu)}\|\eta\|_2.$$

This shows that T_k is a well defined operator on $L^2(X, \mu)$ and $\|T_k\| \leq \|k\|_2$. If $\{\xi_i\}_i$ gives an orthonormal basis for $L^2(X, \mu)$ and $k(x, y) = \sum \alpha_{i,j}\xi_i(x)\xi_j(y)$ is a finite sum then for $\eta \in L^2(X, \mu)$ we have

$$T_k\eta = \sum \alpha_{i,j}\langle \xi, \xi_j\rangle \xi_i = (\sum \alpha_{i,j}\xi_i \otimes \overline{\xi_j})\eta.$$

Thus, $\|T_k\|_2 = \| \sum \alpha_{i,j}\xi_i \otimes \overline{\xi_j}\|_2 = \|k\|_2$, which shows that $k \mapsto T_k$ is a unitary operator.

The same formula above also shows that $T_k^* = T_{k^*}$. ■

3.3 Compact operators

We denote by $(\mathcal{H})_1$ the unit ball in $\mathcal{H}$.

Theorem 3.3.1. *For $x \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:*

(i) $x \in \overline{\mathcal{FR}(\mathcal{H})}^{\|\cdot\|}$.

(ii) *x restricted to $(\mathcal{H})_1$ is continuous from the weak to the norm topology.*

(iii) *$x(\mathcal{H})_1$ is compact in the norm topology.*

(iv) *$x(\mathcal{H})_1$ has compact closure in the norm topology.*

Proof. (i) $\implies$ (ii) Let $\{\xi_\alpha\}_\alpha$ be net in $(\mathcal{H})_1$ which weakly converges to ξ. By hypothesis for every $\varepsilon > 0$ there exists $y \in \mathcal{FR}(\mathcal{H})$ such that $\|x - y\| < \varepsilon$. We then have

$$\|x\xi - x\xi_\alpha\| \leq \|y\xi - y\xi_\alpha\| + 2\varepsilon.$$

Thus, it is enough to consider the case when $x \in \mathcal{FR}(\mathcal{H})$. This case follows easily since then the range of x is then finite dimensional where the weak and norm topologies agree.

(ii) $\implies$ (iii) $(\mathcal{H})_1$ is compact in the weak topology and hence $x(\mathcal{H})_1$ is compact being the continuous image of a compact set.

(iii) $\implies$ (iv) This implication is obvious.

(iv) $\implies$ (i) Let P_α be a net of finite rank projections such that $\|P_\alpha\xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}$. Then $P_\alpha x$ are finite rank and if $\|P_\alpha x - x\| \not\to 0$ then there exists $\varepsilon > 0$, and $\xi_\alpha \in (\mathcal{H})_1$ such that $\|x\xi_\alpha - P_\alpha x\xi_\alpha\| \geq \varepsilon$. By hypothesis we may pass to a subnet and assume that $x\xi_\alpha$ has a limit ξ in the norm topology. We then have

$$\begin{aligned}\varepsilon &\leq \|x\xi_\alpha - P_\alpha x\xi_\alpha\| \leq \|\xi - P_\alpha\xi\| + \|(1 - P_\alpha)(x\xi_\alpha - \xi)\| \\ &\leq \|\xi - P_\alpha\xi\| + \|x\xi_\alpha - \xi\| \to 0,\end{aligned}$$

which gives a contradiction. ■

If any of the above equivalent conditions are satisfied we say that x is a **compact operator**. We denote the space of compact operators by $\mathcal{K}(\mathcal{H})$. Clearly $\mathcal{K}(\mathcal{H})$ is a norm closed two sided ideal in $\mathcal{B}(\mathcal{H})$.

Lemma 3.3.2. *Let $x \in \mathcal{K}(\mathcal{H})$ be a compact operator, then $\sigma_{ap}(x) \setminus \{0\} \subset \sigma_p(x)$.*

Proof. Suppose $\lambda \in \sigma_{ap}(x) \setminus \{0\}$ and let $\xi_n \in \mathcal{H}$ be a sequence of unit vectors such that $\|(x - \lambda)\xi_n\| \to 0$. Since x is compact, by taking a subsequence we may assume that $x\xi_n$ converges to a vector ξ. We then have $x\xi = \lim_{n\to\infty} x^2\xi_n = \lim_{n\to\infty} \lambda x\xi_n = \lambda\xi$. Moreover, ξ is non-zero since $\|\xi\| = \lim_{n\to\infty} \|x\xi_n\| = \lim_{n\to\infty} \|\lambda\xi_n\| = |\lambda| \neq 0$. Hence, $\lambda \in \sigma_p(x)$. ∎

Lemma 3.3.3. *Let $x \in \mathcal{K}(\mathcal{H})$ be a compact operator, then each point in $\sigma(x) \setminus \{0\}$ is isolated.*

Proof. Suppose that $\{\lambda_n\}_n \subset \sigma(x) \setminus \{0\}$ is a sequence of pairwise distinct values such that $\lambda_n \to \lambda$. From Lemma 3.0.19 we have $\sigma(x) = \sigma_{ap}(x) \cup \overline{\sigma_{ap}(x^*)}$, and hence by taking a further subsequence, and replacing x with x^* if necessary, we will assume that $\lambda_n \in \sigma_{ap}(x)$, for each n, and then from the previous lemma we have $\lambda_n \in \sigma_p(x)$, for each n.

Thus, there exists a sequence of unit eigenvectors $\{\xi_n\} \subset \mathcal{H}$, whose corresponding eigenvalues are $\{\lambda_n\}$. Note that since $\{\lambda_n\}$ are distinct values we have that $\{\xi_n\}$ is a linearly independent set. Let $Y_n = \mathrm{sp}\{\xi_1, \dots, \xi_n\}$, and choose unit vectors $\eta_n \in Y_n$, so that $\|P_{Y_{n-1}}(\eta_n)\| = 0$, for all n. Then for $n < m$ we have

$$\|x\eta_n - x\eta_m\| = \|x\eta_n - (x - \lambda_m)\eta_m + \lambda_m\eta_m\|,$$

and since $x\eta_n - (x - \lambda_m)\eta_m \in Y_{m-1}$ we conclude that $\|x\eta_n - x\eta_m\| \geq |\lambda_m|\|\eta_m\|$. Since x is compact, and $\{\eta_n\}$ are pairwise orthogonal unit vectors it then follows that $|\lambda| = \lim_{m\to\infty} |\lambda_m| = 0$, and hence 0 is the only possible accumulation point of $\sigma(x)$. ∎

Theorem 3.3.4 (The Fredholm Alternative)**.** *Let $x \in \mathcal{K}(\mathcal{H})$ be a compact operator, then for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$, either $\lambda \in \sigma_p(x)$, or else $\lambda \in \rho(x)$, i.e., $\sigma(x) \setminus \{0\} \subset \sigma_p(x)$.*

Proof. By the previous lemma, each point in $\sigma(x) \setminus \{0\}$ is isolated. It then follows from Proposition 3.0.18, that $\sigma(x) \setminus \{0\} = (\partial\sigma(x)) \setminus \{0\} \subset \sigma_{ap}(x)$, and then from Lemma 3.3.2 it follows that $\sigma(x) \setminus \{0\} \subset \sigma_p(x)$. ∎

Theorem 3.3.5 (The spectral theorem for compact operators)**.** *Let $x \in \mathcal{K}(\mathcal{H})$ be a normal compact operator. For each eigenvalue λ for x, denote by E_λ the corresponding eigenspace. Then we have $x = \sum_{\lambda \in \sigma(x)\setminus\{0\}} \lambda P_{E_\lambda}$, where the convergence is in the uniform norm.*

Proof. If ξ_n is an orthogonal sequence of unit eigenvectors for x, then $\xi_n \to 0$ weakly, and since x is compact we then have that $x\xi_n \to 0$ in norm. Thus, the eigenvalues corresponding to ξ_n must converge to 0. This shows, in particular, that each eigenspace corresponding to a non-zero eigenvalue must be finite dimensional. Also, since eigenspaces corresponding to distinct eigenvectors are orthogonal by Lemma 3.0.17, it follows that x can have at most finitely many eigenvalues with modulus greater than any fixed positive number. Thus, by Theorem 3.3.4 we have that $\sigma(x)$ is countable and has no non-zero accumulation points.

If we set $y = \sum_{\lambda \in \sigma(x) \setminus \{0\}} \lambda P_{E_\lambda}$ then y is compact since it is a norm limit of finite rank operators. Thus, $x - y$ is compact and normal, and $\mathrm{sp}\{E_\lambda\} \subset \ker(x-y)$. If $\xi \in \mathcal{H}$ is an eigenvector for $x - y$ with non-zero eigenvalue, then ξ must be orthogonal to $\ker(x-y)$, and hence ξ is orthogonal to E_λ for each $\lambda \in \sigma(x) \setminus \{0\}$. Hence, we would have $y\xi = 0$, and so ξ would be an eigenvector for x. But then $\xi \in E_\lambda$ for some $\lambda \in \sigma(x) \setminus \{0\}$ giving a contradiction.

Thus, we conclude that $x - y$ is a compact operator without non-zero eigenvalues. From Theorem 3.3.4 we than have that $\sigma(x - y) = \{0\}$. Since $x - y$ is normal we then have $x = y = \sum_{\lambda \in \sigma(x) \setminus \{0\}} \lambda P_{E_\lambda}$. ■

Theorem 3.3.6 (Alternate form of the spectral theorem for compact operators)**.** *Let $x \in \mathcal{K}(\mathcal{H})$ be a normal compact operator. Then there exists a set X, a function $f \in c_0(X)$, and a unitary operator $U : \ell^2 X \to \mathcal{H}$, such that $x = U M_f U^*$, where $M_f \in \mathcal{B}(\ell^2 X)$ is the operator defined by $M_f \xi = f\xi$.*

Proof. From the previous theorem we may write $x = \sum_{\lambda \in \sigma(x) \setminus \{0\}} \lambda P_{E_\lambda}$. We then have $\ker(x) = \ker(x^*)$, and $\mathcal{H} = \ker(x) \oplus \bigoplus_{\lambda \in \sigma(x) \setminus \{0\}} E_\lambda$. Thus, there exists an orthonormal basis $\{\xi_x\}_{x \in X} \subset \mathcal{H}$, which consists of eigenvectors for x. If we consider the unitary operator $U : \ell^2 X \to \mathcal{H}$, which sends δ_x to ξ_x, and we consider the function $f \in c_0(X)$ by letting $f(x)$ be the eigenvalue corresponding to the eigenvector ξ_x, then it is easy to see that $x = U M_f U^*$. ■

Exercise 3.3.7. Show that the map $\psi : L^1(\mathcal{B}(\mathcal{H})) \to \mathcal{K}(\mathcal{H})^*$ given by $\psi_x(a) = \mathrm{Tr}(ax)$ implements an isometric Banach space isomorphism between $L^1(\mathcal{B}(\mathcal{H}))$ and $\mathcal{K}(\mathcal{H})^*$.

3.4 Locally convex topologies on the space of operators

Let $\mathcal{H}$ be a Hilbert space. On $\mathcal{B}(\mathcal{H})$ we define the following locally convex topologies:

- The **weak operator topology** (WOT) is defined by the family of semi-norms $T \mapsto |\langle T\xi, \eta \rangle|$, for $\xi, \eta \in \mathcal{H}$.
- The **strong operator topology** (SOT) is defined by the family of semi-norms $T \mapsto \|T\xi\|$, for $\xi \in \mathcal{H}$.

Note that the from coarsest to finest topologies we have

$$\text{WOT} \prec \text{SOT} \prec \text{Uniform}.$$

Also note that since an operator T is normal if and only if $\|T\xi\| = \|T^*\xi\|$ for all $\xi \in \mathcal{H}$, it follows that the adjoint is SOT continuous on the set of normal operators.

Lemma 3.4.1. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(i) There exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$ such that $\varphi(T) = \sum_{i=1}^n \langle T\xi_i, \eta_i \rangle$, for all $T \in \mathcal{B}(\mathcal{H})$.

(ii) φ is WOT continuous.

(iii) φ is SOT continuous.

Proof. The implications (i) $\implies$ (ii) and (ii) $\implies$ (iii) are clear and so we will only show (iii) $\implies$ (i). Suppose φ is SOT continuous. Thus, the inverse image of the open ball in $\mathbb{C}$ is open in the SOT and hence by considering the semi-norms which define the topology we have that there exists a constant $K > 0$, and $\xi_1, \ldots, \xi_n \in \mathcal{H}$ such that

$$|\varphi(T)|^2 \leq K \sum_{i=1}^n \|T\xi_i\|^2.$$

If we then consider $\{\oplus_{i=1}^n T\xi_i \mid T \in \mathcal{B}(\mathcal{H})\} \subset \mathcal{H}^{\oplus n}$, and let $\mathcal{H}_0$ be its closure, we have that

$$\oplus_{i=1}^n T\xi_i \mapsto \varphi(T)$$

extends to a well defined, continuous linear functional on $\mathcal{H}_0$ and hence by the Riesz representation theorem there exists $\eta_1, \dots, \eta_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^{n} \langle T\xi_i, \eta_i \rangle,$$

for all $T \in \mathcal{B}(\mathcal{H})$. ∎

Corollary 3.4.2. *Let $K \subset \mathcal{B}(\mathcal{H})$ be a convex set, then the WOT, SOT, and closures of K coincide.*

Proof. By Lemma 3.4.1 the three topologies above give rise to the same dual space, hence this follows from the the Hahn-Banach separation theorem. ∎

If $\mathcal{H}$ is a Hilbert space then the map $\mathrm{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ defined by $(\mathrm{id} \otimes 1)(x) = x \otimes 1$ need not be continuous in either of the locally convex topologies defined above even though it is an isometric C^*-homomorphism with respect to the uniform topology. Thus, on $\mathcal{B}(\mathcal{H})$ we define the following additional locally convex topologies:

- The σ**-weak operator topology** (σ-WOT) is defined by pulling back the WOT of $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ under the map $\mathrm{id} \otimes 1$.
- The σ**-strong operator topology** (σ-SOT) is defined by pulling back the SOT of $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ under the map $\mathrm{id} \otimes 1$.

Note that the σ-weak operator topology can alternately be defined by the family of semi-norms $T \mapsto |\operatorname{Tr}(Ta)|$, for $a \in L^1(\mathcal{B}(\mathcal{H}))$. Hence, under the identification $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$, we have that the weak*-topology on $\mathcal{B}(\mathcal{H})$ agrees with the σ-WOT.

Lemma 3.4.3. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(i) There exists a trace class operator $a \in L^1(\mathcal{B}(\mathcal{H}))$ such that $\varphi(x) = \operatorname{Tr}(xa)$ for all $x \in \mathcal{B}(\mathcal{H})$

(ii) φ is σ-WOT continuous.

(iii) φ is σ-SOT continuous.

Proof. Again, we need only show the implication (iii) $\implies$ (i), so suppose φ is σ-SOT continuous. Then by the Hahn-Banach theorem, considering $\mathcal{B}(\mathcal{H})$ as a subspace of $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$ through the map $\mathrm{id} \otimes 1$, we may extend φ to a SOT continuous linear functional on $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$. Hence by Lemma 3.4.1 there exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}\overline{\otimes}\ell^2\mathbb{N}$ such that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\varphi(x) = \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle.$$

For each $1 \leq i \leq n$ we may define $a_i, b_i \in \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$ as the operators corresponding to ξ_i, η_i in the Hilbert space isomorphism $\mathcal{H} \otimes \ell^2\mathbb{N} \cong \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$. By considering $a = \sum_{i=1}^{n} b_i^* a_i \in L^1(\mathcal{B}(\mathcal{H}))$, it then follows that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned} \mathrm{Tr}(xa) &= \sum_{i=1}^{n} \langle a_i x, b_i \rangle_2 \\ &= \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle = \varphi(x). \end{aligned}$$ ■

Corollary 3.4.4. *The unit ball in $\mathcal{B}(\mathcal{H})$ is compact in the σ-WOT.*

Proof. This follows from Theorem 3.1.7 and the Banach-Alaoglu theorem. ■

Corollary 3.4.5. *The WOT and the σ-WOT agree on bounded sets.*

Proof. The identity map is clearly continuous from the σ-WOT to the WOT. Since both spaces are Hausdorff it follows that this is a homeomorphism from the σ-WOT compact unit ball in $\mathcal{B}(\mathcal{H})$. By scaling we therefore have that this is a homeomorphism on any bounded set. ■

Exercise 3.4.6. Show that the adjoint $T \mapsto T^*$ is continuous in the WOT, and when restricted to the space of normal operators is continuous in the SOT, but is not continuous in the SOT on the space of all bounded operators.

Exercise 3.4.7. Show that operator composition is jointly continuous in the SOT on bounded subsets.

Exercise 3.4.8. Show that the SOT agrees with the σ-SOT on bounded subsets of $\mathcal{B}(\mathcal{H})$.

Exercise 3.4.9. Show that pairing $\langle x, a \rangle = \mathrm{Tr}(a^* x)$ gives an identification between $\mathcal{K}(\mathcal{H})^*$ and $(L^1(\mathcal{B}(\mathcal{H})), \| \cdot \|_1)$.

3.5 Von Neumann algebras and the double commutant theorem

A **von Neumann algebra** (over a Hilbert space $\mathcal{H}$) is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains 1 and is closed in the weak operator topology.

Note that since subalgebras are of course convex, it follows from Corollary 3.4.2 that von Neumann algebras are also closed in the strong operator topology.

If $A \subset \mathcal{B}(\mathcal{H})$ then we denote by $W^*(A)$ the von Neumann subalgebra which is generated by A, i.e., $W^*(A)$ is the smallest von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ which contains A.

Lemma 3.5.1. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $(A)_1$ is compact in the WOT.*

Proof. This follows directly from Corollary 3.4.4. ∎

Corollary 3.5.2. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then $(A)_1$ and $A_{\text{s.a.}}$ are closed in the weak and strong operator topologies.*

Proof. Since taking adjoints is continuous in the weak operator topology it follows that $A_{\text{s.a.}}$ is closed in the weak operator topology, and by the previous result this is also the case for $(A)_1$. ∎

If $B \subset \mathcal{B}(\mathcal{H})$, the **commutant** of B is

$$B' = \{T \in \mathcal{B}(\mathcal{H}) \mid TS = ST, \text{ for all } S \in B\}.$$

We also use the notation $B'' = (B')'$ for the **double commutant**.

Theorem 3.5.3. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint set, then A' is a von Neumann algebra.*

Proof. It is easy to see that A' is a self-adjoint algebra containing 1. To see that it is closed in the weak operator topology just notice that if $x_\alpha \in A'$ is a net such that $x_\alpha \to x \in \mathcal{B}(\mathcal{H})$ then for any $a \in A$, and $\xi, \eta \in \mathcal{H}$, we have

$$\langle [x,a]\xi, \eta\rangle = \langle xa\xi, \eta\rangle - \langle x\xi, a^*\eta\rangle$$

$$= \lim_{\alpha\to\infty} \langle x_\alpha a\xi, \eta\rangle - \langle x_\alpha \xi, a^*\eta\rangle = \lim_{\alpha\to\infty} \langle [x_\alpha, a]\xi, \eta\rangle = 0.$$ ∎

Corollary 3.5.4. *A self-adjoint maximal abelian subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*

Proof. Since A is maximal abelian we have $A = A'$. ∎

Lemma 3.5.5. *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing 1. Then for all $\xi \in \mathcal{H}$, and $x \in A''$ there exists $x_\alpha \in A$ such that $\lim_{\alpha \to \infty} \|(x - x_\alpha)\xi\| = 0$.*

Proof. Consider the closed subspace $\mathcal{K} = \overline{A\xi} \subset \mathcal{H}$, and denote by p the projection onto this subspace. Since for all $a \in A$ we have $a\mathcal{K} \subset \mathcal{K}$, it follows that $ap = pap$. But since A is self-adjoint it then also follows that for all $a \in A$ we have $pa = (a^*p)^* = (pa^*p)^* = pap = ap$, and hence $p \in A'$.

We therefore have that $xp = xp^2 = pxp$ and hence $x\mathcal{K} \subset \mathcal{K}$. Since $1 \in A$ it follows that $\xi \in \mathcal{K}$ and hence also $x\xi \in \overline{A\xi}$. ∎

Theorem 3.5.6 (Von Neumann's double commutant theorem)**.** *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing 1. Then A'' is equal to the weak operator topology closure of A.*

Proof. By Theorem 3.5.3 we have that A'' is closed in the weak operator topology, and we clearly have $A \subset A''$, so we just need to show that $A \subset A''$ is dense in the weak operator topology. For this we use the previous lemma together with a matrix trick.

Let $\xi_1, \ldots, \xi_n \in \mathcal{H}$, $x \in A''$ and consider the subalgebra $\tilde{A}$ of $\mathcal{B}(\mathcal{H}^n) \cong \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$ consisting of diagonal matrices with constant diagonal coefficients contained in A. Then the diagonal matrix whose diagonal entries are all x is easily seen to be contained in $\tilde{A}''$, hence the previous lemma applies and so there exists a net $a_\alpha \in A$ such that $\lim_{\alpha \to \infty} \|(x - a_\alpha)\xi_k\| = 0$, for all $1 \leq k \leq n$. This shows that $A \subset A''$ is dense in the strong operator topology. ∎

We also have the following formulation which is easily seen to be equivalent.

Corollary 3.5.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra. Then A is a von Neumann algebra if and only if $A = A''$.*

Corollary 3.5.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $x \in A$, and consider the polar decomposition $x = v|x|$. Then $v \in A$.*

Proof. Note that $\ker(v) = \ker(|x|)$, and if $a \in A'$ then we have $a \ker(|x|) \subset \ker(|x|)$. Also, we have

$$\|(av - va)|x|\xi\| = \|ax\xi - xa\xi\| = 0,$$

for all $\xi \in \mathcal{H}$. Hence av and va agree on $\ker(|x|) + \overline{R(|x|)} = \mathcal{H}$, and so $v \in A'' = A$. ∎

Proposition 3.5.9. *Let (X, μ) be a σ-finite*[1] *measure space. Consider the Hilbert space $L^2(X, \mu)$, and the map $M : L^\infty(X, \mu) \to \mathcal{B}(L^2(X, \mu))$ defined by $(M_g\xi)(x) = g(x)\xi(x)$, for all $\xi \in L^2(X, \mu)$. Then M is an isometric $*$-isomorphism from $L^\infty(X, \mu)$ onto a maximal abelian von Neumann subalgebra of $\mathcal{B}(L^2(X, \mu))$.*

Proof. The fact that M is a $*$-isomorphism onto its image is clear. If $g \in L^\infty(X, \mu)$ then by definition of $\|g\|_\infty$ we can find a sequence E_n of measurable subsets of X such that $0 < \mu(E_n) < \infty$, and $|g|_{|E_n} \geq \|g\|_\infty - 1/n$, for all $n \in \mathbb{N}$. We then have

$$\|M_g\| \geq \|M_g 1_{E_n}\|_2 / \|1_{E_n}\|_2 \geq \|g\|_\infty - 1/n.$$

The inequality $\|g\|_\infty \leq \|M_g\|$ is also clear and hence M is isometric.

To see that $M(L^\infty(X, \mu))$ is maximal abelian let's suppose $T \in \mathcal{B}(L^2(X, \mu))$ commutes with M_f for all $f \in L^\infty(X, \mu)$. We take $E_n \subset X$ measurable sets such that $0 < \mu(E_n) < \infty$, $E_n \subset E_{n+1}$, and $X = \cup_{n \in \mathbb{N}} E_n$. Define $f_n \in L^2(X, \mu)$ by $f_n = T(1_{E_n})$.

For each $g, h \in L^\infty(X, \mu) \cap L^2(X, \mu)$, we have

$$|\int f_n g \overline{h} d\mu| = |\langle M_g T(1_{E_n}), h\rangle| = |\langle T(g_{|E_n}), h\rangle| \leq \|T\| \|g\|_2 \|h\|_2.$$

Since $L^\infty(X, \mu) \cap L^2(X, \mu)$ is dense in $L^2(X, \mu)$, it then follows from Hölder's inequality that $f_n \in L^\infty(X, \mu)$ with $\|f_n\|_\infty \leq \|T\|$, and that $M_{1_{E_n}} T = M_{f_n}$. Note that for $m \geq n$, $1_{E_m} f_n = 1_{E_m} T(1_{E_n}) = T(1_{E_n}) = f_n$. Hence, $\{f_n\}$ converges almost every where to a measurable function f. Since $\|f_n\|_\infty \leq \|T\|$ for each n, we have $\|f\|_\infty \leq \|T\|$. Moreover, if $g, h \in L^2(X, \mu)$ then we have

$$\int f g \overline{h} d\mu = \lim_{n \to \infty} \int f_n g \overline{h} d\mu = \lim_{n \to \infty} \langle 1_{E_n} T(g), h\rangle = \langle T(g), h\rangle.$$

Thus, $T = M_f$. ∎

[1] For technical reasons we restrict ourselves to σ-finite spaces, although here and throughout most of these notes the proper setting is really that of localizable spaces [?]

Because of the previous result we will often identify $L^\infty(X,\mu)$ with the subalgebra of $\mathcal{B}(L^2(X,\mu))$ as described above. This should not cause any confusion.

Exercise 3.5.10. Let X be an uncountable set, $\mathcal{B}_1$ the set of all subsets of X, $\mathcal{B}_2 \subset B_1$ the set consisting of all sets which are either countable or have countable complement, and μ the counting measure on X. Show that the identity map implements a unitary operator $\mathrm{id} : L^2(X,\mathcal{B}_1,\mu) \to L^2(X,\mathcal{B}_2,\mu)$, and we have $L^\infty(X,\mathcal{B}_2,\mu) \subsetneq L^\infty(X,\mathcal{B}_2,\mu)'' = \mathrm{id}\, L^\infty(X,\mathcal{B}_1,\mu)\,\mathrm{id}^*$.

3.6 Kaplansky's density theorem

Proposition 3.6.1. *If $f \in C(\mathbb{C})$ then $x \mapsto f(x)$ is continuous in the strong operator topology on any bounded set of normal operators in $\mathcal{B}(\mathcal{H})$.*

Proof. By the Stone-Weierstrass theorem we can approximate f uniformly well by polynomials on any compact set. Since multiplication is jointly SOT continuous on bounded sets, and since taking adjoints is SOT continuous on normal operators, the result follows easily. ∎

Proposition 3.6.2 (The Cayley transform)**.** *The map $x \mapsto (x-i)(x+i)^{-1}$ is strong operator topology continuous from the set of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ into the unitary operators in $\mathcal{B}(\mathcal{H})$.*

Proof. Suppose $\{x_k\}_k$ is a net of self-adjoint operators such that $x_k \to x$ in the SOT. By the spectral mapping theorem we have $\|(x_k+i)^{-1}\| \leq 1$ and hence for all $\xi \in \mathcal{H}$ we have

$$\begin{aligned}
&\|(x-i)(x+i)^{-1}\xi - (x_k-i)(x_k+i)^{-1}\xi\| \\
&= \|(x_k+i)^{-1}((x_k+i)(x-i) - (x_k-i)(x+i))(x+i)^{-1}\xi\| \\
&= \|2i(x_k+i)^{-1}(x-x_k)(x+i)^{-1}\xi\| \leq 2\|(x-x_k)(x+i)^{-1}\xi\| \to 0.
\end{aligned}$$ ∎

Corollary 3.6.3. *If $f \in C_0(\mathbb{R})$ then $x \mapsto f(x)$ is strong operator topology continuous on the set of self-adjoint operators.*

Proof. Since f vanishes at infinity, we have that $g(t) = f\left(i\frac{1+t}{1-t}\right)$ defines a continuous function on $\mathbb{T}$ if we set $g(1) = 0$. By Proposition 3.6.1 $x \mapsto g(x)$ is then SOT continuous on the space of unitaries. If $U(z) = \frac{z-i}{z+i}$ is the Cayley transform, then by Proposition 3.6.2 it follows that $f = g \circ U$ is SOT continuous being the composition of two SOT continuous functions. ∎

Theorem 3.6.4 (Kaplansky's density theorem)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ and denote by B the strong operator topology closure of A.*

(i) The strong operator topology closure of $A_{\text{s.a.}}$ is $B_{\text{s.a.}}$.

(ii) The strong operator topology closure of $(A)_1$ is $(B)_1$.

Proof. We may assume that A is a C^*-algebra. If $\{x_k\}_k \subset A$ is a net of elements which converge in the SOT to a self-adjoint element x_k, then since taking adjoints is WOT continuous we have that $\frac{x_k+x_k^*}{2} \to x$ in the WOT. But $A_{\text{s.a.}}$ is convex and so the WOT and SOT closures coincide, showing (a). Moreover, if $\{y_k\}_k \subset A_{\text{s.a.}}$ such that $y_k \to x$ in the SOT then by considering a function $f \in C_0(\mathbb{R})$ such that $f(t) = t$ for $|t| \leq \|x\|$, and $|f(t)| \leq \|x\|$, for $t \in \mathbb{R}$, we have $\|f(y_k)\| \leq \|x\|$, for all k and $f(y_k) \to f(x)$ in the SOT by Corollary 3.6.3. Hence $(A)_1 \cap A_{\text{s.a.}}$ is SOT dense in $(B)_1 \cap B_{\text{s.a.}}$.

Note that $\mathbb{M}_2(A)$ is SOT dense in $\mathbb{M}_2(B) \subset \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Therefore if $x \in (B)_1$ then $\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (\mathbb{M}_2(B))_1$ is self-adjoint. Hence from above there exists a net of operators $\tilde{x}_n \in (\mathbb{M}_2(A))_1$ such that $\tilde{x}_n \to \tilde{x}$ in the SOT. Writing $\tilde{x}_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ we then have that $\|b_n\| \leq 1$ and $b_n \to x$ in the SOT. ■

Corollary 3.6.5. *A self-adjoint unital subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $(A)_1$ is closed in the SOT.*

Corollary 3.6.6. *A self-adjoint unital subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if A is closed in the σ-WOT.*

3.7 The spectral theorem and Borel functional calculus

For $T \in \mathcal{K}(\mathcal{H})$ a compact normal operator, there were two different perspectives we could take when describing the spectral theorem for T. The first (Theorem 3.3.5) was a basis free approach, we considered the eigenvalues $\sigma_p(T)$ for T, and to each eigenvalue λ associated to it the projection $E(\lambda)$

onto the corresponding eigenspace. Since T is normal we have that the $E(\lambda)$'s are pairwise orthogonal and we showed

$$T = \sum_{\lambda \in \sigma(T)} \lambda E(\lambda).$$

The second approach (Theorem 3.3.6) was to use that since T is normal, it is diagonalizable with respect to a given basis, i.e., we produced a set X unitary matrix $U : \ell^2 X \to \mathcal{H}$ such that UTU^* is a multiplication operator corresponding to some function $f \in \ell^\infty X$.

For bounded normal operators there are two similar approaches to the spectral theorem. The first approach is to find a substitute for the projections $E(\lambda)$ and this leads naturally to the notion of a spectral measure. For the second approach, this naturally leads to the interpretation of diagonal matrices corresponding to multiplication by essentially bounded functions on a measure space.

Lemma 3.7.1. *Let $x_\alpha \in \mathcal{B}(\mathcal{H})$ be an increasing net of positive operators such that $\sup_\alpha \|x_\alpha\| < \infty$, then there exists a bounded operator $x \in \mathcal{B}(\mathcal{H})$ such that $x_\alpha \to x$ in the SOT.*

Proof. We may define a quadratic form on $\mathcal{H}$ by $\xi \mapsto \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$. Since $\sup_\alpha \|x_\alpha\| < \infty$ we have that this quadratic form is bounded and hence there exists a bounded positive operator $x \in \mathcal{B}(\mathcal{H})$ such that $\|\sqrt{x}\xi\|^2 = \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$, for all $\xi \in \mathcal{H}$. Note that $x_\alpha \leq x$ for all α, and $\sup_\alpha \|(x - x_\alpha)^{1/2}\| < \infty$. Thus for each $\xi \in \mathcal{H}$ we have

$$\begin{aligned}\|(x - x_\alpha)\xi\|^2 &\leq \|(x - x_\alpha)^{1/2}\|^2 \|(x - x_\alpha)^{1/2}\xi\|^2 \\ &= \|(x - x_\alpha)^{1/2}\|^2 (\|\sqrt{x}\xi\|^2 - \|\sqrt{x_\alpha}\xi\|^2) \to 0.\end{aligned}$$

Hence, $x_\alpha \to x$ in the SOT. ∎

Corollary 3.7.2. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\{p_\iota\}_{\iota \in I} \subset A$ is a collection of pairwise orthogonal projections then $p = \sum_{\iota \in I} p_\iota \in A$ is well defined as a SOT limit of finite sums.*

3.7.1 Spectral measures

Let K be a locally compact Hausdorff space and let $\mathcal{H}$ be a Hilbert space. A **spectral measure** E on K relative to $\mathcal{H}$ is a mapping from the Borel subsets of K to the set of projections in $\mathcal{B}(\mathcal{H})$ such that

(i) $E(\emptyset) = 0, E(K) = 1.$

(ii) $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for all Borel sets B_1 and B_2.

(iii) For all $\xi, \eta \in \mathcal{H}$ the function

$$B \mapsto E_{\xi,\eta}(B) = \langle E(B)\xi, \eta \rangle$$

is a finite Radon measure on K.

Example 3.7.3. If K is a locally compact Hausdorff space and μ is a σ-finite Radon measure on K, then the map $E(B) = 1_B \in L^\infty(K, \mu) \subset \mathcal{B}(L^2(K, \mu))$ defines a spectral measure on K relative to $L^2(K, \mu)$.

We denote by $B_\infty(K)$ the space of all bounded Borel functions on K. This is clearly a C^*-algebra with the sup norm.

For each $f \in B_\infty(K)$ it follows that the map

$$(\xi, \eta) \mapsto \int f \, dE_{\xi,\eta}$$

gives a continuous sesqui-linear form on $\mathcal{H}$ and hence it follows that there exists a bounded operator T such that $\langle T\xi, \eta \rangle = \int f \, dE_{\xi,\eta}$. We denote this operator T by $\int f \, dE$ so that we have the formula $\langle (\int f \, dE)\xi, \eta \rangle = \int f \, dE_{\xi,\eta}$, for each $\xi, \eta \in \mathcal{H}$.

Theorem 3.7.4. *Let K be a locally compact Hausdorff space, let $\mathcal{H}$ be a Hilbert space, and suppose that E is a spectral measure on K relative to $\mathcal{H}$. Then the association*

$$f \mapsto \int f \, dE$$

defines a continuous unital $*$-homomorphism from $B_\infty(K)$ to $\mathcal{B}(\mathcal{H})$. Moreover, the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, and if $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty$, then $\int f_n dE \to \int f dE$ in the SOT.

Proof. It is easy to see that this map defines a linear contraction which preserves the adjoint operation. If $A, B \subset K$ are Borel subsets, and $\xi, \eta \in \mathcal{H}$, then denoting $x = \int 1_A dE$, $y = \int 1_B dE$, and $z = \int 1_{A \cap B} dE$ we have

$$\begin{aligned} \langle xy\xi, \eta \rangle &= \langle E(A)y\xi, \eta \rangle = \langle E(B)\xi, E(A)\eta \rangle \\ &= \langle E(B \cap A)\xi, \eta \rangle = \langle z\xi, \eta \rangle. \end{aligned}$$

Hence $xy = z$, and by linearity we have that $(\int f\, dE)(\int g\, dE) = \int fg\, dE$ for all simple functions $f, g \in B_\infty(K)$. Since every function in $B_\infty(K)$ can be approximated uniformly by simple functions this shows that this is indeed a $*$-homomorphism.

To see that the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, note that if a commutes with all operators of the form $\int f\, dE$ for $f \in C(K)$ then for all $\xi, \eta \in \mathcal{H}$ we have

$$0 = \langle (a(\int f\, dE) - (\int f\, dE)a)\xi, \eta\rangle = \int f\, dE_{\xi, a^*\eta} - \int f\, dE_{a\xi,\eta}.$$

Thus $E_{\xi,a^*\eta} = E_{a\xi,\eta}$ and hence we have that a also commutes with operators of the form $\int g\, dE$ for any $g \in B_\infty(K)$. Therefore by Theorem 3.5.6 $\int g\, dE$ is contained in the von Neumann algebra generated by the image of $C(K)$.

Now suppose $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty(K)$. For each $\xi, \eta \in \mathcal{H}$ we have

$$\int f_n\, dE_{\xi,\eta} \to \int f\, dE_{\xi,\eta},$$

hence $\int f_n\, dE$ converges in the WOT to $\int f\, dE$. However, since $\int f_n\, dE$ is an increasing sequence of bounded operators with $\| \int f_n\, dE\| \leq \|f\|_\infty$, Lemma 3.7.1 shows that $\int f_n\, dE$ converges in the SOT to some operator $x \in \mathcal{B}(\mathcal{H})$ and we must then have $x = \int f\, dE$. ■

The previous theorem shows, in particular, that if A is an abelian C^*-algebra, and E is a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$, then we obtain a unital $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ by the formula

$$\pi(x) = \int \Gamma(x) dE.$$

We next show that in fact every unital $*$-representation arises in this way.

Theorem 3.7.5 (The spectral theorem). *Let A be an abelian C^*-algebra, $\mathcal{H}$ a Hilbert space and $\pi : A \to \mathcal{B}(\mathcal{H})$ a $*$-representation, which is non-degenerate in the sense that $\xi = 0$ if and only if $\pi(x)\xi = 0$ for all $x \in A$. Then there is a unique spectral measure E on $\sigma(A)$ relative to $\mathcal{H}$ such that for all $x \in A$ we have*

$$\pi(x) = \int \Gamma(x) dE.$$

Proof. For each $\xi, \eta \in \mathcal{H}$ we have that $f \mapsto \langle \pi(\Gamma^{-1}(f))\xi, \eta \rangle$ defines a bounded linear functional on $\sigma(A)$ and hence by the Riesz representation therorem there exists a Radon measure $E_{\xi,\eta}$ such that for all $f \in C(\sigma(A))$ we have

$$\langle \pi(\Gamma^{-1}(f))\xi, \eta \rangle = \int f dE_{\xi,\eta}.$$

Since the Gelfand transform is a $*$-homomorphism we verify easily that $f dE_{\xi,\eta} = dE_{\pi(\Gamma^{-1}(f))\xi,\eta} = dE_{\xi,\pi(\Gamma^{-1}(\overline{f}))\eta}$.

Thus for each Borel set $B \subset \sigma(A)$ we can consider the sesquilinear form $(\xi, \eta) \mapsto \int 1_B dE_{\xi,\eta}$. We have $|\int f dE_{\xi,\eta}| \leq \|f\|_\infty \|\xi\| \|\eta\|$, for all $f \in C(\sigma(A))$ and hence this sesquilinear form is bounded and there exists a bounded operator $E(B)$ such that $\langle E(B)\xi, \eta \rangle = \int 1_B dE_{\xi,\eta}$, for all $\xi, \eta \in \mathcal{H}$. For all $f \in C(\sigma(A))$ we have

$$\langle \pi(\Gamma^{-1}(f))E(B)\xi, \eta \rangle = \int 1_B dE_{\xi,\pi(\Gamma^{-1}(\overline{f}))\eta} = \int 1_B f dE_{\xi,\eta}.$$

Thus it follows that $E(B)^* = E(B)$, and $E(B')E(B) = E(B' \cap B)$, for any Borel set $B' \subset \sigma(A)$. In particular, $E(B)$ is a projection and since π is non-degenerate it follows easily that $E(\sigma(A)) = 1$, thus E gives a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$. The fact that for $x \in A$ we have $\pi(x) = \int \Gamma(x) dE$ follows easily from the way we constructed E. ∎

If $\mathcal{H}$ is a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$ is a normal operator, then by applying the previous theorem to the C^*-subalgebra A generated by x and 1, and using the identification $\sigma(A) = \sigma(x)$ we obtain a homomorphism from $B_\infty(\sigma(x))$ to $\mathcal{B}(\mathcal{H})$ and hence for $f \in B_\infty(\sigma(x))$ we may define

$$f(x) = \int f dE.$$

Note that it is straight forward to check that considering the function $f(z) = z$ we have

$$x = \int z dE(z).$$

We now summarize some of the properties of this functional calculus which follow easily from the previous results.

Theorem 3.7.6 (Borel functional calculus)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $x \in A$ is a normal operator, then the Borel functional calculus defined by $f \mapsto f(x)$ satisfies the following properties:*

(i) $f \mapsto f(x)$ *is a continuous unital* $*$*-homomorphism from* $B_\infty(\sigma(x))$ *into* A.

(ii) *If* $f \in B_\infty(\sigma(x))$ *then* $\sigma(f(x)) \subset f(\sigma(x))$.

(iii) *If* $f \in C(\sigma(x))$ *then* $f(x)$ *agrees with the definition given by continuous functional calculus.*

Corollary 3.7.7. *Let* $A \subset \mathcal{B}(\mathcal{H})$ *be a von Neumann algebra, then* A *is the uniform closure of the span of its projections.*

Proof. By decomposing an operator into its real and imaginary parts it is enough to check this for self-adjoint operators in the unit ball, and this follows from the previous theorem by approximating the function $f(t) = t$ uniformly by simple functions on $[-1, 1]$. ■

Corollary 3.7.8. *Let* $A \subset \mathcal{B}(\mathcal{H})$ *be a von Neumann algebra, then the unitary group* $\mathcal{U}(A)$ *is path connected in the uniform topology.*

Proof. If $u \in \mathcal{U}(A)$ is a unitary and we consider a branch of the log function $f(z) = \log z$, then from Borel functional calculus we have $u = e^{ix}$ where $x = -if(u)$ is self-adjoint. We then have that $u_t = e^{itx}$ is a uniform norm continuous path of unitaries such that $u_0 = 1$ and $u_1 = u$. ■

Corollary 3.7.9. *If* $\mathcal{H}$ *is an infinite dimensional separable Hilbert space, then* $\mathcal{K}(\mathcal{H})$ *is the unique non-zero proper norm closed two sided ideal in* $\mathcal{B}(\mathcal{H})$.

Proof. If $I \subset \mathcal{B}(\mathcal{H})$ is a norm closed two sided ideal and $x \in I \setminus \{0\}$, then for any $\xi \in R(x^*x)$, $\|\xi\| = 1$ we can consider $y = (\xi \otimes \overline{\xi})x^*x(\xi \otimes \overline{\xi}) \in I$ which is a rank one self-adjoint operator with $R(y) = \mathbb{C}\xi$. Thus y is a multiple of $(\xi \otimes \overline{\xi})$ and hence $(\xi \otimes \overline{\xi}) \in I$. For any $\zeta, \eta \in \mathcal{H}$, we then have $\zeta \otimes \overline{\eta} = (\zeta \otimes \overline{\xi})(\xi \otimes \overline{\xi})(\xi \otimes \overline{\eta}) \in I$ and hence I contains all finite rank operators. Since I is closed we then have that $\mathcal{K}(\mathcal{H}) \subset I$.

If $x \in I$ is not compact then for some $\varepsilon > 0$ we have that $\dim(1_{[\varepsilon,\infty)}(x^*x)\mathcal{H}) = \infty$. If we let $u \in \mathcal{B}(\mathcal{H})$ be an isometry from $\mathcal{H}$ onto $1_{[\varepsilon,\infty)}(x^*x)\mathcal{H}$, then we have that $\sigma(u^*x^*xu) \subset [\varepsilon, \infty)$. Hence, $u^*x^*xu \in I$ is invertible which shows that $I = \mathcal{B}(\mathcal{H})$. ■

Exercise 3.7.10. Suppose that K is a compact Hausdorff space and E is a spectral measure for K relative to a Hilbert space $\mathcal{H}$, show that if $f \in B_\infty(K)$, and we have a decomposition of K into a countable union of pairwise disjoint Borel sets $K = \cup_{n\in\mathbb{N}} B_n$ then we have that

$$\int f dE = \sum_{n\in\mathbb{N}} \int_{B_n} f dE,$$

where the convergence of the sum is in the weak operator topology.

3.8 Abelian von Neumann algebras

Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and suppose $\xi \in \mathcal{H}$ is a non-zero vector. Then ξ is said to be **cyclic** for A if $A\xi$ is dense in $\mathcal{H}$. We say that ξ is **separating** for A if $x\xi \neq 0$, for all $x \in A$, $x \neq 0$.

Proposition 3.8.1. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then a non-zero vector $\xi \in \mathcal{H}$ is cyclic for A if and only if ξ is separating for A'.*

Proof. Suppose ξ is cyclic for A, and $x \in A'$ such that $x\xi = 0$. Then $xa\xi = ax\xi = 0$ for all $a \in A$, and since $A\xi$ is dense in $\mathcal{H}$ it follows that $x\eta = 0$ for all $\eta \in \mathcal{H}$. Conversely, if $A\xi$ is not dense, then the orthogonal projection p onto its complement is a nonzero operator in A' such that $p\xi = 0$. ∎

Corollary 3.8.2. *If $A \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra and $\xi \in \mathcal{H}$ is cyclic, then ξ is also separating.*

Proof. Since ξ being separating passes to von Neumann subalgebras and $A \subset A'$ this follows. ∎

Infinite dimensional von Neumann algebras are never separable in the norm topology. For this reason we will say that a von Neumann algebra A is **separable** if A is separable in the SOT. Equivalently, A is separable if its predual A_* is separable.

Proposition 3.8.3. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable von Neumann algebra. Then there exists a separating vector for A.*

Proof. Since A is separable, it follows that there exists a countable collection of vectors $\{\xi_k\}_k \subset \mathcal{H}$ such that $x\xi_k = 0$ for all k only if $x = 0$. Also, since A is separable we have that $\mathcal{H}_0 = \overline{\text{sp}}(A\{\xi_k\}_k)$ is also separable. Thus, restricting A to $\mathcal{H}_0$ we may assume that $\mathcal{H}$ is separable.

By Zorn's lemma we can find a maximal family of non-zero unit vectors $\{\xi_\alpha\}_\alpha$ such that $A\xi_\alpha \perp A\xi_\beta$, for all $\alpha \neq \beta$. Since $\mathcal{H}$ is separable this family must be countable and so we may enumerate it $\{\xi_n\}_n$, and by maximality we have that $\{A\xi_n\}_n$ is dense in $\mathcal{H}$.

If we denote by p_n the orthogonal projection onto the closure of $A\xi_n$ then we have that $p_n \in A'$, hence, setting $\xi = \sum_n \frac{1}{2^n}\xi$ if $x \in A$ such that $x\xi = 0$, then for every $n \in \mathbb{N}$ we have $0 = 2^n p_n x\xi = 2^n x p_n \xi = x\xi_n$ and so $x = 0$ showing that ξ is a separating vector for A. ■

Corollary 3.8.4. *Suppose $\mathcal{H}$ is separable, if $A \subset \mathcal{B}(\mathcal{H})$ is a maximal abelian self-adjoint subalgebra (masa), then there exists a cyclic vector for A.*

Proof. By Propostion 3.8.3 there exists a non-zero vector $\xi \in \mathcal{H}$ which is separating for A, and hence by Proposition 3.8.1 is cyclic for $A' = A$. ■

The converse of the previous corollary also holds (without the separability hypothesis), which follows from Proposition 3.5.9, together with the following theorem.

Theorem 3.8.5. *Let $A \subset \mathcal{B}(\mathcal{H})$ be an abelian von Neumann algebra and suppose $\xi \in \mathcal{H}$ is a cyclic vector. Then for any SOT dense C^*-subalgebra $A_0 \subset A$ there exists a Radon probability measure μ on $K = \sigma(A_0)$ with* $\operatorname{supp}(\mu) =$ *K, and a unitary $U : L^2(K,\mu) \to \mathcal{H}$ such that $U^*AU = L^\infty(K,\mu) \subset \mathcal{B}(L^2(X,\mu))$, and such that $\int U^*aU \, \mathrm{d}\mu = \langle a\xi, \xi\rangle$ for all $x \in A$.*

Proof. Fix a SOT dense C^*-algebra $A_0 \subset A$, then by the Riesz representation theorem we obtain a finite Radon measure μ on $K = \sigma(A_0)$ such that $\langle \Gamma(f)\xi, \xi\rangle = \int f d\mu$ for all $f \in C(K)$. Since the Gelfand transform takes positive operator to positive functions we see that μ is a probability measure.

We define a map $U_0 : C(K) \to \mathcal{H}$ by $f \mapsto \Gamma(f)\xi$, and note that $\|U_0(f)\|^2 = \langle \Gamma(\overline{f}f)\xi, \xi\rangle = \int \overline{f} f d\mu = \|f\|_2$. Hence U_0 extends to an isometry $U : L^2(K,\mu) \to \mathcal{H}$. Since ξ is cyclic we have that $A_0\xi \subset U(L^2(K,\mu))$ is dense and hence U is a unitary. If the support of μ were not K then there would exist a non-zero continuous function $f \in C(K)$ such that $0 = \int |f^2| d\mu = \|\Gamma(f)\xi\|^2$, but since by Corollary 3.8.2 we know that ξ is separating and hence this cannot happen.

If $f \in C(K) \subset \mathcal{B}(L^2(K,\mu))$, and $g \in C(K) \subset L^2(K,\mu)$ then we have

$$U^*\Gamma(f)Ug = U^*\Gamma(f)\Gamma(g)\xi = fg = M_f g.$$

Since $C(K)$ is $\|\cdot\|_2$-dense in $L^2(K,\mu)$ it then follows that $U^*\Gamma(f)U = M_f$, for all $f \in C(K)$ and thus $U^*A_0U \subset L^\infty(K,\mu)$. Since A_0 is SOT dense in A we then have that $U^*AU \subset L^\infty(K,\mu)$. But since $x \mapsto U^*xU$ is WOT continuous and $(A)_1$ is compact in the WOT it follows that $U^*(A)_1U = (L^\infty(K,\mu))_1$ and hence $U^*AU = L^\infty(K,\mu)$. This similarly shows that we have $\int U^*aU \, \mathrm{d}\mu = \langle a\xi, \xi\rangle$ for all $x \in A$. ∎

In general, if $A \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann algebra and $\xi \in \mathcal{H}$ is a non-zero vector, then we can consider the projection p onto the $\mathcal{K} = \overline{A\xi}$. We then have $p \in A'$, and $Ap \subset \mathcal{B}(\mathcal{H})$ is an abelian von Neumann for which ξ is a cyclic vector, thus by the previous result Ap is $*$-isomorphic to $L^\infty(X,\mu)$ for some probability space (X,μ). An application of Zorn's Lemma can then be used to show that A is $*$-isomorphic to $L^\infty(Y,\nu)$ were (Y,ν) is a measure space which is a disjoint union of probability spaces. In the case when A is separable an even more concrete classification will be given below.

Theorem 3.8.6. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there exists a separable compact Hausdorff space K with a Radon probability measure μ on K such that A and $L^\infty(K,\mu)$ are $*$-isomorphic. Moreover, if φ is a normal faithful state on A, then the isomorphism $\theta : A \to L^\infty(K,\mu)$ may be chosen so that $\varphi(a) = \int \theta(a) \, \mathrm{d}\mu$ for all $a \in A$.*

Proof. By Proposition 3.8.3 there exists a non-zero vector $\xi \in \mathcal{H}$ which is separating for A. Thus if we consider $\mathcal{K} = \overline{A\xi}$ we have that restricting each operator $x \in A$ to $\mathcal{K}$ is a C^*-algebra isomorphism and $\xi \in \mathcal{K}$ is then cyclic. Thus, the result follows from Theorem 3.8.5.

If φ is a normal faithful state on A, then considering the GNS-construction we may representation A on $L^2(A,\varphi)$ with a cyclic vector $\hat{1}$ which satisfies $\varphi(a) = \langle a\hat{1}, \hat{1}\rangle$. The result then follows as above. ∎

If $x \in \mathcal{B}(\mathcal{H})$ is normal such that $A = W^*(x)$ is separable (e.g., if $\mathcal{H}$ is separable), then we may let A_0 be the C^*-algebra generated by x. We then obtain the following alternate version of the spectral theorem.

Corollary 3.8.7. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $x \in A$ is normal such that $W^*(x)$ is separable, then there exists a Radon probability measure μ on $\sigma(x)$ and a $*$-homomorphism $f \mapsto f(x)$ from $L^\infty(\sigma(x),\mu)$ into A which agrees with Borel functional calculus. Moreover, we have that $\sigma(f(x))$ is the essential range of f.*

Note that $W^*(x)$ need not be separable in general. For example, $\ell^\infty([0,1]) \subset \mathcal{B}(\ell^2([0,1]))$ is generated by the multiplication operator corresponding to the function $t \mapsto t$.

Lemma 3.8.8. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra, then there exists a self-adjoint operator $x \in A$ such that $A = \{x\}''$.*

Proof. Since A is separable we have that A is countably generated as a von Neumann algebra. Indeed, just take a countable family in A which is dense in the SOT. By functional calculus we can approximate any self-adjoint element by a linear combination of projections and thus A is generated by a countable collection of projections $\{p_k\}_{k=0}^\infty$.

Define a sequence of self adjoint elements $x_n = \sum_{k=0}^n 4^{-k} p_k$, and let $x = \sum_{k=0}^\infty 4^{-k} p_k$. We denote by $A_0 = \{x\}''$. Define a continuous function $f : [-1,2] \to \mathbb{R}$ such that $f(t) = 1$ if $t \in [1 - \frac{1}{3}, 1 + \frac{1}{3}]$ and $f(t) = 0$ if $t \leq \frac{1}{3}$, then we have that $f(x_n) = p_0$ for every n and hence by continuity of continuous functional calculus we have $p_0 = f(x) \in A_0$. The same argument shows that $p_1 = f(4(x - p_0)) \in A_0$ and by induction it follows easily that $p_k \in A_0$ for all $k \geq 0$, thus $A_0 = A$. ■

Theorem 3.8.9. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a separable abelian von Neumann algebra and φ a normal faithful state on A. Then there is there exists a probability space (X, μ) and an isomorphism $\theta : A \to L^\infty(X, \mu)$, such that $\varphi(a) = \int \theta(a)\, \mathrm{d}\mu$ for all $a \in A$.*

Moreover, (X, μ) may be taken to be of one of the following forms:

(i) (K, ν)*, where K is countable;*

(ii) $(K, \nu) \times ([0, c_0], \lambda)$ *where K is countable, $0 < c_0 < 1$, and λ is Lebesgue measure;*

(iii) $([0,1], \lambda)$ *where λ is Lebesgue measure.*

Proof. Since A is separable we have from Lemma 3.8.8 that as a von Neumann algebra A is generated by a single self-adjoint element $x \in A$.

We define $K = \{a \in \sigma(x) \mid 1_{\{a\}}(x) \neq 0\}$. Since the projections corresponding to elements in K are pairwise orthogonal it follows that K is countable. Further, if we denote by $p_K = \sum_{a \in K} 1_{\{a\}}$ then we have that $Ap_K \cong \ell^\infty K$, and restricting φ to $c_0(K)$ under this isomorphism gives a positive measure on K which is taken to $\varphi_{|Ap_K}$ under this isomorphism.

Thus, by considering $(1 - p_K)A$ we may assume that $p_K = 0$, and it is enough to show that in this case there exists an isomorphism $\theta : A \to L^\infty([0,1], \lambda)$, such that $\varphi(a) = \int \theta(a)\,\mathrm{d}\lambda$ for all $a \in A$.

Thus, we suppose that $\sigma(x)$ has no isolated points. We may then inductively define a sequence of partitions $\{A_k^n\}_{k=1}^{2^n}$ of $\sigma(x)$ such that $A_k^n = A_{2k-1}^{n+1} \cup A_{2k}^{n+1}$, and A_k^n has non-empty interior, for all $n > 0$, $1 \leq k \leq 2^n$. If we then consider the elements $y_n = \sum_{k=1}^\infty \frac{k}{2^n} 1_{A_k}(x)$ then we have that $y_n \to y$ where $0 \leq y \leq 1$, $\{x\}'' = \{y\}''$ and every dyadic rational is contained in the spectrum of y (since the space of invertible operators is open in the norm topology), hence $\sigma(y) = [0,1]$.

By Theorem 3.8.6 there exists an isomorphism $\theta_0 : \{y\}'' \to L^\infty([0,1], \mu)$ for some Radon measure μ on $[0,1]$ which has full support, no atoms and such that $\varphi(a) = \int \theta_0(a)\,\mathrm{d}\mu$ for all $a \in A$. If we define the function $\theta : [0,1] \to [0,1]$ by $\theta(t) = \mu([0,t])$ then θ gives a continuous bijection of $[0,1]$, and we have $\theta_*\mu = \lambda$, since both are Radon probability measures such that for intervals $[a,b]$ we have $\theta_*\mu([a,b]) = \mu([\theta^{-1}(a), \theta^{-1}(b)]) = \lambda([a,b])$. The map $\theta^* : L^\infty([0,1], \lambda) \to L^\infty([0,1], \mu)$ given by $\theta^*(f) = f \circ \theta^{-1}$ is then easily seen to be the desired $*$-isomorphism. ■

3.9 Standard probability spaces

A topological space X is a **Polish space** if X is homeomorphic to a separable complete metric space. A σ-algebra $(X, \mathcal{B})$ is a **standard Borel space** if $(X, \mathcal{B})$ is isomorphic to the σ-algebra of Borel subsets of a Polish space. A measure space (X, μ) is a **standard measure space** if it is σ-finite, and its underlying σ-algebra is a standard Borel space, and a **standard probability space** if it is also a probability space.

Theorem 3.9.1. *Let X be a Polish space, and $\{E_n\}_{n \in \mathbb{N}}$ a countable collection of Borel subsets, then there exists a finer Polish topology on X with the same Borel sets, such that for each $n \in \mathbb{N}$, E_n is clopen in this new topology.*

Proof. We first consider the case of a single Borel subset $E \subset X$. We let $\mathcal{A}$ denote the set of subsets which satisfy the conclusion of the theorem and we let $\mathcal{B}$ be the σ-algebra of Borel subsets of X. Since X is Polish there exists a complete metric d on X such that (X, d) gives the topology on X.

If $A \subset X$ is closed, then it's not hard to see that the metric $d'(x,y) = d(x,y) + |\frac{1}{d(x,A)} - \frac{1}{d(y,A)}|$ gives a finer Polish topology on X with the same

Borel σ-algebra such that A becomes clopen in this new topology. Thus, $\mathcal{A}$ contains all closed subsets of (X, d).

It is also clear that $\mathcal{A}$ is closed under taking complements. Thus, to conclude that $\mathcal{B} \subset \mathcal{A}$ it is then enough to show that $\mathcal{A}$ is closed under countable intersections. Suppose therefore that $A_n \in \mathcal{A}$, and let d_n be metrics giving a finer Polish topology on X such that the Borel structures all agree with $\mathcal{B}$, and such that A_n is clopen in (X, d_n) for each $n \in \mathbb{N}$.

Note that a metric ρ is complete if and only if the metric $\frac{\rho}{1+\rho}$ is complete, thus we may assume that each d_n assigns X a diameter at most 1. We may then define a new metric $\tilde{d}$ on X given by $\tilde{d}(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} d_n(x,y)$. Note that $(X, \tilde{d})$ is a finer Polish topology than (X, d), and since for each $n \in \mathbb{N}$, the metric space (X, d_n) has the same Borel structure as $\mathcal{B}$, it is easy to see that the Borel structure on $(X, \tilde{d})$ also agrees with $\mathcal{B}$.

We then have that A_n is closed in $(X, \tilde{d})$ for each n, and hence $\cap_{n \in \mathbb{N}} A_n$ is also closed in this topology. From above there then exists a finer polish topology with the same Borel structure such that $\cap_{n \in \mathbb{N}} A_n$ is clopen in this new topology.

Consider now a countable collection of Borel subsets $\{E_n\}_{n \in \mathbb{N}}$. Then from above, there exists a metric d_n on X such that the corresponding topology generates the Borel structure $\mathcal{B}$, and such that E_n is clopen in this topology. Then just as above, we may assume that X has a diameter at most 1 with respect to d_n, and considering the new metric $\tilde{d}(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} d_n(x,y)$ we have that this metic generates the Borel structure $\mathcal{B}$, and E_n is then clopen for each $n \in \mathbb{N}$. ■

Corollary 3.9.2. *Let $(X, \mathcal{B})$ be a standard Borel space, and $E \in \mathcal{B}$ a Borel subset, then $(E, \mathcal{B}|_E)$ is a standard Borel space.*

Proof. By the previous theorem we may assume X is Polish and $E \subset X$ is clopen, and hence Polish. We then have that $\mathcal{B}|_E$ is the associated Borel structure on E and hence $(E, \mathcal{B}|_E)$ is standard. ■

Corollary 3.9.3. *Let X be a standard Borel space, Y a Polish space, and $f : X \to Y$ a Borel map, then there exists a Polish topology on X which generates the same Borel structure and such that f is continuous with respect to this topology.*

Proof. Let $\{E_n\}$ be a countable basis for the topology on Y. By Theorem 3.9.1 there exists a Polish topology on X which generates the same

Borel structure and such that $f^{-1}(E_n)$ is clopen for each $n \in \mathbb{N}$. Hence, in this topology f is continuous. ∎

We let $\mathbb{N}^{<\mathbb{N}}$ denote the set of finite sequences of natural numbers, i.e., $\mathbb{N}^{<\mathbb{N}}$ consists of the empty set, together with the disjoint union of $\mathbb{N}^n$, for $n \in \mathbb{N}$. If $s = (s_1, \ldots, s_n) \in \mathbb{N}^{<\mathbb{N}}$, and $k \in \mathbb{N}$, we denote by $s\hat{\ }k$ the sequence $(s_1, \ldots, s_n, k)$. If $s \in \mathbb{N}^{\mathbb{N}}$, and $n \in \{0\} \cup \mathbb{N}$ then we denote by $s|n$ the sequence which consists of the first n entires of s. A **Souslin scheme** on a set X is a family of subsets $\{E_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$.

Lemma 3.9.4. *Let X be a Polish space, then there exists a Souslin scheme $\{E_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ consisting of Borel subsets such that the the following conditions are satisfied:*

(i) $E_\emptyset = X$.

(ii) *For each* $s \in \mathbb{N}^{<\mathbb{N}}$, $E_s = \sqcup_{k \in \mathbb{N}} E_{s\hat{\ }k}$.

(iii) *For each* $s \in \mathbb{N}^{\mathbb{N}}$ *the set* $\cap_{n \in \mathbb{N}} \overline{E_{s|n}}$ *consists of at most one element.*

(iv) *For each* $s \in \mathbb{N}^{\mathbb{N}}$, $\cap_{n \in \mathbb{N}} \overline{E_{s|n}} = \{x\} \neq \emptyset$ *if and only if* $E_{s|n} \neq \emptyset$ *for all* $n \in \mathbb{N}$, *and in this case for any sequence* $x_n \in E_{s|n}$ *we have* $x_n \to x$.

Proof. Let d be a complete metric on X which generates the Polish topology on X. By replacing d with $\frac{d}{1+d}$ we may assume that the diameter of X is at most 1. We will inductively construct $\{E_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ so that for $s \in \mathbb{N}^n$ the diameter of E_s is at most 2^{-n}. First, we set $E_\emptyset = X$. Now suppose E_s has been constructed for each $s \in \{\emptyset\} \cup_{n=1}^{k} \mathbb{N}^n$. If $s \in \mathbb{N}^k$, let $\{x_n\}_{n \in \mathbb{N}}$ be a countable dense subset of E_s (note that any subspace of a separable metric space is again separable).

We define $E_{s\hat{\ }i} = E_s \cap (B_{2^{-k-1}}(x_i) \setminus \cup_{j<i} B_{2^{-k-1}}(x_j))$, where $B_r(x)$ denotes the open ball of radius r centered at x. It is then easy to see that for each $s \in \mathbb{N}^{<\mathbb{N}}$, we have $E_s = \sqcup_{k \in \mathbb{N}} E_{s\hat{\ }k}$. Moreover, for each $s \in \mathbb{N}^{\mathbb{N}}$, we have that $E_{s|n}$ has diameter at most 2^{-n}, hence $\cap_{n \in \mathbb{N}} \overline{E_{s|n}}$ contains at most one element. Finally, if $\overline{E_{s|n}} \neq \emptyset$ for all $n \in \mathbb{N}$, then as the diameter of $E_{s|n}$ converges to 0, it follows from completeness, that there exists $x \in \cap_{n \in \mathbb{N}} \overline{E_{s|n}}$, and for each sequence $x_n \in E_{s|n}$ we have $x_n \to x$. ∎

If X is a standard Borel space and $A, B \subset X$ are disjoint, then we say that A and B are **Borel separated** if there exists a Borel subset $E \subset X$ such that $A \subset E$, and $B \subset X \setminus E$.

Lemma 3.9.5. *Let X be a standard Borel space and suppose that $A = \cup_{n\in\mathbb{N}} A_n$, and $B = \cup_{m\in\mathbb{N}} B_m$, are such that A_n and B_m are Borel separated for each $n, m \in \mathbb{N}$, then A and B are Borel separated.*

Proof. Suppose $E_{n,m}$ is a Borel subset which separates A_n and B_m for each $n, m \in \mathbb{N}$. Then $E = \cup_{n\in\mathbb{N}} \cap_{m\in\mathbb{N}} E_{n,m}$ separates A and B. ■

If X is a Polish space, a subset $E \subset X$ is **analytic** if there exits a Polish space Y and a continuous function $f : Y \to X$ such that $E = f(Y)$. Note that it follows from Corollary 3.9.3 that if $f : Y \to X$ is Borel then $f(Y)$ is analytic. In particular, it follows that all Borel sets are analytic. If X is a standard Borel space then a subset $E \subset X$ is **analytic** if it is analytic for some (and hence all) Polish topologies on X which give the Borel structure.

Theorem 3.9.6 (The Lusin Separation Theorem)**.** *Let X be a standard Borel space, and $A, B \subset X$ two disjoint analytic sets, then A and B are Borel separated.*

Proof. We may assume that X is a Polish space, and that there are Polish spaces Y_1, and Y_2, and continuous functions $f_i : Y_i \to X$ such that $A = f_1(Y_1)$ and $B = f_2(Y_2)$.

Let $\{E_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ (resp. $\{F_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$) be a Souslin scheme for Y_1 (resp. Y_2) which satisfies the conditions in Lemma 3.9.4. If A and B are not Borel separated then by Lemma 3.9.5 we may recursively define sequences $\{s_n\}_n, \{r_n\}_n \in \mathbb{N}^{\mathbb{N}}$ such that $f_1(E_{s|n})$ and $f_2(F_{r|n})$ are not Borel separated for each $n \in \mathbb{N}$. In particular, we have that $E_{s|n}$ and $F_{r|n}$ are non-empty for each $n \in \mathbb{N}$, hence there exists $a \in Y_1$, $b \in Y_2$ such that $\cap_{n\in\mathbb{N}} \overline{E_{s|n}} = \{a\}$, $\cap_{n\in\mathbb{N}} \overline{F_{r|n}} = \{b\}$.

If $V, W \subset X$ are disjoint open subsets with $f_1(a) \in V$, and $f_2(b) \in W$, then by continuity of f_i, for large enough n we have $f_1(E_{x|n}) \subset V$, and $f_2(F_{y|n}) \subset W$. Hence V separates $E_{x|n}$ from $F_{y|n}$ for large enough n, a contradiction. ■

Corollary 3.9.7. *If X is a standard Borel space then a subset $E \subset X$ is Borel if and only if both E and $X \setminus E$ are analytic.*

Corollary 3.9.8. *let X be a standard Borel space, and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint analytic subsets, then there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of disjoint Borel subsets such that $A_n \subset E_n$ for each $n \in \mathbb{N}$.*

Proof. It is easy to see that the countable union of analytic sets is analytic. Hence, by Lusin's separation theorem we may inductively define a sequence of Borel subsets $\{E_n\}_{n\in\mathbb{N}}$ such that $A_n \subset E_n$, while $(\cup_{k>n} A_k) \cup (\cup_{k<n} E_k) \subset X \setminus A_n$. ∎

Theorem 3.9.9 (Lusin-Souslin)**.** *Let X and Y be standard Borel spaces, and $f : X \to Y$ an injective Borel map, then $f(X)$ is Borel, and f implements an isomorphism of standard Borel spaces between X and $f(X)$.*

Proof. We first show that $f(X)$ is Borel. By Corollary 3.9.3 we may assume that X and Y are Polish spaces and f is continuous. Let $\{E_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ be a Souslin scheme for X which satisfies the conditions of Lemma 3.9.4. Then $\{f(E_s)\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ gives a Souslin scheme of analytic sets for Y, and since f is injective it follows that for each $s \in \mathbb{N}^n$ we have that $\{f(E_{s\hat{}k})\}_{k\in\mathbb{N}}$ are pairwise disjoint. Thus, by Corollary 3.9.8 there exist pairwise disjoint Borel subsets $\{Y_{s\hat{}k}\}_{k\in\mathbb{N}}$ such that $f(E_{s\hat{}k}) \subset Y_{s\hat{}k}$ for each $k \in \mathbb{N}$.

We inductively define a new Souslin scheme $\{C_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ for Y by setting $C_\emptyset = Y$, and $C_{s\hat{}k} = C_s \cap \overline{f(E_{s\hat{}k})} \cap Y_{s\hat{}k}$ for all $s \in \mathbb{N}^{<\mathbb{N}}$, and $k \in \mathbb{N}$. Then for each $s \in \mathbb{N}^{<\mathbb{N}}$ we have that C_s is Borel, and also

$$f(E_s) \subset C_s \subset \overline{f(E_s)}.$$

We claim that $f(X) = \cap_{k\in\mathbb{N}} \cup_{s\in\mathbb{N}^k} C_s$, from which it then follows that $f(X)$ is Borel.

If $y \in f(X)$, then let $x \in X$ be such that $f(x) = y$. There exists $s \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \cap_{k\in\mathbb{N}} E_{s|k}$, and hence $y \in \cap_{k\in\mathbb{N}} f(E_{s|k})$. Thus, $y \in \cap_{k\in\mathbb{N}} C_{s|k} \subset \cap_{k\in\mathbb{N}} \cup_{s\in\mathbb{N}^k} C_s$. Conversely, if $y \in \cap_{k\in\mathbb{N}} \cup_{s\in\mathbb{N}^k} C_s$, then there exists $s \in \mathbb{N}^{\mathbb{N}}$ such that $y \in C_{s|k} \subset \overline{f(E_{s|k})}$ for each $k \in \mathbb{N}$. Hence $E_{s|k} \neq \emptyset$ for each $k \in \mathbb{N}$ and thus $\cap_{k\in\mathbb{N}} \overline{E_{s|k}} = \{x\}$ for some $x \in X$. We must then have that $f(x) = y$, since if this were not the case there would exists an open neighborhood U of $f(x)$ such that $y \notin \overline{U}$. By continuity of f we would then have that $f(E_{s|k}) \subset U$ for large enough k, and hence $y \in \cap_{k\in\mathbb{N}} \overline{f(E_{s|k})} \subset \overline{U}$, a contradiction.

Having established that $f(X)$ is Borel, the rest of the theorem follows easily. We have that f gives a bijection from X to $f(X)$ which is Borel, and if $E \subset X$ is Borel, then from Corollary 3.9.2 and the argument above we have that $f(E)$ is again Borel. Thus, f^{-1} is a Borel map. ∎

Corollary 3.9.10. *Suppose X and Y are standard Borel spaces such that there exists injective Borel maps $f : X \to Y$, and $g : Y \to X$, then X and Y are isomorphic.*

Proof. Suppose $f : X \to Y$, and $g : Y \to X$ are injective Borel maps. From Theorem 3.9.9 we have that f and g are Borel isomorphisms onto their image and hence we may apply an argument used for the Cantor-Bernstein theorem. Specifically, if we set $B = \cup_{n \in \mathbb{N}} (f \circ g)^n (Y \setminus f(X))$, and we set $A = X \setminus g(B)$, then we have $g(B) = X \setminus A$, and

$$f(A) = f(X) \setminus (f \circ g)(B) = Y \setminus ((Y \setminus f(X)) \cup (f \circ g)(B)) = Y \setminus B.$$

Hence if we define $\theta : X \to Y$ by $\theta(x) = f(x)$, if $x \in A$, and $\theta(x) = g^{-1}(x)$, if $x \in Y \setminus A = g(B)$, then we have that theta is a bijective Borel map whose inverse is also Borel. ∎

Theorem 3.9.11 (Kuratowski)**.** *Any two uncountable standard Borel spaces are isomorphic. In particular, two standard Borel spaces X and Y are isomorphic if and only if they have the same cardinality.*

Proof. Let X be an uncountable standard Borel space, we'll show that X is isomorphic as Borel spaces to the Polish space $\mathcal{C} = 2^{\mathbb{N}}$. Note that by Corollary 3.9.10 it is enough to show that there exist injective Borel maps $f : X \to \mathcal{C}$, and $g : \mathcal{C} \to X$.

To construct f, fix a metric d on X such that d gives the Borel structure to X and such that the diameter of X is at most 1. Let $\{x_n\}$ be a countable dense subset of (X, d), and define $f_0 : X \to [0,1]^{\mathbb{N}}$ by $(f_0(x))(n) = d(x, x_n)$. The function f_0, is clearly injective and continuous, thus to construct f it is enough to construct an injective Borel map from $[0,1]^{\mathbb{N}}$ to $\mathcal{C}$, and since $\mathcal{C}^{\mathbb{N}}$ is homeomorphic to $\mathcal{C}$, it is then enough to construct an injective Borel map from $[0,1]$ to $\mathcal{C}$, and this is easily done. For example, if $y \in [0,1)$ then we may consider its dyadic expansion $y = \sum_{k=1}^{\infty} b_k 2^{-k}$, where in the case when y is a dyadic rational we take the expansion such that b_k is eventually 0. Then it is easy to see that $[0,1) \ni y \mapsto \{b_k\}_k \in \mathcal{C}$ gives an injective function which is continuous except at the countable family of dyadic rational, hence is Borel. We may then extend this map to $[0,1]$ by sending 1 to $(1,1,1,\cdots) \in \mathcal{C}$.

To construct g, we again endow X with a compatible metric d such that X has diameter at most 1. We let $Z \subset X$ denote the subset of X consisting of all points x such that every neighborhood of x is uncountable. Then $X \setminus Z$ has a countable dense subset, and each point in this subset has a neighborhood with only countably many points, hence it follows that $X \setminus Z$ is countable and so Z is uncountable. By induction on n we may define sets F_s for $s \in \{0,1\}^n$, with the following properties:

(i) $F_\emptyset = Z$,

(ii) For each $s \in \{0,1\}^n$ we have that $F_{s\hat{}0}$ and $F_{s\hat{}1}$ are disjoint subsets of F_s.

(iii) For each $s \in \{0,1\}^n$ we have that F_s is a closed ball of diameter at most 2^{-n}.

Thus, for each $s \in 2^{\mathbb{N}}$ we have that $\{F_{s|n}\}_{n\in\mathbb{N}}$ is a decreasing sequence of closed balls with diameter tending to 0. Since (X,d) is complete there then exists a unique element $g(s) \in \cap_{n\in\mathbb{N}} F_{s|n}$. It is then easy to see that $g : 2^{\mathbb{N}} \to X$ is an injective function which is continuous and hence also Borel. ∎

If (X,μ) and (Y,ν) are σ-finite measure spaces and $\pi : X \to Y$ is a measurable map such that $\pi_*\mu \prec \nu$, then we obtain a unital $*$-homomorphism $\pi^* : L^\infty(Y,\nu) \to L^\infty(X,\mu)$ given by $\pi^*(f) = f \circ \pi$. Note that π^* is well defined since $\pi^*\mu \prec \nu$. Note also that if (Y,ν) is standard, and if $\tilde{\pi} : X \to Y$ were another such map, then we would have $\pi^* = \tilde{\pi}^*$ if and only if $\pi(x) = \tilde{\pi}(x)$ for almost every $x \in X$.

We also have that π^* is normal. Indeed, the predual of $L^\infty(X,\mu)$ may naturally be identified with $M(X,\mu)$ the set of finite measures η on X such that $\eta \prec \mu$. The push forward of π then defines a bounded linear map $\pi_* : M(X,\mu) \to M(Y,\nu)$, and it is then easy to see that π^* is the dual map to π_*.

When the measure spaces are standard every normal endomorphism arrises in this way.

Theorem 3.9.12 (von Neumann)**.** *Suppose (X,μ) and (Y,ν) are σ-finite measure spaces such that Y is standard, and suppose that $\alpha : L^\infty(Y,\nu) \to L^\infty(X,\mu)$ is a normal unital $*$-homomorphism, then there exists a measurable map $\pi : X \to Y$ such that $\pi_*\mu \prec \nu$, and such that $\alpha = \pi^*$. Moreover, if X and Y are both standard and have the same cardinality and if α is an isomorphism then π can be chosen to be bijective.*

Proof. If Y is countable then the result is easy and so we only consider the case when Y is uncountable. Also, by replacing μ and ν with equivalent measures, we may assume that (X,μ) and (Y,ν) are probability spaces.

By Theorem 3.9.11 we may assume that Y is a separable compact Hausdorff space. We then have that $C(Y)$ is also separable and hence there exists a countable $\mathbb{Q}[i]$-algebra $A_0 \subset C(Y)$ which is dense in $C(Y)$. If for each

$f \in A_0$ we chose a measurable function on X which realizes $\alpha(f)$, then as α is a unital $*$-homomorphism and since A_0 is countable it follows that for almost every $x \in X$ the functional $A_0 \ni f \mapsto \alpha(f)(x)$ extends to a continuous unital $*$-homomorphism on $C(Y)$. Thus, for almost every $x \in X$, there exists a unique point $\pi(x) \in Y$ such that $\alpha(f)(x) = f(\pi(x))$ for all $f \in A_0$. We let $E \subset X$ denote a conull set of points for which this is the case. We then fix $y_0 \in Y$ and set $\pi(x) = y_0$ for $x \in X \setminus E$.

If $K \subset Y$ is closed, then let $A_n = \{f \in A_0 \mid |f(y)| \leq 1/n \text{ for all } y \in K\}$. If $x \in E$ then since $A \subset C(Y)$ is dense we have that $\pi(x) \in K$ if and only if $|f(\pi(x))| \leq 1/n$ for all $f \in A_n$, and $n \in \mathbb{N}$. Hence $E \cap \pi^{-1}(K) = E \cap \cap_{n\in\mathbb{N}} \cap_{f \in A_n} \alpha(f)^{-1}(B(1/n))$ is measurable, where $B(1/n) \subset \mathbb{C}$ denotes the closed ball with center 0 and radius $1/n$. We therefore have that π is measurable and since A_0 is dense in $C(Y)$, we have that $\int f \, \mathrm{d}\pi_*\mu = \int \alpha(f) \, \mathrm{d}\mu$ for all $f \in C(Y)$. If $F \subset Y$ is measurable such that $\pi_*\mu(F) > 0$, then there exist $f_n \in C(Y)$ uniformly bounded so that $f_n \to 1_F$ almost everywhere with respect to $\pi_*\mu$. We then have $\pi_*\mu(F) = \lim_{n\to\infty} \int \alpha(f_n) \, \mathrm{d}\mu = \int \alpha(1_F) \, \mathrm{d}\mu$ by the bounded convergence theorem. Thus $\alpha(1_F) \neq 0$ and hence $\nu(F) \neq 0$. Therefore, $\pi_*\mu \prec \nu$, and since A_0 is weakly dense in $L^\infty(Y, \nu)$ we have $\pi^* = \alpha$.

If X is also standard and uncountable, then there exists a countable collection of Borel sets $\{E_n\}_{n\in\mathbb{N}}$ which separates points. If α is a $*$-isomorphism then there exist $F_n \subset Y$ measurable sets such that $\mu(E_n \Delta \pi^{-1}(F_n))$. We set $X_0 = \cup_{n\in\mathbb{N}}(E_n \Delta \pi^{-1}(F_n))$ then we have $\mu(X_0) = 0$, and $\pi^{-1}(F_n)$ separates points on $X \setminus X_0$. Hence π is injective when restricted to $X \setminus X_0$. Since X is uncountable, it has an uncountable Borel set with measure 0, and hence we may assume that X_0 is uncountable.

Similarly, there exists an uncountable Borel set $Y_0 \subset Y$ and an injective Borel map $\tilde{\pi} : Y \setminus Y_0 \to X$ such that $\nu(Y_0) = 0$, and $\tilde{\pi}^* = \alpha^{-1}$. By again removing a null set we may assume that the range of π agrees with the domain of $\tilde{\pi}$ and vice versa. We then have that $(\tilde{\pi} \circ \pi)^* = \mathrm{id}$, $(\pi \circ \tilde{\pi})^* = \mathrm{id}$ an hence $\tilde{\pi} \circ \pi$ and $\pi \circ \tilde{\pi}$ must agree almost everywhere with the identity map. Thus, by enlarging X_0 and Y_0 with sets of measure 0 we may assume that $\pi^{-1} = \tilde{\pi}$ on $Y \setminus Y_0$. Since X_0 and Y_0 are uncountable there exists a Borel bijection between them and hence we can extend π from $X \setminus X_0$ to a Borel bijection between X and Y such that $\pi^* = \alpha$. ■

Corollary 3.9.13. *Let (X, μ) be a standard probability space, then (X, μ) is isomorphic to a probability space of one of the following forms:*

(*i*) *(K, ν) where K is countable;*

(ii) $(K \sqcup [0,1], \nu)$, *where* K *is countable, and* $\nu([0,1]) = 0$.

(iii) $(K, \nu_0) \times ([0, c_0], \lambda)$ *where* K *is countable,* $0 < c_0 < 1$, *and* λ *is Lebesgue measure;*

(iv) $([0,1], \lambda)$ *where* λ *is Lebesgue measure.*

Proof. Since (X, μ) is a standard probability space, then by Theorem 3.8.9 there exists a probability space (Y, ν) of one of the forms above, and a normal isomorphism $\theta : L^\infty(X, \mu) \to L^\infty(Y, \nu)$, such that $\int f \, d\mu = \int \theta(f) \, d\nu$, for all $f \in L^\infty(X, \mu)$. The result then follows easily from the previous theorem. ■

3.10 Normal linear functionals

Proposition 3.10.1. *Let* $A \subset \mathcal{B}(\mathcal{H})$ *be a von Neumann algebra, and let* $A_* \subset A^*$ *be the subspace of* σ*-WOT continuous linear functionals, then* $(A_*)^* = A$ *and under this identification the weak***-topology on* A *agrees with the* σ*-WOT.*

Proof. By the Hahn-Banach Theorem, and Lemma 3.4.3 we can identify A_* with $L^1(\mathcal{B}(\mathcal{H}))/A_\perp$, where $A_\perp$ is the pre-annihilator

$$A_\perp = \{x \in L^1(\mathcal{B}(\mathcal{H})) \mid \mathrm{Tr}(ax) = 0, \text{ for all } a \in A\}.$$

From the general theory of Banach spaces it follows that $(L^1(\mathcal{B}(\mathcal{H}))/A_\perp)^*$ is canonically isomorphic to the weak* closure of A, which is equal to A by Corollary 3.6.6. The fact that the weak*-topology on A agrees with the σ-WOT is then obvious. ■

If $A \subset \mathcal{B}(\mathcal{H})$ and $B \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras, then a linear map $\Phi : A \to B$ is said to be **normal** if it is continuous from the σ-WOT of A to the σ-WOT of B. Equivalently, $\Phi : A \to B$ is normal if the dual map $\Phi^* : B^* \to A^*$ given by $\Phi^*(\psi)(a) = \psi(\Phi(a))$ satisfies $\Phi^*(B_*) \subset A_*$, in this case we denote $\Phi_* = \Phi^*|_{B_*}$.

Lemma 3.10.2. *Suppose* φ *and* ψ *are positive linear functionals on a von Neumann algebra* M, *and* $p \in \mathcal{P}(M)$ *such that* $p \cdot \psi \cdot p$ *is normal and* $\varphi(p) < \psi(p)$, *then there exists a non-zero projection* $q \in \mathcal{P}(M)$, $q \leq p$ *such that* $\varphi(x) < \psi(x)$ *for all* $x \in qMq$, $x > 0$.

Proof. Consider the set $\mathcal{C}$ of all operators $0 \leq x \leq p$ such that $\varphi(x) \geq \psi(x)$. If x_i is any increasing family in $\mathcal{C}$ then since $p \cdot \psi \cdot p$ is normal we have $\psi(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} \psi(x_i)$ and since for each i we have $\varphi(\lim_{j\to\infty} x_j) \geq \varphi(x_i)$ it follows that $\lim_{i\to\infty} x_i \in \mathcal{C}$. Thus, by Zorn's lemma there exists a maximal operator $x_0 \in \mathcal{C}$. Moreover, $x_0 \neq p$ since $\varphi(p) < \psi(p)$.

Since $x_0 \neq p$, there exists $\varepsilon > 0$ such that $q = 1_{[\varepsilon,1]}(p - x_0) \neq 0$. We then have $q \leq p$ and if $0 < y \leq \varepsilon q$ then $x_0 < x_0 + y \leq x_0 + \varepsilon q \leq p$, hence $\varphi(x_0 + y) < \psi(x_0 + y) \leq \varphi(x_0) + \psi(y)$, and so $\varphi(y) < \psi(y)$. ■

Proposition 3.10.3. *Let φ be a positive linear functional on a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

(*i*) *φ is normal.*

(*ii*) *There is a positive trace class operator A such that $\varphi(x) = \mathrm{Tr}(xA)$, for all $x \in M$.*

(*iii*) *If $x_i \in M$ is any bounded increasing net, then we have $\varphi(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} \varphi(x_i)$.*

(*iv*) *If $\{p_i\}_i$ is any family of pairwise orthogonal projections in M, then $\varphi(\sum_i p_i) = \sum_i \varphi(p_i)$.*

Proof. (i) $\Longrightarrow$ (ii) If φ is normal then there exist Hilbert-Schmidt operators B and C such that $\varphi(x) = \langle xB, C\rangle_2$ for all $x \in A$. If we set $\psi(x) = \frac{1}{4}\langle x(B+C), (B+C)\rangle_2$, then ψ is a positive linear functional and for $x \in M$, $x \geq 0$ we have

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\langle xB, C\rangle_2 + \frac{1}{2}\overline{\langle xB, C\rangle_2} \\ &= \frac{1}{4}\langle x(B+C), (B+C)\rangle_2 - \frac{1}{4}\langle x(B-C), (B-C)\rangle_2 \leq \psi(x).\end{aligned}$$

By Proposition 2.3.8 there exists $T \in (M\overline{\otimes}\mathbb{C})' \subset \mathcal{B}(\mathcal{H}\overline{\otimes}\overline{\mathcal{H}})$ such that $0 \leq T \leq 1$ and $\varphi(x) = \frac{1}{4}\langle xT^{1/2}(B+C), T^{1/2}(B+C)\rangle_2$, for all $x \in M$. The result then follows.

(ii) $\Longrightarrow$ (iii) This follows easily since $x \mapsto \mathrm{Tr}(xA)$ is SOT continuous and since $x_i \to \lim_{i\to\infty} x_i$ in the SOT topology.

(iii) $\Longrightarrow$ (iv) This is obvious.

(iv) $\Longrightarrow$ (i) If $p \in \mathcal{P}(M)$ is a non-zero projection, then we can consider $\xi \in \mathcal{H}$ such that $\varphi(p) < \langle p\xi, \xi \rangle$. By Lemma 3.10.2 there then exists a non-zero projection $q \leq p$ such that $\varphi(x) < \langle x\xi, \xi \rangle$ for all $x \in qMq$. By the Cauchy-Schwarz inequality for any $x \in M$ we then have

$$|\varphi(xq)|^2 \leq \varphi(qx^*xq)\varphi(1) \leq \langle qx^*xq\xi, \xi \rangle \varphi(1) = \|xq\xi\|^2 \varphi(1).$$

Thus $q \cdot \varphi$ is SOT continuous, and hence normal.

By Zorn's lemma we may consider a maximal family $\{p_i\}_{i \in I}$ of pairwise orthogonal projections such that $p_i \cdot \varphi$ is SOT continuous for all $i \in I$. From the previous paragraph we have that $\sum_i p_i = 1$. By hypothesis, for any $\varepsilon > 0$ there exists a finite subcollection $J \subset I$ such that if $p = \sum_{j \in J} p_j$ then $\varphi(p) > \varphi(1) - \varepsilon$, but then for $x \in M$ we have

$$|(\varphi - p \cdot \varphi)(x)|^2 \leq \varphi(xx^*)\varphi(1 - p) \leq \|x\|^2 \varphi(1)(\varphi(1) - \varphi(p)),$$

hence $\|\varphi - p \cdot \varphi\|^2 < \varphi(1)\varepsilon$. Therefore the finite partial sums of $\sum_{i \in I} p_i \cdot \varphi$ converge to φ in norm, and since each $p_i \cdot \varphi$ is normal it then follows that φ is normal. ■

Corollary 3.10.4. *If φ is a normal state on a von Neumann algebra M then the GNS-representation $(\pi_\varphi, L^2(M, \varphi), 1_\varphi)$ is a normal representation.*

Proof. From the implication (i) $\Longrightarrow$ (ii) of the previous proposition we have that φ is of the form $x \mapsto \langle xT, T \rangle$ for some Hilbert-Schmidt operator T. By uniqueness of the GNS-construction it then follows that the GNS-representation $(\pi_\varphi, L^2(M, \varphi), 1_\varphi)$ is equivalent to a subrepresentation of the normal representation $x \mapsto x \otimes 1 \in \mathcal{B}(\mathcal{H} \overline{\otimes} \overline{\mathcal{H}})$. ■

Corollary 3.10.5. *Every $*$-isomorphism between von Neumann algebras is normal.*

Proof. If $\theta : M \to N$ is a $*$-isomorphism, then for any bounded increasing net $x_i \in M$ we have $\theta(\lim_{i \to \infty} x_i) \geq \lim_{i \to \infty} \theta(x_i)$, and applying θ^{-1} gives the reverse inequality as well. Hence from Proposition 3.10.3 it follows that $\varphi \circ \theta$ is normal whenever φ is. Hence, θ is normal. ■

3.11 Polar and Jordan decomposition

Lemma 3.11.1. *Let M be a von Neumann algebra and $I \subset M$ a left ideal which is closed in the WOT, then there exists a projection $p \in \mathcal{P}(M)$ such* *that $I = Mp$. If, in addition, I is a two sided ideal then p is central. If $V \subset M_*$ is a closed left invariant subspace (i.e., $x \cdot \varphi \in V$ for all $x \in M$, $\varphi \in V$), then there exists a projection $q \in \mathcal{P}(M)$ such that $V = M_* q$. If, in addition, V is also right invariant then q is central.*

Proof. By Theorem 2.1.1 any closed left ideal $I \subset M$ has a right approximate identity. Since I is closed in the WOT it then follows that I has a right identity p. Since p is positive and $p^2 = p$ we have that p is a projection, and $Mp = Ip = I$.

If I is a two sided ideal then p is also a left identity, hence for all $x \in M$ we have $xp = pxp = px$, and so $p \in \mathcal{Z}(M)$.

If $V \subset M_*$ is a closed left invariant subspace then $V^0 = \{x \in M \mid \varphi(x) = 0, \text{ for all } \varphi \in V\}$ is a right ideal which is closed in the WOT. Hence there exists $q \in \mathcal{P}(M)$ such that $V^0 = qM$. and then it is easy to check that $V = M_* q$. If V is also right invariant then V^0 will be a two sided ideal and hence q will be central. ∎

If $\varphi : M \to \mathbb{C}$ is a normal positive linear functional, then $\{x \in M \mid \varphi(x^* x) = 0\}$ is a left ideal which is closed in the WOT, thus by the previous lemma there exists a projection $p \in \mathcal{P}(M)$ such that $\varphi(x^* x) = 0$ if and only if $x \in Mp$. We denote by $s(\varphi) = 1 - p$ the **support projection** of φ. Note that if $q = s(\varphi)$ then $\varphi(xq) = \varphi(qx) = \varphi(x)$ for all $x \in M$, and moreover, φ will be faithful when restricted to qMq.

Theorem 3.11.2 (Polar decomposition)**.** *Suppose M is a von Neumann algebra and $\varphi \in M_*$, then there exists a unique partial isometry $v \in M$ and positive linear functional $\psi \in M_*$ such that $\varphi = v \cdot \psi$ and $v^* v = s(\psi)$.*

Proof. We will assume that $\|\varphi\| = 1$. Since $(M_*)^* = M$, if $\varphi \in M_*$ there exists $a \in M$, $\|a\| \leq 1$, such that $\varphi(a) = \|\varphi\|$. Consider $a^* = v|a^*|$, the polar decomposition of a^*. Then if $\psi = v^* \cdot \varphi$ we have $\psi(|a^*|) = \varphi(a) = \|\varphi\| = 1$. Since $0 \leq |a^*| \leq 1$, we have $\||a^*| + e^{i\theta}(1 - |a^*|)\| \leq 1$ for every $\theta \in \mathbb{R}$. If we fix $\theta \in \mathbb{R}$ such that $e^{i\theta}\psi(1 - |a^*|) \geq 0$ then we have

$$\psi(|a^*|) \leq \psi(|a^*|) + e^{i\theta}\psi(1 - |a^*|) = \psi(|a^*| + e^{i\theta}(1 - |a^*|)) \leq \|\psi\| = \psi(|a^*|).$$

Thus $\psi(1) = \psi(|a^*|) = \|\psi\|$ and hence ψ is a positive linear functional.

Set $p = v^* v$. By replacing a with $avs(\psi)v^*$ we may assume that $p \leq s(\psi)$, and for $x \in M$ such that $\|x\| \leq 1$, we have that $\psi(|a^*| + (1-p)x^* x(1-p)) \leq \|\psi\| = \varphi(|a^*|)$ which shows that $\psi((1-p)x^* x(1-p)) = 0$ and hence $p \geq s(\psi)$.

To see that $\varphi = v \cdot \psi$ it suffices to show that $\varphi(x(1-p)) = 0$ for all $x \in M$. Suppose that $\|x\| = 1$ and $\varphi(x(1-p)) = \beta \geq 0$. Then for $n \in \mathbb{N}$ we have

$$\begin{aligned} n + \beta = \varphi(na + x(1-p)) &\leq \|na + x(1-p)\| \\ &= \|(na + x(1-p))(na + x(1-p))^*\|^{1/2} \\ &= \|n^2|a^*|^2 + x(1-p)x^*\|^{1/2} \leq \sqrt{n^2+1}, \end{aligned}$$

which shows that $\beta = 0$.

To see that this decomposition is unique, suppose that $\varphi = v_0 \cdot \psi_0$ gives another decomposition, and set $p_0 = v_0^* v_0 = s(\psi_0)$. Then for $x \in M$ we have

$$\psi(x) = \varphi(xv^*) = \psi_0(xv^*v_0) = \psi_0(p_0 x v^* v_0).$$

Setting $x = 1 - p_0$ we then have $p = s(\psi) \leq p_0$, and by symmetry we have $p_0 \leq p$.

In particular we have $v_0^* v \in pMp$ and so we may write $v_0^* v = h + ik$ where h and k are self-adjoint elements in pMp. Then $\psi(h) + i\psi(k) = \psi(v_0^* v) = \psi_0(p) = \|\psi_0\| = \|\varphi\|$.

Hence, $\psi(h) = \|\varphi\|$ and $\psi(k) = 0$. We then have $p - h \geq 0$ and $\psi(p-h) = 0$, thus since ψ is faithful on pMp it follows that $h = p$, and we must then also have $k = 0$ since $\|v_0^* v\| \leq 1$. Hence, $v_0^* v = p$ and taking adjoints gives $v^* v_0 = p$.

Thus, $v = vp = vv^* v_0$ and so $vv^* \leq v_0 v_0^*$. Similarly, we have $v_0 v_0^* \leq vv^*$ from which it then follows that $v = v_0$. Therefore, $\psi = v^* \cdot \varphi = v_0^* \cdot \varphi = \psi_0$. ■

If $\varphi \in M_*$ as in the previous theorem then we denote by $|\varphi| = \psi$ the **absolute value** of φ. We also denote by $s_r(\varphi) = v^* v$ the **right support projection** of φ, and $s_l(\varphi) = vv^*$ the **left support projection** of φ, so that $s_l(\varphi) \cdot \varphi \cdot s_r(\varphi) = \varphi$.

Theorem 3.11.3 (Jordan decomposition)**.** *Suppose M is a von Neumann algebra and φ is a normal Hermitian linear functional, then there exist unique normal positive linear functionals φ_+, φ_- such that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. As in the previous theorem, we will take $a \in M$, $\|a\| \leq 1$, such that $\varphi(a) = \|\varphi\|$. Note that since φ is Hermitian we may assume that $a^* = a$, and hence if we consider the polar decomposition $a = |a|v$ we have that $v^* = v$ and hence $v = p - q$ for orthogonal projections $p, q \in M$. For $\psi = |\varphi|$, since

$\varphi = v \cdot \psi = (v \cdot \psi)^* = \psi \cdot v$ it follows that any spectral projection of v will commute with ψ, and hence $p \cdot \psi$ and $q \cdot \psi$ will both be positive.

Since $p \cdot \psi$ and $q \cdot \psi$ have orthogonal supports and since $p \cdot \varphi - q \cdot \varphi = \psi$ it follows that $\varphi_+ = p \cdot \varphi$ and $\varphi_- = -q \cdot \varphi$ are both positive. Thus, $\varphi = v^2 \cdot \varphi = \varphi_+ - \varphi_-$, and

$$\|\varphi\| = \psi(1) = \varphi_+(1) + \varphi_-(1) = \|\varphi_+\| + \|\varphi_-\|.$$

To see that this decomposition is unique, suppose that $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are positive, and such that $\|\varphi\| = \|\varphi_1\| + \|\varphi_2\|$. Then $\|\varphi_+\| = \varphi(s(\varphi_+)) \leq \varphi_1(s(\varphi_+)) \leq \|\varphi_1\|$, and similarly $\|\varphi_-\| \leq \|\varphi_2\|$. However, $\|\varphi_+\| + \|\varphi_-\| = \|\varphi_1\| + \|\varphi_2\|$ and so we have

$$\|\varphi_+\| = \varphi_1(s(\varphi_+)) = \|\varphi_1\|;$$

$$\|\varphi_-\| = \varphi_2(s(\varphi_-)) = \|\varphi_2\|.$$

Thus, $s(\varphi_1)$ and $s(\varphi_2)$ are orthogonal and hence $\varphi = (s(\varphi_1) - s(\varphi_2))(\varphi_1 + \varphi_2)$.

By the uniqueness for polar decomposition we then have $s(\varphi_1) - s(\varphi_2) = s(\varphi_+) - s(\varphi_-)$ from which it follows that $s(\varphi_1) = s(\varphi_+)$ and $s(\varphi_2) = s(\varphi_-)$. Therefore, $\varphi_1 = s(\varphi_1) \cdot \varphi = \varphi_+$, and $\varphi_2 = s(\varphi_2) \cdot \varphi = \varphi_-$. ■

Corollary 3.11.4. *Let M be a von Neumann algebra, then M_* is spanned by normal positive linear functionals.*

By combining the previous corollary with Proposition 3.10.3 we obtain the following corollaries.

Corollary 3.11.5. *Let $\varphi : M \to \mathbb{C}$ be a continuous linear functional on a von Neumann algebra M, then φ is normal if and only if for any family $\{p_i\}_i$ of pairwise orthogonal projections we have $\varphi(\sum_i p_i) = \sum_i \varphi(p_i)$.*

Corollary 3.11.6. *Let $\varphi : M \to \mathbb{C}$ be a continuous linear functional on a von Neumann algebra M, then φ is normal if and only if φ is normal when restricted to any abelian von Neumann subalgebra.*

We conclude this section with the following application for linear functionals on an arbitrary C^*-algebra.

Theorem 3.11.7. *Let A be a C^*-algebra and $\varphi \in A^*$, then there is a $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$, and vectors $\xi, \eta \in \mathcal{H}$ such that $\varphi(x) = \langle \pi(x)\xi, \eta \rangle$, for all $x \in A$, and such that $\|\varphi\| = \|\xi\|\|\eta\|$. Moreover, if A is a von Neumann algebra and φ is normal, then π may be taken to be a normal representation.*

Proof. Let $\varphi \in A^*$ be given, and assume that $\|\varphi\| = 1$. From Corollary 2.3.15 there exists a representation $\pi_0 : A \to \mathcal{B}(\mathcal{H}_0)$, and vectors $\xi_0, \eta_0 \in \mathcal{H}_0$ such that $\varphi(a) = \langle \pi_0(a)\xi_0, \eta_0 \rangle$ for all $a \in A$.

We set $M = \pi_0(A)''$, and we consider the normal linear functional on M given by $\tilde{\varphi}(x) = \langle x\xi_0, \eta_0 \rangle$, for $x \in M$. If $x \in (M)_1$, then by Kaplansky's density theorem there exists a sequence $a_n \in (A)_1$ such that $\pi_0(a_n) \to x$ in the strong operator topology. Hence, we have $|\tilde{\varphi}(x)| = \lim_{n\to\infty} |\varphi(a_n)| \leq \limsup_{n\to\infty} \|a_n\| \leq 1$, and so $\|\tilde{\varphi}\| = 1$.

We may then consider the polar decomposition $\tilde{\varphi} = v \cdot |\tilde{\varphi}|$. Considering the GNS-representation, we then obtain a normal representation $\rho : M \to \mathcal{B}(L^2(M, |\tilde{\varphi}|))$, such that $\psi(x) = \langle \rho(x)1_{|\tilde{\varphi}|}, 1_{|\tilde{\varphi}|} \rangle$, for all $x \in M$. If we set $\xi = 1_{|\tilde{\varphi}|}$, and $\eta = v^*1_{|\tilde{\varphi}|}$, then we have $\|\eta\|^2 = |\tilde{\varphi}|(vv^*) = 1 = \|\xi\|^2$, and for all $a \in A$ we have $\varphi(a) = \langle \rho \circ \pi(a)\xi, \eta \rangle$.

For the case when A is a von Neumann algebra and φ is normal, we may set $\tilde{\varphi} = \varphi$ and proceed as in the previous paragraph. ∎

Chapter 4

Unbounded operators

4.1 Definitions and examples

Let $\mathcal{H}$, and $\mathcal{K}$ Hilbert spaces. An **linear operator** $T : \mathcal{H} \to \mathcal{K}$ consists of a linear subspace $D(T) \subset \mathcal{H}$ together with a linear map from $D(T)$ to $\mathcal{K}$ (which will also be denoted by T). A linear operator $T : \mathcal{H} \to \mathcal{K}$ is bounded if there exists $K \geq 0$ such that $\|T\xi\| \leq K\|\xi\|$ for all $\xi \in D(T)$.

The **graph** of T is the subspace

$$\mathcal{G}(T) = \{\xi \oplus T\xi \mid \xi \in D(T)\} \subset \mathcal{H} \oplus \mathcal{K},$$

T is said to be **closed** if its graph $\mathcal{G}(T)$ is a closed subspace of $\mathcal{H}\oplus\mathcal{K}$, and T is said to be **closable** if there exists an unbounded closed operator $S : \mathcal{H} \to \mathcal{K}$ such that $\overline{\mathcal{G}(T)} = \mathcal{G}(S)$. If T is closable we denote the operator S by $\overline{T}$ and call it the **closure** of T. A linear operator T is **densely defined** if $D(T)$ is a dense subspace. We denote by $\mathcal{C}(\mathcal{H}, \mathcal{K})$ the set of closed, densely defined linear operators from $\mathcal{H}$ to $\mathcal{K}$, and we also write $\mathcal{C}(\mathcal{H})$ for $\mathcal{C}(\mathcal{H}, \mathcal{H})$. Note that we may consider $\mathcal{B}(\mathcal{H}, \mathcal{K}) \subset \mathcal{C}(\mathcal{H}, \mathcal{K})$.

If $T, S : \mathcal{H} \to \mathcal{K}$ are two linear operators, then we say that S is an **extension** of T and write $S \sqsubseteq T$ if $D(S) \subset D(T)$ and $T_{|D(S)} = S$. Also, if $T : \mathcal{H} \to \mathcal{K}$, and $S : \mathcal{K} \to \mathcal{L}$ are linear operators, then the composition $ST : \mathcal{H} \to \mathcal{L}$ is the linear operator with domain

$$D(ST) = \{\xi \in D(T) \mid T\xi \in D(S)\},$$

defined by $(ST)(\xi) = S(T(\xi))$, for all $\xi \in D(ST)$. We may similarly define addition of linear operators as

$$D(S + T) = D(S) \cap D(T),$$

and $(S+T)\xi = S\xi + T\xi$, for all $\xi \in D(S+T)$. Even if S and T are both densely defined this need not be the case for ST or $S+T$. Both composition and addition are associative operations, and we still have the right distributive property $(R+S)T = (RT) + (ST)$, although note that in general we only have $T(R+S) \sqsupseteq (TR) + (TS)$.

If $S \in \mathcal{C}(\mathcal{H})$, and $T \in \mathcal{B}(\mathcal{H})$ then ST is still closed, although it may not be densely defined. Similarly, TS will be densely defined, although it may not be closed. If T also has a bounded inverse, then both ST and TS will be closed and densely defined.

If $T : \mathcal{H} \to \mathcal{K}$ is a densely defined linear operator, and $\eta \in \mathcal{K}$ such that the linear functional $\xi \mapsto \langle T\xi, \eta \rangle$ is bounded on $D(T)$, then by the Riesz representation theorem there exists a unique vector $T^*\eta \in \mathcal{H}$ such that for all $\xi \in D(T)$ we have

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle.$$

We denote by $D(T^*)$ the linear subspace of all vectors η such that $\xi \mapsto \langle T\xi, \eta \rangle$ is bounded, and we define the linear operator $\eta \mapsto T^*\eta$ to be the **adjoint** of T. Note that T^* is only defined for operators T which are densely defined.

A densely defined operator $T : \mathcal{H} \to \mathcal{H}$ is **symmetric** if $T \sqsubseteq T^*$, and is **self-adjoint** if $T = T^*$.

Example 4.1.1. Let $A = (a_{i,j}) \in M_{\mathbb{N}}(\mathbb{C})$ be a matrix, for each $n \in \mathbb{N}$ we consider the finite rank operator $T_n = \sum_{i,j \leq n} a_{i,j} \delta_i \otimes \overline{\delta_j}$, so that we may think of T_n as changing the entries of A to 0 whenever $i > n$, or $j > n$.

We set $D = \{\xi \in \ell^2\mathbb{N} \mid \lim_{n\to\infty} T_n\xi \text{ exists.}\}$, and we define $T_A : D \to \ell^2\mathbb{N}$ by $T_A\xi = \lim_{n\to\infty} T_n\xi$.

Suppose now that for each $j \in \mathbb{N}$ we have $\{a_{i,j}\}_i \in \ell^2\mathbb{N}$. Then we have $\mathbb{C}\mathbb{N} \subset D$ and so T_A is densely defined. If $\eta \in D(T_A^*)$ then it is easy to see that if we denote by P_n the projection onto the span of $\{\delta_i\}_{i \leq n}$, then we have $P_n T_A^* \eta = T_n^* \eta$, hence $\eta \in D(T_{A^*})$ where A^* is the Hermitian transpose of the matrix A. It is also easy to see that $D(T_{A^*}) \subset D(T_A^*)$, and so $T_A^* = T_{A^*}$.

In particular, if $\{a_{i,j}\}_i \in \ell^2\mathbb{N}$, for every $j \in \mathbb{N}$, and if $\{a_{i,j}\}_j \in \ell^2\mathbb{N}$, for every $i \in \mathbb{N}$, then $T_A \in \mathcal{C}(\ell^2 N)$.

Example 4.1.2. Let (X, μ) be a σ-finite measure space and $f \in \mathcal{M}(X, \mu)$ a measurable function. We define the linear operator $M_f : L^2(X,\mu) \to L^2(X,\mu)$ by setting $D(M_f) = \{g \in L^2(X,\mu) \mid fg \in L^2(X,\mu)\}$, and $M_f(g) = fg$ for $g \in D(M_f)$. It's easy to see that each M_f is a closed operator, and we have $M_{\overline{f}} = M_f^*$. Also, if $f, g \in M(X,\mu)$ then we have $M_{f+g} \sqsupseteq M_f + M_g$, and $M_{fg} \sqsupseteq M_f M_g$.

Example 4.1.3. Let $D \subset L^2[0,1]$ denote the space of absolutely continuous functions $f : [0,1] \to \mathbb{C}$, such that $f(0) = f(1) = 0$, and $f' \in L^2[0,1]$. Then D is dense in $L^2[0,1]$, and we may consider the densely defined operator $T : L^2[0,1] \to L^2[0,1]$ with domain D, given by $T(f) = if'$. Note that the constant functions are orthogonal to the range of T. Moreover, if $g \in L^2[0,1]$ is such that $\int_0^1 g = 0$, then setting $G(x) = \int_0^x g(t)dt$ we have that $G \in D$ and $\langle TG, g\rangle = \|g\|_2^2$. Thus, if $g \in L^2[0,1]$ is any function which is orthogonal to the range of T then we have that g agrees almost everywhere with the constant function $\int_0^1 g$, i.e., $R(T)^\perp$ equals the constant functions.

If $g \in D(T^*)$, and $h = T^*g$, then set $H(x) = \int_0^x h(t)\,\mathrm{d}t$. For every $f \in D$, integration by parts gives

$$i\int_0^1 f'\overline{g} = \langle Tf, g\rangle = \langle f, h\rangle = \int_0^1 f\overline{H'} = -\int_0^1 f'\overline{H}.$$

Thus, $\langle f', H - ig\rangle = 0$ for all $f \in D$, so that $H - ig \in R(T)^\perp$, and so $H - ig$ is a constant function. In particular, we see that g is absolutely continuous, and $g' = ih \in L^2[0,1]$. Conversely, if $g : [0,1] \to \mathbb{C}$ is absolutely continuous and $g' \in L^2[0,1]$ then it is equally easy to see that $g \in D(T^*)$, and $T^*g = ig'$.

In particular, this shows that T is symmetric, but not self-adjoint. Note that the range of T^* is dense, and so the same argument above shows that if we take $g \in D(T^{**})$, $h = T^{**}g$, and $H(x) = \int_0^x h(t)dt$, then we have $H - ig \in R(T^*)^\perp = \{0\}$. Thus, $g = iH$ is absolutely continuous, $T^{**}g = h = ig'$, and we also have $g(1) = H(1) = \int_0^1 h(t)\,\mathrm{d}t = \langle 1, T^*g\rangle = \langle T(1), g\rangle = 0 = g(0)$. Thus, we conclude that $T^{**} = T$ (We'll see in Proposition 4.1.6 below that this implies that T is closed).

If we consider instead the space $\tilde{D}$ consisting of all absolutely continuous functions $f : [0,1] \to \mathbb{C}$, such that $f(0) = f(1)$, and if we define $S : \tilde{D} \to L^2[0,1]$ by $S(f) = if'$, then a similar argument shows that S is self-adjoint. Thus, we have the following sequence of extensions:

$$T^{**} = T \sqsubseteq S = S^* \sqsubseteq T^*.$$

Lemma 4.1.4. *Let $T : \mathcal{H} \to \mathcal{K}$ be a densely defined operator, and denote by $J : \mathcal{H} \oplus \mathcal{K} \to \mathcal{K} \oplus \mathcal{H}$ the isometry defined by $J(\xi \oplus \eta) = -\eta \oplus \xi$. Then we have $\mathcal{G}(T^*) = J(\mathcal{G}(T))^\perp$.*

Proof. If $\eta, \zeta \in \mathcal{K}$, the $\eta \oplus \zeta \in J(\mathcal{G}(T))^\perp$ if and only if for all $\xi \in D(T)$ we have

$$0 = \langle -T\xi \oplus \xi, \eta \oplus \zeta\rangle = \langle \xi, \zeta\rangle - \langle T\xi, \eta\rangle.$$

Which, since $\mathcal{H} = \overline{D(T)}$, is also if and only if $\eta \in D(T^*)$ and $\zeta = T^*\eta$. ■

Corollary 4.1.5. *For any densely defined operator* $T : \mathcal{H} \to \mathcal{K}$*, the operator* T^* *is closed. In particular, self-adjoint operators are closed, and symmetric operators are closable.*

Proposition 4.1.6. *A densely defined operator* $T : \mathcal{H} \to \mathcal{K}$ *is closable if and only if* T^* *is densely defined, and if this is the case then we have* $\overline{T} = (T^*)^*$.

Proof. Suppose first that T^* is densely defined. Then by Lemma 4.1.4 we have

$$\mathcal{G}((T^*)^*) = -J^*(J(\mathcal{G}(T))^{\perp})^{\perp} = (\mathcal{G}(T)^{\perp})^{\perp} = \overline{\mathcal{G}(T)},$$

hence T is closable and $(T^*)^* = \overline{T}$.

Conversely, if T is closable then take $\zeta \in D(T^*)^{\perp}$.

For all $\eta \in D(T^*)$ we have

$$0 = \langle \zeta, \eta \rangle = \langle 0 \oplus \zeta, -T^*\eta \oplus \eta \rangle,$$

and hence $0 \oplus \zeta \in (-J^*\mathcal{G}(T^*))^{\perp} = \overline{\mathcal{G}(T)}$. Since T is closable we then have $\zeta = 0$. ■

We leave the proof of the following lemma to the reader.

Lemma 4.1.7. *Suppose* $T : \mathcal{H} \to \mathcal{K}$*, and* $R, S : \mathcal{K} \to \mathcal{L}$ *are densely defined operators such that* ST *(resp.* $R + S$*) is also densely defined, then* $T^*S^* \sqsubseteq (ST)^*$ *(resp.* $R^* + S^* \sqsubseteq (R + S)^*$*).*

4.1.1 The spectrum of a linear operator

Let $T : \mathcal{H} \to \mathcal{K}$ be an injective linear operator. The **inverse** of T is the linear operator $T^{-1} : \mathcal{K} \to \mathcal{H}$ with domain $D(T^{-1}) = R(T)$, such that $T^{-1}(T\xi) = \xi$, for all $\xi \in D(T^{-1})$.

The **resolvent** set of an operator $T : \mathcal{H} \to \mathcal{H}$ is

$$\rho(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is injective and } (T - \lambda)^{-1} \in \mathcal{B}(\mathcal{H})\}.$$

The **spectrum** of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

If $\sigma \in \mathcal{U}(\mathcal{H} \oplus \mathcal{K})$ is given by $\sigma(\xi \oplus \eta) = \eta \oplus \xi$, and if $T : \mathcal{H} \to \mathcal{K}$ is injective then we have that $\mathcal{G}(T^{-1}) = \sigma(\mathcal{G}(T))$. Hence, if $T : \mathcal{H} \to \mathcal{H}$ is not closed then $\sigma(T) = \mathbb{C}$. Also, note that if $T \in \mathcal{C}(\mathcal{H})$ then by the closed graph theorem shows that $\lambda \in \rho(T)$ if and only if $T - \lambda$ gives a bijection between $D(T)$ and $\mathcal{H}$.

Lemma 4.1.8. *Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ be injective with dense range, then $(T^*)^{-1} = (T^{-1})^*$. In particular, for $T \in \mathcal{C}(\mathcal{H})$ we have $\sigma(T^*) = \{\overline{z} \mid z \in \sigma(T)\}$.*

Proof. If we consider the unitary operators J, and σ from above then we have

$$\mathcal{G}((T^*)^{-1}) = \sigma(\mathcal{G}(T^*)) = \sigma J(\mathcal{G}(T))^{\perp}$$
$$= J^*(\sigma\mathcal{G}(T))^{\perp} = J^*(\mathcal{G}(T^{-1}))^{\perp} = \mathcal{G}((T^{-1})^*).$$ ■

Lemma 4.1.9. *If $T \in \mathcal{C}(\mathcal{H})$, then $\sigma(T)$ is a closed subset of $\mathbb{C}$.*

Proof. We will show that $\rho(T)$ is open by showing that whenever $\lambda \in \rho(T)$ with $|\alpha - \lambda| < \|(T - \lambda)^{-1}\|^{-1}$, then $\alpha \in \rho(T)$. Thus, suppose $\lambda \in \rho(T)$ and $\alpha \in \mathbb{C}$ such that $|\lambda - \alpha| < \|(T - \lambda)^{-1}\|^{-1}$. Then for all $\xi \in \mathcal{H}$ we have

$$\|\xi - (T - \alpha)(T - \lambda)^{-1}\xi\| = \|(\alpha - \lambda)(T - \lambda)^{-1}\xi\| < \|\xi\|.$$

Hence, by Lemma 1.1.1, $S = (T - \alpha)(T - \lambda)^{-1}$, is bounded, everywhere defined operator with a bounded everywhere defined inverse $S^{-1} \in \mathcal{B}(\mathcal{H})$. We then have $(T - \lambda)^{-1}S^{-1} \in \mathcal{B}(\mathcal{H})$, and it's easy to see that $(T - \lambda)^{-1}S^{-1} = (T - \alpha)^{-1}$. ■

Note that an unbounded operator may have empty spectrum. Indeed, if $S \in \mathcal{B}(\mathcal{H})$ has a densely defined inverse, then for each $\lambda \in \sigma(S^{-1}) \setminus \{0\}$ we have $(S - \lambda^{-1})\lambda(\lambda - S)^{-1}S^{-1} = S(\lambda - S^{-1})(\lambda - S)^{-1}S^{-1} = \mathrm{id}$. Hence $\sigma(S^{-1}) \setminus \{0\} \subset (\sigma(S) \setminus \{0\})^{-1}$. Thus, it is enough to find a bounded operator $S \in \mathcal{B}(\mathcal{H})$ such that S is injective but not surjective, and has dense range with $\sigma(S) = \{0\}$. For example, the compact operator $S \in \mathcal{B}(\ell^2\mathbb{Z})$ given by $(S\delta_n) = \frac{1}{|n|+1}\delta_{n+1}$ is injective with dense range, but is not surjective, and $\|S^{2n}\| \leq 1/n!$, so that $r(S) = 0$ and hence $\sigma(S) = \{0\}$.

4.1.2 Quadratic forms

A **quadratic form** $q : \mathcal{H} \to \mathbb{C}$ on a Hilbert space $\mathcal{H}$ consists of a linear subspace $D(q) \subset \mathcal{H}$, together with a sesquilinear form $q : D(q) \times D(q) \to \mathbb{C}$. We say that q is **densely defined** if $D(q)$ is dense. If $\xi \in D(q)$ then we write $q(\xi)$ for $q(\xi, \xi)$; note that we have the polarization identity $q(\xi, \eta) = \frac{1}{4}\sum_{k=0}^{3} i^k q(\xi + i^k\eta)$, and in general, a function $q : D \to \mathcal{H}$ defines a sesquilinear from through the polarization identity if and only if it satisfies the parallelogram identity $q(\xi + \eta) + q(\xi - \eta) = 2q(\xi) + 2q(\eta)$ for all $\xi, \eta \in D(q)$. A quadratic form q is **non-negative definite** if $q(\xi) \geq 0$ for all $\xi \in \mathcal{H}$.

If q is a non-negative definite quadratic form and we denote by $\mathcal{H}_q$ the separation and completion of $D(q)$ with respect to q, then we may consider the identity map $I : D(q) \to \mathcal{H}_q$, and note that for $\xi, \eta \in D(q)$ we have $\langle \xi, \eta \rangle_q := \langle \xi, \eta \rangle + q(\xi, \eta)$ coincides with the inner-product coming from the graph of I. The quadratic form q is **closed** if I is closed, i.e., $(D(q), \langle \cdot, \cdot \rangle_q)$ is complete. We'll say that q is **closable** if I is closable, and in this case we denote by $\overline{q}$ the closed quadratic form given by $D(\overline{q}) = D(\overline{I})$, and $\overline{q}(\xi, \eta) = \langle \overline{I}\xi, \overline{I}\eta \rangle$.

Theorem 4.1.10. *Let $q : \mathcal{H} \to [0, \infty)$ be a non-negative definite quadratic form, then the following conditions are equivalent:*

(*i*) *q is closed.*

(*ii*) *There exists a Hilbert space $\mathcal{K}$, and a closed linear operator $T : \mathcal{H} \to \mathcal{K}$ with $D(T) = D(q)$ such that $q(\xi, \eta) = \langle T\xi, T\eta \rangle$ for all $\xi, \eta \in D(T)$.*

(*iii*) *q is lower semi-continuous, i.e., for any sequence $\xi_n \in D(q)$, such that $\xi_n \to \xi$, and $\liminf_{n\to\infty} q(\xi_n) < \infty$, we have $\xi \in D(q)$ and $q(\xi) \leq \liminf_{n\to\infty} q(\xi_n)$.*

Proof. The implication (i) $\implies$ (ii) follows from the discussion preceding the theorem. For (ii) $\implies$ (iii) suppose that $T : \mathcal{H} \to \mathcal{K}$ is a closed linear operator such that $D(T) = D(q)$, and $q(\xi, \eta) = \langle T\xi, T\eta \rangle$ for all $\xi, \eta \in D(T)$. If $\xi_n \in D(T)$ is a sequence such that $\xi_n \to \xi \in \mathcal{H}$, and $K = \liminf_{n\to\infty} \|T\xi_n\|^2 < \infty$, then by taking a subsequence we may assume that $K = \lim_{n\to\infty} \|T\xi_n\|^2$, and $T\xi_n \to \eta$ weakly for some $\eta \in \mathcal{H}$. Taking convex combinations we may then find a sequence ξ'_n such that $\xi'_n \to \xi \in \mathcal{H}$, $T\xi'_n \to \eta$ strongly, and $\|\eta\| = \lim_{n\to\infty} \|T\xi'_n\|^2 \leq K$. Since T is closed we then have $\xi \in D(T)$, and $T\xi = \eta$, so that $\|T\xi\|^2 \leq K$.

We show (iii) $\implies$ (i) by contraposition, so suppose that $\mathcal{H}_q$ is the separation and completion of $D(q)$ with respect to q, and that $I : D(q) \to \mathcal{H}_q$ is not closed. If I were closable, then there would exist a sequence $\xi_n \in D(q)$ such that $\xi_n \to \eta \in \mathcal{H}$, and $I(\xi_n)$ is Cauchy, but $\eta \notin D(q)$. However, if $I(\xi_n)$ is Cauchy then in particular we have that $q(\xi_n)$ is bounded, hence this sequence would show that (iii) does not hold.

Thus, we may assume that I is not closable, so that there exists a sequence $\xi_n \in D(q)$ such that $\|\xi_n\| \to 0$, and $I(\xi_n) \to \eta \neq 0$. Since, $D(q)$ is dense in $\mathcal{H}_q$ there exists $\eta_0 \in D(q)$ such that the square distance from η_0 to η in $\mathcal{H}_q$ is less than $q(\eta_0)$. We then have that $\eta_0 - \xi_n \to \eta_0 \in \mathcal{H}$, and by the triangle inequality, $\lim_{n\to\infty} q(\eta_0 - \xi_n) < q(\eta_0)$. Thus, the sequence $\eta_0 - \xi_n$ shows that (iii) does not hold in this case also. ■

Corollary 4.1.11. *Let $q_n : \mathcal{H} \to [0, \infty)$ be a sequence of closed non-negative definite quadratic forms, and assume that this sequence is increasing, i.e., $q_n(\xi)$ is an increasing sequence for all $\xi \in \cap_{n\in\mathbb{N}} D(q_n)$. Then there exists a closed quadratic form $q : \mathcal{H} \to [0, \infty)$ with domain*

$$D(q) = \{\xi \in \cap_{n\in\mathbb{N}} D(q_n) \mid \lim_{n\to\infty} q_n(\xi) < \infty\}$$

such that $q(\xi) = \lim_{n\to\infty} q_n(\xi)$, for all $\xi \in D(q)$.

Proof. If we define q as above then note that since each q_n satisfies the parallelogram identity then so does q, and hence q has a unique sesquilinear extension on $D(q)$. That q is closed follows easily from condition (iii) of Theorem 4.1.10. ■

4.2 Symmetric operators and extensions

Lemma 4.2.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a densely defined operator, then T is symmetric if and only if $\langle T\xi, \xi\rangle \in \mathbb{R}$, for all $\xi \in D(T)$.*

Proof. If T is symmetric then for all $\xi \in D(T)$ we have $\langle T\xi, \xi\rangle = \langle \xi, T\xi\rangle = \overline{\langle T\xi, \xi\rangle}$. Conversely, if $\langle T\xi, \xi\rangle = \langle \xi, T\xi\rangle$ for all $\xi \in D(T)$, then the polarization identity shows that $D(T) \subset D(T^*)$ and $T^*\xi = T\xi$ for all $\xi \in D(T)$. ■

Proposition 4.2.2. *Let $T \in \mathcal{C}(\mathcal{H})$ be a symmetric operator, then for all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, we have $\ker(T - \lambda) = \{0\}$, and $R(T - \lambda)$ is closed.*

Proof. Fix $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, and set $\lambda = \alpha + i\beta$. For $\xi \in D(T)$ we have

$$\begin{aligned}\|(T-\lambda)\xi\|^2 &= \|(T-\alpha)\xi\|^2 + \|\beta\xi\|^2 - 2\mathrm{Re}(\langle (T-\alpha)\xi, i\beta\xi\rangle) \\ &= \|(T-\alpha)\xi\|^2 + \|\beta\xi\|^2 \geq \beta^2\|\xi\|^2. \end{aligned} \tag{4.1}$$

Thus, $\ker(T - \lambda) = \{0\}$, and if $\xi_n \in D(T)$ such that $(T - \lambda)\xi_n$ is Cauchy, then so is ξ_n, and hence $\xi_n \to \eta$ for some $\eta \in \mathcal{H}$. Since T is closed we have $\eta \in D(T)$ and $(T - \lambda)\eta = \lim_{n\to\infty}(T - \lambda)\xi_n$. Hence, $R(T - \lambda)$ is closed. ■

Lemma 4.2.3. *Let $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{H}$ be two closed subspaces such that $\mathcal{K}_1 \cap \mathcal{K}_2^\perp = \{0\}$, then $\dim \mathcal{K}_1 \leq \dim \mathcal{K}_2$.*

Proof. Let P_i be the orthogonal projection onto $\mathcal{K}_i$. Then by hypothesis we have that P_2 is injective when viewed as an operator from $\mathcal{K}_1$ to $\mathcal{K}_2$, hence if we let v be the partial isometry in the polar decomposition of $P_{2|\mathcal{K}_1}$ then v is an isometry and so $\dim \mathcal{K}_1 \leq \dim \mathcal{K}_2$. ∎

Theorem 4.2.4. *If* $T \in \mathcal{C}(\mathcal{H})$ *is symmetric, then* $\dim \ker(T^* - \lambda)$ *is a constant function for* $\operatorname{Im} \lambda > 0$*, and for* $\operatorname{Im} \lambda < 0$*.*

Proof. Note that the result will follow easily if we show that for all $\lambda, \alpha \in \mathbb{C}$ such that $|\alpha - \lambda| < |\operatorname{Im} \lambda|/2$, then we have $\dim \ker(T^* - \lambda) = \dim \ker(T^* - \alpha)$. And this in turn follows easily if we show that for all $\lambda, \alpha \in \mathbb{C}$ such that $|\alpha - \lambda| < |\operatorname{Im} \lambda|$, then we have $\dim \ker(T^* - \alpha) \leq \dim \ker(T^* - \alpha)$.

Towards this end, suppose we have such $\alpha, \lambda \in \mathbb{C}$. If $\xi \in \ker(T^* - \alpha) \cap (\ker(T^* - \lambda))^\perp$ such that $\|\xi\| = 1$, then since $R(T - \overline{\lambda})$ is closed we have $\xi \in (\ker(T^* - \lambda))^\perp = R(T - \overline{\lambda})$ and so $\xi = (T - \overline{\lambda})\eta$ for some $\eta \in D(T)$. Since, $\xi \in \ker(T^* - \alpha)$ we then have

$$0 = \langle (T^* - \alpha)\xi, \eta \rangle = \langle \xi, (T - \overline{\lambda})\eta \rangle + \langle \xi, \overline{\lambda - \alpha}\eta \rangle = \|\xi\|^2 + (\lambda - \alpha)\langle \xi, \eta \rangle.$$

Hence, $1 = \|\xi\|^2 = |\lambda - \alpha||\langle \xi, \eta \rangle| < |\operatorname{Im} \lambda|\|\eta\|$. However, by (4.1) we have $|\operatorname{Im} \lambda|^2 \|\eta\|^2 \leq \|(T - \overline{\lambda})\eta\|^2 = 1$, which gives a contradiction.

Thus, we conclude that $\ker(T^* - \alpha) \cap (\ker(T^* - \lambda))^\perp = \{0\}$, and hence $\dim \ker(T^* - \alpha) \leq \dim \ker(T^* - \lambda)$ by Lemma 4.2.3. ∎

Corollary 4.2.5. *If* $T \in \mathcal{C}(\mathcal{H})$ *is symmetric, then one of the following occurs:*

(*i*) $\sigma(T) = \mathbb{C}$.

(*ii*) $\sigma(T) = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \geq 0\}$.

(*iii*) $\sigma(T) = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \leq 0\}$.

(*iv*) $\sigma(T) \subset \mathbb{R}$.

Proof. For $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \neq 0$ then by (4.1) we have that $T - \lambda$ is injective with closed range. Thus, $\lambda \in \rho(T)$ if and only if $T - \lambda$ is surjective, or equivalently, if $T^* - \overline{\lambda}$ is injective. By the previous theorem if $T^* - \overline{\lambda}$ is injective for some λ with $\operatorname{Im} \lambda > 0$, then $T^* - \overline{\lambda}$ is injective for all λ with $\operatorname{Im} \lambda > 0$. Hence, either $\sigma(T) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \leq 0\}$ or $\{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda > 0\} \subset \sigma(T)$.

Since $\sigma(T)$ is closed, it is then easy to see that only one of the four possibilities can occur. ∎

Theorem 4.2.6. *If $T \in \mathcal{C}(\mathcal{H})$ is symmetric, then the following are equivalent:*

(i) T *is self-adjoint.*

(ii) $\ker(T^* - i) = \ker(T^* + i) = \{0\}$.

(iii) $\sigma(T) \subset \mathbb{R}$.

Proof. (i) $\implies$ (ii) follows from Proposition 4.2.2, while (ii) $\Leftrightarrow$ (iii) follows from the previous corollary. To see that (ii) $\implies$ (i) suppose that $\ker(T^* - i) = \ker(T^* + i) = \{0\}$. Then by Proposition 4.2.2 we have that $R(T+i) = \ker(T^* - i)^\perp = \mathcal{H}$. Thus, $T + i$ is the only injective extension of $T + i$. Since $T^* + i$ is an injective extension of $T + i$ we conclude that $T^* + i = T + i$ and hence $T^* = T$. ■

The subspaces $\mathcal{L}_+ = \ker(T^* - i) = R(T + i)^\perp$ and $\mathcal{L}_- = \ker(T^* + i) = R(T - i)^\perp$ are the **deficiency subspaces** of the symmetric operator $T \in \mathcal{C}(\mathcal{H})$, and $n_\pm = \dim \mathcal{L}_\pm$ is the **deficiency indices**.

4.2.1 The Cayley transform

Recall from Section 3.6 that the Cayley transform $t \mapsto (t-i)(t+i)^{-1}$ and its inverse $t \mapsto i(1+t)(1-t)^{-1}$ give a bijection between self-adjoint operators $x = x^* \in \mathcal{B}(\mathcal{H})$ and unitary operators $u \in \mathcal{U}(\mathcal{H})$ such that $1 \not\in \sigma(u)$. Here, we extend this correspondence to the setting of unbounded operators.

If $T \in C(\mathcal{H})$ is symmetric with deficiency subspaces $\mathcal{L}_\pm$, then the **Cayley transform** of T is the operator $U : \mathcal{H} \to \mathcal{H}$ given by $U_{|\mathcal{L}_+} = 0$, and

$$U\xi = (T - i)(T + i)^{-1}\xi$$

for all $\xi \in \mathcal{L}_+^\perp = R(T + i)$. If $\eta \in D(T)$ then by (4.1) we have that $\|(T + i)\eta\|^2 = \|T\eta\|^2 + \|\eta\|^2 = \|(T - i)\eta\|^2$, hence it follows that U is a partial isometry with initial space $\mathcal{L}_+^\perp$ and final space $\mathcal{L}_-^\perp$. Moreover, if $\xi \in D(T)$ then $(1 - U)(T + i)\xi = (T + i)\xi - (T - i)\xi = 2i\xi$. Since $R(T + i) = \mathcal{L}_+^\perp$ it follows that $(1 - U)(\mathcal{L}_+^\perp) = D(T)$ is dense.

Conversely, if $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry with $(1-U)(U^*U\mathcal{H})$ dense, then we also have that $(1 - U)$ is injective. Indeed, if $\xi \in \ker(1 - U)$ then $\|\xi\| = \|U\xi\|$ so that $\xi \in UU^*\mathcal{H}$. Hence, $\xi = U^*U\xi = U^*\xi$ and so $\xi \in \ker(1 - U^*) = R(1 - U)^\perp = \{0\}$.

We define the **inverse Cayley transform** of U to be the densely defined operator with domain $D(T) = (1 - U)(U^*U\mathcal{H})$ given by

$$T = i(1 + U)(1 - U)^{-1}.$$

Note that T is densely defined, and

$$\mathcal{G}(T) = \{(1 - U)\xi \oplus i(1 + U)\xi \mid \xi \in U^*U\mathcal{H}\}.$$

If $\xi_n \in U^*U\mathcal{H}$ such that $(1-U)\xi_n \oplus i(1+U)\xi_n$ is Cauchy, then both $(1-U)\xi_n$ and $(1+U)\xi_n$ is Cauchy and hence so is ξ_n. Thus, $\xi_n \to \xi$ for some $\xi \in U^*U\mathcal{H}$, and we have $(1 - U)\xi_n \oplus i(1 + U)\xi_n \to (1 - U)\xi \oplus i(1 + U)\xi \in \mathcal{G}(T)$. Hence, T is a closed operator.

Note also that for all $\xi, \zeta \in U^*U\mathcal{H}$ we have

$$\begin{aligned}
&\langle (1 - U)\xi \oplus i(1 + U)\xi, -i(1 + U)\zeta \oplus (1 - U)\zeta\rangle \\
&= i\langle (1 - U)\xi, (1 + U)\zeta\rangle + i\langle (1 + U)\xi, (1 - U)\zeta\rangle \\
&= 2i\langle \xi, \zeta\rangle - 2i\langle U\xi, U\zeta\rangle = 0
\end{aligned}$$

Thus, by Lemma 4.1.4 we have $\mathcal{G}(T) \subset J(\mathcal{G}(T))^\perp = \mathcal{G}(T^*)$, and hence T is symmetric.

Theorem 4.2.7. *The Cayley transform and its inverse give a bijective correspondence between densely defined closed symmetric operators $T \in \mathcal{C}(\mathcal{H})$, and partial isometries $U \in \mathcal{B}(\mathcal{H})$ such that $(1 - U)(U^*U\mathcal{H})$ is dense. Moreover, self-adjoint operators correspond to unitary operators.*

Also, if $S, T \in \mathcal{C}(\mathcal{H})$ are symmetric, and $U, V \in \mathcal{B}(\mathcal{H})$ their respective Cayley transforms then we have $S \sqsubseteq T$ if and only if $U^*U\mathcal{H} \subset V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$.

Proof. We've already seen above that the Cayley transform of a densely defined closed symmetric operator T is a partial isometry U with $(1-U)(U^*U\mathcal{H})$ dense. And conversely, the inverse Cayley transform of a partial isometry U with $(1 - U)(U^*U\mathcal{H})$ dense, is a densely defined closed symmetric operator. Moreover, it is easy to see from construction that these are inverse operations.

We also see from construction that the deficiency subspaces of T are $\mathcal{L}_+ = \ker(U)$ and $\mathcal{L}_- = \ker(U^*)$ respectively. By Theorem 4.2.6 T is self-adjoint if and only if $\mathcal{L}_+ = \mathcal{L}_- = \{0\}$, which is if and only if U is a unitary.

Suppose now that $S, T \in \mathcal{C}(\mathcal{H})$ are symmetric and $U, V \in \mathcal{B}(\mathcal{H})$ are the corresponding Cayley transforms. If $S \sqsubseteq T$ then for all $\xi \in D(S) \subset D(T)$ we have $(S+i)\xi = (T+i)\xi$ and hence

$$U(S+i)\xi = (S-i)\xi = (T-i)\xi = V(S+i)\xi.$$

Therefore, $U^*U\mathcal{H} = R(S+i) \subset R(T+i) = V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$. Conversely, if $U^*U\mathcal{H} \subset V^*V\mathcal{H}$ and $V\xi = U\xi$ for all $\xi \in U^*U\mathcal{H}$, then

$$D(S) = R((1-U)(U^*U)) = R((1-V)(U^*U)) \subset R((1-V)(V^*V)) = D(T),$$

and for all $\xi \in U^*U\mathcal{H}$ we have

$$S(1-U)\xi = i(1+U)\xi = i(1+V)\xi = T(1-V)\xi = T(1-U)\xi,$$

hence $S \sqsubseteq T$. ■

The previous theorem in particular shows us that if $T \in \mathcal{C}(\mathcal{H})$ is a symmetric operator, and U its Cayley transform, then symmetric extensions of T are in bijective correspondence with partial isometries which extend U. Since the latter are in bijective correspondence with partial isometries from $(UU^*\mathcal{H})^\perp$ to $(U^*U\mathcal{H})^\perp$, simply translating this via the inverse Cayley transform gives the following, whose details we leave to the reader.

Theorem 4.2.8. *Let $T \in \mathcal{C}(\mathcal{H})$ be a symmetric operator, and $\mathcal{L}_\pm$ its deficiency spaces. For each partial isometry $W : \mathcal{L}_+ \to \mathcal{L}_-$, denote the operator T_W by*

$$D(T_W) = \{\xi + \eta + W\eta \mid \xi \in D(T), \eta \in W^*W(\mathcal{L}_+)\},$$

and

$$T_W(\xi + \eta + W\eta) = T\xi + i\eta - iW\eta.$$

Then T_W is a symmetric extension of T with

$$\mathcal{G}(T_W^*) = \mathcal{G}(T_W) + (\mathcal{L}_+ \ominus W^*W(\mathcal{L}_+)) + (\mathcal{L}_- \ominus WW^*(\mathcal{L}_-)).$$

Moreover, every symmetric extension arises in this way, and T_W is self-adjoint if and only if W is unitary.

Corollary 4.2.9. *If $T \in \mathcal{C}(\mathcal{H})$ is symmetric, then T has a self-adjoint extension if and only if $n_+ = n_-$.*

Exercise 4.2.10. show that for any pair $(n_+, n_-) \in (\mathbb{N} \cup \{0\} \cup \{\infty\})^2$ there exists a densely defined closed symmetric operator $T \in \mathcal{C}(\ell^2\mathbb{N})$ such that n_+ and n_- are the deficiency indices for T.

4.3 Functional calculus for normal operators

If $T : \mathcal{H} \to \mathcal{K}$ is a closed operator, then a subspace $D \subset D(T)$ is a **core** for T if $\mathcal{G}(T) = \overline{\mathcal{G}(T_{|D})}$.

4.3.1 Positive operators

Theorem 4.3.1. *Suppose $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, then*

(i) $D(T^*T)$ is a core for T.

(ii) T^*T is self-adjoint.

(iii) $\sigma(T^*T) \subset [0, \infty)$.

Proof. We start by showing that $-1 \in \rho(T^*T)$. Since $\mathcal{K} \oplus \mathcal{H} = J\mathcal{G}(T) + \mathcal{G}(T^*)$, if $\xi \in \mathcal{H}$ then there exists $\eta \in \mathcal{H}$, $\zeta \in \mathcal{K}$ such that

$$0 \oplus \xi = -T\eta \oplus \eta + \zeta \oplus T^*\zeta.$$

Hence, $\zeta = T\eta$ and $\xi = \eta + T^*\zeta = (1 + T^*T)\eta$, showing that $(1 + T^*T)$ is onto.

If $\xi \in D(T^*T)$ then

$$\|\xi + T^*T\xi\|^2 = \|\xi\|^2 + 2\|T\xi\|^2 + \|T^*T\xi\|^2.$$

Hence, we see that $1 + T^*T$ is injective. This also shows that if $\xi_n \in D(T^*T)$ is a sequence which is Cauchy in the graph norm of $1 + T^*T$, then we must have that $\{\xi_n\}_n$, $\{T\xi_n\}_n$, and $\{T^*T\xi_n\}$ are all Cauchy. Since T and T^* are closed it then follows easily that $1 + T^*T$ is also closed. Thus, $(1 + T^*T)^{-1}$ is an everywhere defined closed operator and hence is bounded, showing that $-1 \in \rho(T^*T)$.

To see that $D(T^*T)$ is a core for T consider $\xi \oplus T\xi \in \mathcal{G}(T)$ such that $\xi \oplus T\xi \perp \{\eta \oplus T\eta \mid \eta \in D(T^*T)\}$. Then for all $\eta \in D(T^*T)$ we have

$$0 = \langle \xi \oplus T\xi, \eta \oplus T\eta \rangle = \langle \xi, \eta \rangle + \langle T\xi, T\eta \rangle = \langle \xi, (1 + T^*T)\eta \rangle.$$

Since $(1 + T^*T)$ is onto, this shows that $\xi = 0$.

In particular, T^*T is densely defined and we have $-1 \in \rho(T^*T)$. Note that by multiplying by scalars we see that $(-\infty, 0) \subset \rho(T)$. If $\xi = (1+T^*T)\eta$ for $\eta \in D(T^*T)$ then we have

$$\langle (1 + T^*T)^{-1}\xi, \xi \rangle = \langle \eta, (1 + T^*T)\eta \rangle = \|\eta\|^2 + \|T\eta\|^2 \geq 0.$$

Thus $(1+T^*T)^{-1} \geq 0$ and hence it follows from Lemma 4.1.8 that $1+T^*T$ and hence also T^*T is self-adjoint. By Theorem 4.2.6 this shows that $\sigma(T^*T) \subset \mathbb{R}$, and hence $\sigma(T^*T) \subset [0, \infty)$. ■

An operator $T \in \mathcal{C}(\mathcal{H})$ is **positive** if $T = S^*S$ for some densely defined closed operator $S : \mathcal{H} \to \mathcal{H}$.

4.3.2 Borel functional calculus

Suppose K is a locally compact Hausdorff space, and E is a spectral measure on K relative to $\mathcal{H}$. We let $B(K)$ denote the space of Borel functions on K. For each $f \in B(K)$ we define a linear operator $T = \int f \, \mathrm{d}E$ by setting

$$D(T) = \{\xi \in \mathcal{H} \mid \eta \mapsto \int f \, \mathrm{d}E_{\xi,\eta} \text{ is bounded.}\},$$

and letting $T\xi$ be the unique vector such that $\int f \, \mathrm{d}E_{\xi,\eta} = \langle T\xi, \eta \rangle$, for all $\eta \in \mathcal{H}$.

If $B \subset K$ is any Borel set such that $f_{|B}$ is bounded, then for all $\xi, \eta \in \mathcal{H}$ we have that $1_B E_{\xi,\eta} = E_{E(B)\xi,\eta}$ and hence $|\int f \, \mathrm{d}E_{E(B)\xi,\eta}| = |\int f_{|B} \, \mathrm{d}E_{\xi,\eta}| \leq \|f_{|B}\|_\infty \|\xi\| \|\eta\|$, and so $E(B)\mathcal{H} \subset D(T)$. Taking $B_n = \{x \in K \mid |f(x)| \leq n\}$ we then have that $\cup_{n \in \mathbb{N}} E(B_n)\mathcal{H} \subset D(T)$ and this is dense since $E(B_n)$ converges strongly to 1. Thus T is densely defined.

If $S = \int \overline{f} \, \mathrm{d}E$, then for all $\xi \in D(T)$ and $\eta \in D(S)$ we have

$$\langle T\xi, \eta \rangle = \int f \, \mathrm{d}E_{\xi,\eta} = \overline{\int \overline{f} \, \mathrm{d}E_{\eta,\xi}} = \overline{\langle S\eta, \xi \rangle} = \langle \xi, S\eta \rangle.$$

A similar argument shows that $D(T^*) \subset D(S)$, so that in fact we have $S = T^*$ and $T^* = S$. In particular, T is a closed operator, and is self-adjoint if f is real valued. It is equally easy to see that $T^*T = TT^* = \int |f|^2 \, \mathrm{d}E$.

It is easy to see that if $f, g \in B(K)$ then $\int f \, \mathrm{d}E + \int g \, \mathrm{d}E \sqsubseteq \int (f+g) \, \mathrm{d}E$, and $(\int f \, \mathrm{d}E)(\int g \, \mathrm{d}E) \sqsubseteq \int fg \, \mathrm{d}E$, and in both cases the domains on the left are cores for the operators on the right. In fact, if $f_1, \ldots, f_n \in B(K)$ is any finite collection of Borel functions, then we have that $\cap_{i=1}^n D(\int f_n \, \mathrm{d}E)$ is a common core for each operator $\int f_i \, \mathrm{d}E$. In particular, on the set of all operators of the form $\int f \, \mathrm{d}E$ we may consider the operations $\hat{+}$, and $\hat{\circ}$ given by $S \hat{+} T = \overline{S + T}$, and $S \hat{\circ} T = \overline{S \circ T}$, and under these operations we have that $f \mapsto \int f \, \mathrm{d}E$ is a unital $*$-homomorphism from $B(K)$ into $\mathcal{C}(\mathcal{H})$.

We also note that $\sigma(\int f\,\mathrm{d}E)$ is contained in the closure of the range of f, for each $f \in B(K)$.

An operator $T \in \mathcal{C}(\mathcal{H})$ is **normal** if $T^*T = TT^*$. Note that equality here implies also $D(T^*T) = D(TT^*)$. We would like to associate a spectral measure for each normal operator as we did for bounded normal operators. However, our approach for bounded operators, Theorem 3.7.5, does not immediately apply since we used there that a bounded normal operator generated an abelian C^*-algebra. Our approach therefore will be to reduce the problem to the case of bounded operators.

Lemma 4.3.2. *Suppose $T \in \mathcal{C}(\mathcal{H})$, then $R = T(1 + T^*T)^{-1}$ and $S = (1 + T^*T)^{-1}$ are bounded contractions. If T is normal then we have $SR = RS$.*

Proof. If $\xi \in \mathcal{H}$, fix $\eta \in D(T^*T)$ such that $(1 + T^*T)\eta = \xi$. Then

$$\|\xi\|^2 = \|(1+T^*T)\eta\|^2 = \|\eta\|^2+2\|T\eta\|^2+\|T^*T\eta\|^2 \geq \|\eta\|^2 = \|(1+T^*T)^{-1}\xi\|^2.$$

Hence $\|S\| \leq 1$. Similarly, $\|\xi\|^2 \geq \|T\eta\|^2 = \|R\xi\|^2$, hence also $\|R\| \leq 1$.

Suppose now that T is normal and $\xi \in D(T)$. Since $\eta \in D(T^*T)$ and $\xi = (1 + T^*T)\eta \in D(T)$ we have that $T\eta \in D(TT^*) = D(T^*T)$. Hence, $T\xi = T(1 + T^*T)\eta = (1 + TT^*)T\eta = (1 + T^*T)T\eta$. Thus, $ST\xi = TS\xi$ for all $\xi \in D(T)$.

Suppose now that $\xi \in \mathcal{H}$ is arbitrary. Since $\eta \in D(T^*T) \subset D(T)$, we have $SR\xi = ST\eta = TS\eta = RS\xi$. ■

Theorem 4.3.3. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator, then $\sigma(T) \neq \emptyset$ and there exists a unique spectral measure E for $\sigma(T)$ relative to $\mathcal{H}$ such that*

$$T = \int t\,\mathrm{d}E(t).$$

Proof. Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator. For each $n \in \mathbb{N}$ we denote by $P_n = 1_{(\frac{1}{n+1},\frac{1}{n}]}(S)$, where $S = (1 + T^*T)^{-1}$. Notice that since S is a positive contraction which is injective, we have that P_n are pairwise orthogonal projections and $\sum_{n\in\mathbb{N}} P_n = 1$, where the convergence of the sum is in the strong operator topology. Note, also that if $\mathcal{H}_n = R(P_n)$ then we have $S\mathcal{H}_n = \mathcal{H}_n$ and restricting S to $\mathcal{H}_n$ we have that $\frac{1}{n+1} \leq S_{|\mathcal{H}_n} \leq \frac{1}{n}$. In particular, we have that $\mathcal{H}_n \subset R(S) = D(T^*T)$, $(1 + T^*T)$ maps $\mathcal{H}_n$ onto itself for each $n \in \mathbb{N}$, and $\sigma((1 + T^*T)_{|\mathcal{H}_n}) \subset \{\lambda \in \mathbb{C} \mid n \leq |\lambda| \leq n + 1\}$.

By Lemma 4.3.2 $R = T(1+T^*T)^{-1}$ commutes with S and since S is self-adjoint we then have that R commutes with any of the spectral projections P_n. Since we've already established that $(1+T^*T)$ give a bijection on $\mathcal{H}_n$ it then follows that $T\mathcal{H}_n \subset \mathcal{H}_n$ for all $n \in \mathbb{N}$. Note that since T is normal, by symmetry we also have that $T^*\mathcal{H}_n \subset \mathcal{H}_n$ for all $n \in \mathbb{N}$. Hence, restricting to $\mathcal{H}_n$ we have $(TP_n)^*(TP_n) = P_n(T^*T)P_n = P_n(TT^*)P_n = (TP_n)(TP_n)^*$ for all $n \in \mathbb{N}$.

Let $I = \{n \in \mathbb{N} \mid P_n \neq 0\}$, and note that $I \neq \emptyset$ since $\sum_{n\in I} P_n = 1$. For $n \in I$, restricting to $\mathcal{H}_n$, we have that TP_n is a bounded normal operator with spectrum $\sigma(TP_n) \subset \{\lambda \in \mathbb{C} \mid n-1 \leq |\lambda| \leq n\}$. Let E_n denote the unique spectral measure on $\sigma(TP_n)$ so that $T_{|\mathcal{H}_n} = \int t \, \mathrm{d}E_n(t)$.

We let E be the spectral measure on $\overline{\cup_{n\in I}\sigma(TP_n)} = \cup_{n\in I}\sigma(TP_n)$ which is given by $E(F) = \sum_{n\in I} E_n(F)$ for each Borel subset $F \subset \cup_{n\in I}\sigma(TP_n)$. Since the $E_n(F)$ are pairwise orthogonal it is easy to see that E is indeed a spectral measure. We set $\tilde{T}$ to be the operator $\tilde{T} = \int t \, \mathrm{d}E(t)$.

We claim that $\tilde{T} = T$. To see this, first note that if $\xi \in \mathcal{H}_n$ then $\tilde{T}\xi = TP_n\xi = T\xi$. Hence, $\tilde{T}$ and T agree on $\mathcal{K}_0 = \cup_{n\in I}\mathcal{H}_n$. Since both operators are closed, and since $\mathcal{K}_0$ is clearly a core for $\tilde{T}$, to see that they are equal it is then enough to show that $\mathcal{K}_0$ is also core for T. If we suppose that $\xi \in D(T^*T)$, and write $\xi_n = P_n\xi$ for $n \in I$, then $\lim_{N\to\infty} \sum_{n\leq N} \xi_n = \xi$, and setting $\eta = (1+T^*T)\xi$ we have

$$\begin{aligned} \sum_{n\in I} \|T\xi_n\|^2 &= \sum_{n\in I} \langle T^*T\xi_n, \xi_n\rangle \\ &= -\|\xi\|^2 + \sum_{n\in I} \langle (1+T^*T)\xi_n, \xi_n\rangle \\ &\leq \|\xi\|\|\eta\| < \infty. \end{aligned}$$

Since T is closed we therefore have $\lim_{N\to\infty} T(\sum_{n\leq N} \xi_n) = T\xi$. Thus, $\overline{\mathcal{G}(T_{|\mathcal{K}_0})} = \overline{\mathcal{G}(T_{|D(T^*T)})} = \mathcal{G}(T)$.

Since $\sigma(T) = \sigma(\tilde{T}) = \cup_{n\in I}\sigma(TP_n)$, this completes the existence part of the proof. For the uniqueness part, if $\tilde{E}$ is a spectral measure on $\sigma(T)$ such that $T = \int t \, \mathrm{d}\tilde{E}(t)$ then by uniqueness of the spectral measure for bounded normal operators it follows that for every $F \subset \sigma(T)$ Borel, and $n \in I$, we have $P_nE(F) = P_n\tilde{E}(F)$, and hence $E = \tilde{E}$. ∎

If $T = \int t \, \mathrm{d}E(t)$ as above, then for any $f \in B(\sigma(T))$ we define $f(T)$ to be the operator $f(T) = \int f(t) \, \mathrm{d}E(t)$.

Corollary 4.3.4. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator. Then for any $*$-polynomial $p \in \mathbb{C}[t, t^*]$ we have that $p(T)$ is densely defined and closable, and in fact $D(p(T))$ is a core for T.*

Proposition 4.3.5. *Let $T \in \mathcal{C}(\mathcal{H})$ be a normal operator, and consider the abelian von Neumann algebra $W^*(T) = \{f(T) \mid f \in B_\infty(\sigma(T))\}'' \subset \mathcal{B}(\mathcal{H})$. If $u \in \mathcal{U}(\mathcal{H})$, then $u \in \mathcal{U}(W^*(T)')$ if and only if $uTu^* = T$.*

Proof. Suppose that $u \in \mathcal{U}(\mathcal{H})$ and $T \in \mathcal{C}(\mathcal{H})$ is normal. We let E be the spectral measure on $\sigma(T)$ such that $T = \int t \, \mathrm{d}E(t)$, and consider the spectral measure $\tilde{E}$ given by $\tilde{E}(F) = uE(F)u^*$ for all $F \subset \sigma(T)$ Borel. We then clearly have $uTu^* = \int t \, \mathrm{d}\tilde{E}(t)$ from which the result follows easily. ∎

If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T : \mathcal{H} \to \mathcal{H}$ is a linear operator, then we say that T is **affiliated with** M and write $T \eta M$ if $uTu^* = T$ for all $u \in \mathcal{U}(M')$, (note that this implies $uD(T) = D(T)$ for all $u \in U(M')$). The previous proposition shows that any normal linear operator is affiliated with an abelian von Neumann algebra.

Corollary 4.3.6. *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T \in \mathcal{C}(\mathcal{H})$ is normal, then $T \eta M$ if and only if $f(T) \in M$ for all $f \in B_\infty(\sigma(T))$.*

Proposition 4.3.7. *Suppose M is a von Neumann algebra and $T, S : \mathcal{H} \to \mathcal{H}$ are linear operators such that $T, S \eta M$. Then $TS, (T + S)\eta M$. Moreover, if T is densely defined then $T^* \eta M$, and if S is closable then $\overline{S} \eta M$.*

Proof. Since $T, S \eta M$, for all $u \in \mathcal{U}(M')$ we have

$$\begin{aligned} uD(TS) &= \{\xi \in \mathcal{H} \mid u^*\xi \in D(S), S(u^*\xi) \in D(T)\} \\ &= \{\xi \in \mathcal{H} \mid \xi \in D(S), u^*S\xi \in D(T)\} \\ &= \{\xi \in \mathcal{H} \mid \xi \in D(S), S\xi \in D(T)\} = D(TS). \end{aligned}$$

Also, for $\xi \in D(TS)$ we have $u^*TSu\xi = (u^*Tu)(u^*Su)\xi = TS\xi$, hence $TS\eta M$. The proof that $(T + S)\eta M$ is similar.

If T is densely defined, then for all $u \in \mathcal{U}(M')$ we have

$$\begin{aligned} uD(T^*) &= \{\xi \in \mathcal{H} \mid \eta \mapsto \langle T\eta, u^*\xi\rangle \text{ is bounded.}\} \\ &= \{\xi \in \mathcal{H} \mid \eta \mapsto \langle T(u\eta), \xi\rangle \text{ is bounded.}\} = D(T^*), \end{aligned}$$

and for $\xi \in D(T^*)$, and $\eta \in D(T)$ we have $\langle T\eta, u^*\xi\rangle = \langle Tu, \xi\rangle$, from which it follows that $T^*u^*\eta = u^*T^*\eta$, and hence $T^*\eta M$.

If S is closable, then in particular we have that $u\overline{D(S)} = \overline{D(S)}$ for all $u \in \mathcal{U}(M')$. Hence if p denotes the orthogonal projection onto $\overline{D(S)}$ then $p \in M'' = M$, and $S\eta pMp \subset \mathcal{B}(p\mathcal{H})$. Hence, we may assume that S is densely defined in which case we have $S\eta M \implies S^*\eta M \implies \overline{S} = S^{**}\eta M$. ■

4.3.3 Polar decomposition

For $T \in \mathcal{C}(\mathcal{H})$ the **absolute value** of T is the positive operator $|T| = \sqrt{T^*T} \in \mathcal{C}(\mathcal{H})$.

Theorem 4.3.8 (Polar decomposition). *Fix $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$. Then $D(|T|) = D(T)$, and there exists a unique partial isometry $v \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\ker(v) = \ker(T) = \ker(|T|)$, and $T = v|T|$.*

Proof. By Theorem 4.3.1 we have that $D(T^*T)$ is a core for both $|T|$ and T. We define the map $V_0 : \mathcal{G}(|T|_{|D(T^*T)}) \to \mathcal{G}(T)$ by $V_0(\xi \oplus |T|\xi) = \xi \oplus T\xi$. Since, for $\xi \in D(T^*T)$ we have $\|\xi\|^2 + \||T|\xi\|^2 = \|\xi\|^2 + \|T\xi\|^2$ this shows that V_0 is isometric, and since $D(T^*T)$ is a core for both $|T|$ and T we then have that V_0 extends to an isometry from $\mathcal{G}(|T|)$ onto $\mathcal{G}(T)$, and we have $D(|T|) = P_{\mathcal{H}}(\mathcal{G}(|T|)) = P_{\mathcal{H}}(V\mathcal{G}(|T|)) = P_{\mathcal{H}}(\mathcal{G}(T)) = D(T)$.

Moreover, this also shows that the map $v_0 : R(|T|) \to R(T)$ given by $v_0(|T|\xi) = T\xi$, is well defined and extends to a partial isometry $v \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\ker(v) = R(T)^\perp = \ker(T)$. From the definition of v we clearly have that $T = v|T|$. Uniqueness follows from the fact that any other partial isometry w which satisfies $T = w|T|$ must agree with v on $\overline{R(|T|)} = \ker(|T|)^\perp = \ker(T)^\perp$. ■

Proposition 4.3.9. *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $T \in \mathcal{C}(\mathcal{H})$ has polar decomposition $T = v|T|$, then $T\eta M$ if and only if $v \in M$ and $|T|\eta M$.*

Proof. If $T\eta M$, then $T^*T\eta M$ by Proposition 4.3.7. By Corollary 4.3.6 we then have that $|T|\eta M$. Hence, for any $u \in M'$ if $\xi \in R(|T|)$ say $\xi = |T|\eta$ for $\eta \in D(|T|) = D(T)$, then $uv\xi = uv|T|\eta = uT\eta = Tu\eta = v|T|u\eta = vu\xi$, hence $v \in M'' = M$.

Conversely, if $v \in M$ and $|T|\eta M$, then $T = (v|T|)\eta M$ by Proposition 4.3.7. ■

4.4 Semigroups and infinitesimal generators.

4.4.1 Contraction semigroups

A **one-parameter contraction semigroup** consists of a family of positive contractions $\{S_t\}_{t\geq 0}$ such that $S_0 = 1$, and $S_t S_s = S_{t+s}$, $s, t \geq 0$. The semigroup is **strongly continuous** if for all $t_0 \geq 0$ we have $\lim_{t\to t_0} S_t = S_{t_0}$, in the strong operator topology.

If $A \in \mathcal{C}(\mathcal{H})$ is a normal operator such that $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$, then the operator $S_t = e^{-tA}$ is normal and satisfies $\sigma(S_t) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. Hence, $\{S_t\}_{t\geq 0}$ defines a one-parameter contraction semigroup of normal operators. Moreover, for all $\xi \in \mathcal{H}$ and $t_0 > 0$, we have $\lim_{t\to t_0} \|(S_t - S_{t_0})\xi\|^2 = \lim_{t\to t_0} \int |e^{-ta} - e^{t_0 a}|^2 \mathrm{d}E_{\xi,\xi}(a) = 0$. Thus, the semigroup $\{S_t\}_{t\geq 0}$ is strongly continuous. The operator A is the **infinitesimal generator** of the semigroup $\{S_t\}_{t\geq 0}$.

Theorem 4.4.1 (Hille-Yosida). *Let $\{S_t\}_{t\geq 0}$ be a strongly continuous one-parameter contraction semigroup of normal operators, then there exists a unique normal operator $A \in \mathcal{C}(\mathcal{H})$ satisfying $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$ such that A is the infinitesimal generator of the semigroup.*

Proof. Let $\{S_t\}_{t\geq 0}$ be a strongly continuous one-parameter semigroup of normal contractions. We define the linear operator $A : \mathcal{H} \to \mathcal{H}$ by setting $A\xi = \lim_{t\to 0} \frac{1}{t}(1 - S_t)\xi$, where the domain of A is the subspace of all vectors ξ such that this limit exists. We claim that A is the unique densely defined normal operator which generates this semigroup.

To see that A is densely defined we introduce for each $\alpha > 0$ the operator $R_\alpha = \int_0^\infty e^{-\alpha t} S_t \mathrm{d}t$, where this integral is understood as a Riemann integral in the strong operator topology. This is a well defined bounded normal operator since each S_t is normal and $t \mapsto S_t$ is strong operator continuous and uniformly bounded. If $s, t > 0$ we have

$$\begin{aligned}\frac{1}{t}(1 - S_t)R_\alpha &= \frac{1}{t}\int_0^\infty (1 - S_t)e^{-\alpha r} S_r \mathrm{d}r \\ &= \frac{1}{t}\int_0^\infty e^{-\alpha r} S_r \mathrm{d}r - \frac{1}{t}\int_t^\infty e^{-\alpha(r-t)} S_r \mathrm{d}r \\ &= \frac{1}{t}\int_0^t e^{-\alpha r} S_r \mathrm{d}r + \int_t^\infty e^{-r\alpha}\frac{1 - e^{t\alpha}}{t} S_r \mathrm{d}r. \qquad (4.2)\end{aligned}$$

If $\alpha > 0$, and $\xi \in \mathcal{H}$, then taking a limit as t tends to 0 in Equation (4.2) gives

$$\lim_{t\to\infty} \frac{1}{t}(1 - S_t)R_\alpha\xi = \xi - \alpha R_\alpha\xi.$$

Thus, we see that $R_\alpha\mathcal{H} \subset D(A)$, and we have $AR_\alpha = 1-\alpha R_\alpha$, or equivalently $(A+\alpha)R_\alpha = 1$, for all $\alpha > 0$.

If $\xi \in \mathcal{H}$, then since $\int_0^\infty \alpha e^{-\alpha t}\mathrm{d}t = 1$ for all $\alpha > 0$ it follows that

$$(\alpha R_\alpha - 1)\xi = \alpha\int_0^\infty e^{-\alpha t}(S_t - 1)\xi\mathrm{d}t. \tag{4.3}$$

If we fix $\varepsilon > 0$ and take $\delta > 0$ such that $\|(S_t - 1)\xi\| < \varepsilon$ for all $0 < t \leq \delta$, then using the triangle inequality in Equation (4.3) it follows that

$$\|(\alpha R_\alpha - 1)\xi\| \leq \alpha\int_0^\delta e^{-\alpha t}\varepsilon\mathrm{d}t + \alpha\int_\delta^\infty e^{-\alpha t}2\|\xi\| \leq \varepsilon + 2\|\xi\|e^{-\delta\alpha}.$$

As ε was arbitrary, and this holds for all $\alpha > 0$ it then follows that $\lim_{\alpha\to\infty}\alpha R_\alpha\xi = \xi$. Thus, A is densely defined since $R_\alpha\mathcal{H} \subset D(A)$, and αR_α converges strongly to 1 as α tends to ∞.

For all $\alpha, t > 0$ we have

$$\begin{aligned}\frac{1}{t}(1 - S_t)R_\alpha &= \frac{1}{t}(1 - S_t)\int_0^\infty e^{-\alpha r}S_r\mathrm{d}r\\ &= \int_0^\infty \frac{e^{-\alpha r}}{t}(S_r - S_{r+t})\mathrm{d}r\\ &= R_\alpha\frac{1}{t}(1 - S_t).\end{aligned}$$

Thus, for $\xi \in D(A)$ we have $\lim_{t\to0}\frac{1}{t}(1 - S_t)R_\alpha\xi = \lim_{t\to0}R_\alpha\frac{1}{t}(1 - S_t)\xi = R_\alpha A\xi$. Hence, we have that $R_\alpha D(A) \subset D(A)$, and $AR_\alpha = R_\alpha A$.

We then have that $A + \alpha$ and R_α commute, and since $A + \alpha$ is a left inverse for R_α it then follows that $A = R_\alpha^{-1} - \alpha$, for all $\alpha > 0$. In particular, since R_α is bounded and normal it then follows that A is a closed normal operator.

Finally, note that if $\xi \in D(A)$, then the function $r \mapsto S_r\xi$ is differentiable with derivative $-AS_r$, i.e., $-AS_r\xi = \lim_{h\to0}\frac{1}{h}(S_{r+h} - S_r)\xi$. If we consider the semigroup $\tilde{S}_r = e^{-rA}$, then for all $\xi \in D(A)$ we similarly have $\frac{d}{dr}e^{-rA} = -Ae^{-rA}$. If we now fix $t > 0$, and $\xi \in D(A)$, and we consider the function

$[0, t] \ni r \mapsto S_r e^{-(t-r)A}\xi$, then the chain rule shows that this is differentiable and has derivative

$$\frac{d}{dr}(S_r e^{-(t-r)A}\xi) = AS_r e^{-(t-r)A}\xi - S_r A e^{-(t-r)A}\xi = 0.$$

Thus, this must be a constant function and if we consider the cases $s = 0$ and $s = t$ we have $e^{-tA}\xi = S_t\xi$. Since $D(A)$ is dense and these are bounded operators it then follows that $e^{-tA} = S_t$ for all $t \geq 0$. ■

4.4.2 Stone's Theorem

A **one-parameter group of unitaries** consists of a family of unitary operators $\{u_t\}_{t\in\mathbb{R}}$ such that $u_t u_s = u_{t+s}$. The one-parameter group of unitaries is **strongly continuous** if for all $t_0 \in \mathbb{R}$ we have $\lim_{t\to t_0} u_t = u_{t_0}$ where the limit is in the strong operator topology. Note that by multiplying on the left by $u_{t_0}^*$ we see that strong continuity is equivalent to strong continuity for $t_0 = 0$. Also, since the strong and weak operator topologies agree on the space of unitaries we see that this is equivalent to $\lim_{t\to 0} u_t = u_0$ in the weak operator topology.

If $A \in \mathcal{C}(\mathcal{H})$, $A = A^*$, then we have $\sigma(A) \subset \mathbb{R}$ and hence $\sigma(e^{itA}) \subset \mathbb{T}$ for all $t \in \mathbb{R}$. Thus, $\{e^{itA}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitaries.

Theorem 4.4.2 (Stone)**.** *Let $\{u_t\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter group of unitaries, then there exists a closed densely defined self-adjoint operator $A \in \mathcal{C}(\mathcal{H})$ which is the infinitesimal generator of $\{u_t\}_{t\in\mathbb{R}}$.*

Proof. Suppose that $\{u_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitaries. From the Hille-Yosida Theorem there exists a closed densely defined normal operator $A \in \mathcal{C}(\mathcal{H})$ such that $u_t = e^{itA}$, for all $t \geq 0$. Note that for $t < 0$ we have $u_t = u_{-t}^* = (e^{-itA})^* = e^{itA^*}$. Thus, it suffices to show that A is self-adjoint. To see this, note that for $\xi \in D(A^*A) = D(AA^*)$ we have

$$iA^*\xi = \lim_{t\to 0^+} \frac{1}{-t}(1 - u_t^*)\xi = \lim_{t\to 0^+} \frac{1}{t}(1 - u_t)u_t^*\xi = \lim_{t\to 0^+} \frac{1}{t}(1 - u_t)\xi = iA\xi.$$

Since $D(A^*A)$ is a core for A^* we then have $A \sqsubseteq A^*$. By symmetry we also have $A^* \sqsubseteq (A^*)^* = A$, hence $A = A^*$. ■

4.4.3 Dirichlet forms

Let (X, μ) be a σ-finite measure space. A **Dirichlet form** on (X, μ) is a densely defined non-negative definite quadratic form q on $L^2(X, \mu)$ such that the following two conditions are satisfied:

1. If $f \in D(q)$, then $|f| \in D(q)$ and $q(|f|) \leq q(f)$.
2. If $f \in D(q)$, $f \geq 0$, then $f \vee 1 \in D(q)$, and $q(f \vee 1) \leq q(f)$.

A Dirichlet form q is closable (resp. closed) if it is closable (resp. closed) as a quadratic form. Note that if q is a closable Dirichlet form then by lower semi-continuity it follows that its closure $\overline{q}$ is again a Dirichlet form. A Dirichlet form q is **symmetric** if the domain $D(q)$ is closed under complex conjugation and we have $q(f) = q(\overline{f})$ for all $f \in D(q)$.

Dirichlet forms and Markov semigroups

Suppose (X, μ) is a σ-finite measure space, and $1 \leq p \leq \infty$. A linear operator $S : L^p(X, \mu) \to L^p(X, \mu)$ is **Markov** (or **positive**) if $Sf \geq 0$, whenever $f \geq 0$.

Lemma 4.4.3. *Suppose $S : L^\infty(X, \mu) \to L^\infty(X, \mu)$ is Markov, and suppose we also have $S(1) = 1$, and $\int S(f) \mathrm{d}\mu = \int f \mathrm{d}\mu$ for all $f \geq 0$. Then $S(L^\infty(X, \mu) \cap L^2(X, \mu)) \subset L^\infty(X, \mu) \cap L^2(X, \mu)$, and $S_{|L^\infty(X,\mu) \cap L^2(X,\mu)}$ extends to a self-adjoint Markov operator on $L^2(X, \mu)$. Every self-adjoint Markov operator on $L^2(X, \mu)$ arises in this way.*

Suppose q is a closed non-negative definite quadratic form on $L^2(X, \mu)$. We let $\Delta : L^2(X, \mu) \to L^2(X, \mu)$ be the associated positive operator so that $D(\Delta^{1/2}) = D(q)$ and $q(\xi) = \|\Delta^{1/2}\xi\|^2$, for all $\xi \in D(q)$. We also let $\{S_t\}_{t \geq 0}$ be the associated strongly continuous contraction semigroup given by $S_t = e^{-t\Delta}$.

Theorem 4.4.4. *Using the notation above, the semigroup $\{S_t\}_{t \geq 0}$ is Markov if and only if the quadratic form q is Dirichlet.*

Dirichlet forms and derivations

Suppose (X, μ) is a σ-finite measure space and consider the space $X \times X$ together with its product σ-algebra, we denote by $p_l, p_r : X \times X \to X$ the maps given by $p_l(x, y) = x$, and $p_r(x, y) = y$, for all $x, y \in X$. Suppose ν is a

σ-finite measure on $X \times X$ such that if $E \subset X$ is a Borel set which satisfies $\mu(E) = 0$ then we also have $\nu(p_l^{-1}(E)) = \nu(p_r^{-1}(E)) = 0$.

We then have two normal representations π, ρ of $L^\infty(X, \mu)$ into $L^\infty(X \times X, \nu)$ given by $(\pi(f))(x, y) = f(x)$, and $\rho(f)(x, y) = f(y)$. If $\xi \in L^2(X \times X, \nu)$ then we denote $f \cdot \xi = \pi(f)\xi$, and $\xi \cdot f = \rho(f)\xi$, and in this way we view $\mathcal{H} = L^2(X \times X, \nu)$ as a bimodule over $L^\infty(X, \mu)$.

A **derivation** δ of $L^\infty(X, \mu)$ into $L^2(X \times X, \nu)$ consists of a weakly dense unital $*$-subalgebra $A_0 = D(\delta) \subset L^\infty(X, \mu)$, together with a linear map $\delta : A_0 \to L^2(X \times X, \nu)$ which satisfies the Leibniz property $\delta(fg) = \delta(f) \cdot g + f \cdot \delta(g)$, for all $f, g \in L^\infty(X, \nu)$. The derivation is **closable** if it is closable as an unbounded operator from $L^2(X, \mu) \to L^2(X \times X, \nu)$.

Proposition 4.4.5. *Suppose $\delta : L^2(X, \mu) \to L^2(X \times X, \nu)$ is a closable derivation, then $D(\overline{\delta}) \cap L^\infty(X, \mu)$ is a $*$-subalgebra and $\overline{\delta}_{|D(\overline{\delta}) \cap L^\infty(X,\mu)}$ is again a derivation.*

Proof. ∎

Proposition 4.4.6. *Suppose $\delta : L^2(X, \mu) \to L^2(X \times X, \nu)$ is a closable derivation, then the associated closed quadratic form $q(\xi) = \|\overline{\delta}(\xi)\|^2$ is a Dirichlet form.*

Part I

Topics in abstract harmonic analysis

Chapter 5

Basic concepts in abstract harmonic analysis

5.1 Polish groups

A **topological group** is a group G which is also a Hausdorff topological space such that inversion is continuous, and multiplication is jointly continuous. Two topological groups are isomorphic if they there exists a group isomorphism between them which is also a homeomorphism of topological spaces.

Example 5.1.1. (i) Any group is a topological group when endowed with the discrete topology.

(ii) If V is a topological vector space, then V is an abelian topological group under the operation of addition. If V is a separable Fréchet space then V is also Polish.

(iii) If A is a Banach algebra then $G(A)$ is a topological group with the uniform topology. If A is a C^*-algebra then $\mathcal{U}(A) \subset G(A)$ is a closed subgroup. These groups are Polish if A is separable.

(iv) Let X be a compact metrizable space and consider $\mathrm{Homeo}(X)$ the group of homeomorphisms from X to itself. Then $\mathrm{Homeo}(X)$ is a Polish group when given the topology of uniform convergence with respect to some (and hence any) compatible metric.

If d is a compatible metric on X, then any isometry θ on X is bijective. Indeed, if $x \in X$ and $c_0 = d(x_0, \theta(X))$, then for all $n \in \mathbb{N}$ we have $\theta^n(x) \in \theta^n(X)$, and $c_0 = d(\theta^n(x_0), \theta^{n+1}(X))$. Thus, $\{\theta^n(x_0)\}_{n \in \mathbb{N}}$ forms a set such that each pair of points are at least distance c_0 apart. Since X is compact we must have $c_0 = 0$.

Thus, $\mathrm{Isom}(X, d)$ forms a subgroup of $\mathrm{Homeo}(X)$, and it is even a compact subgroup. Indeed, suppose $\{\theta_i\}_i$ is a net of isometries. Since X^X is compact there exists a cluster point $\theta : X \to X$ in the topology of pointwise convergence. Since $\{f \in X^X \mid d(f(x), f(y)) = d(x, y), x, y \in X\} \subset X^X$ is closed it follows that θ is also an isometry. Moreover, since θ_i converges pointwise to θ, we must also have that θ_i converge uniformly to θ.

(v) Let (X, d) be a complete metric space and consider $\mathrm{Isom}(X, d)$ the group of isometric automorphisms of (X, d). Then $\mathrm{Isom}(X, d)$ is a topological group when endowed with the topology of pointwise convergence (which is the same as uniform convergence on compact subsets). When X is separable we have that $\mathrm{Isom}(X, d)$ is Polish.

If X is locally compact and connected, then we have that $\mathrm{Isom}(X, d)$ is locally compact.

If $\mathcal{H}$ is a Hilbert space then $\mathcal{U}(\mathcal{H})$ is a closed subgroup of the group of isometries, and the topology on $\mathcal{U}(\mathcal{H})$ then agrees with the strong operator topology (which also agrees with the weak operator topology on this set). Moreover, if $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra then $\mathcal{U}(M)$ is a closed subgroup of $\mathcal{U}(\mathcal{H})$ in this topology.

(vi) If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra we denote by $\mathrm{Aut}(M)$ the group of all normal $*$-automorphisms[1] of M. Each normal automorphism $\alpha \in \mathrm{Aut}(M)$ defines an isometry α_* on M_*, given by $\alpha_*(\varphi)(x) = \varphi(\alpha(x))$. We endow $\mathrm{Aut}(M)$ with the topology of pointwise convergence on M_*. Then $\mathrm{Aut}(M)$ is a topological group which is Polish when M is separable. If φ is a normal faithful state on M, then we denote by $\mathrm{Aut}(M, \varphi)$ the subgroup consisting of automorphisms which preserve φ. This is a closed subgroup.

(vii) (X, μ) is a standard probability space, we denote by $\mathrm{Aut}^*(X, \mu)$ the group of Borel isomorphisms $\theta : X \to X$, such that $\alpha_* \mu \sim \mu$, where

[1] We'll see later that, in fact, $*$-automorphisms are automatically normal.

we identify two transformations if they agree almost everywhere. By Theorem 3.9.12 we have a group isomorphism between $\mathrm{Aut}^*(X,\mu)$ and $\mathrm{Aut}(L^\infty(X,\mu))$ given by $\alpha \mapsto \alpha^*$. We endow $\mathrm{Aut}^*(X,\mu)$ with the Polish topology making this a homeomorphism. Explicitly, we have $\alpha_i \to \alpha$ if for each $f \in L^1(X,\mu)$ we have

$$\left\| \frac{\mathrm{d}\alpha_{i*}\mu}{\mathrm{d}\mu} f \circ \alpha_i^{-1} - \frac{\mathrm{d}\alpha_*\mu}{\mathrm{d}\mu} f \circ \alpha^{-1} \right\|_1 \to 0.$$

To each $\alpha \in \mathrm{Aut}^*(X,\mu)$ we can associate a unitary operator $U_\alpha \in \mathcal{U}(L^2(X,\mu))$ given by $U_\alpha(f)(x) = f \circ \alpha^{-1}(x) \left(\frac{\mathrm{d}\alpha_*\mu}{\mathrm{d}\mu}(x)\right)^{1/2}$. Note that, viewing $L^\infty(X,\mu)$ as a subspace of $\mathcal{B}(L^2(X,\mu))$, for all $f \in L^\infty(X,\mu)$ we have

$$U_\alpha M_f U_\alpha^* = M_{\alpha^*(f)}.$$

We also have that the representation $U : \mathrm{Aut}^*(X,\mu) \to \mathcal{U}(L^2(X,\mu))$ is continuous. Indeed, for $a, b \geq 0$ we have $(a-b)^2 \leq |a^2 - b^2|$, hence if $\xi \in L^2(X,\mu)$ is positive then we have

$$\|U_{\alpha_i}\xi - U_\alpha \xi\|_2^2 \leq \left\| \frac{\mathrm{d}\alpha_{i*}\mu}{\mathrm{d}\mu} \xi^2 \circ \alpha_i^{-1} - \frac{\mathrm{d}\alpha_*\mu}{\mathrm{d}\mu} \xi^2 \circ \alpha^{-1} \right\|_1 \to 0,$$

hence $U_{\alpha_i} \to U_\alpha$ in the strong operator topology.

If $U_{\alpha_i} \to V$ in the strong operator topology then it follows that $V L^\infty(X,\mu) V^* = L^\infty(X,\mu)$ and hence conjugation by V implements an automorphism α of $L^\infty(X,\mu)$, and we have $V = U_\alpha$. Moreover, a similar argument as above shows that $\alpha_i \to \alpha$ in $\mathrm{Aut}^*(X,\mu)$. Hence, the representation $U : \mathrm{Aut}^*(X,\mu) \to \mathcal{U}(L^2(X,\mu))$ has closed range, and is a homeomorphism onto its image. This is the **Koopman representation** of $\mathrm{Aut}^*(X,\mu)$.

We also denote by $\mathrm{Aut}(X,\mu)$ the subgroup of $\mathrm{Aut}^*(X,\mu)$ consisting measure-preserving transformations. This is a closed subgroup, and by Theorem 3.9.12, the map $\pi \mapsto \pi^*$ gives an isomorphism between $\mathrm{Aut}(X,\mu)$ and $\mathrm{Aut}(L^\infty(X,\mu), \int \cdot \, \mathrm{d}\mu)$.

(viii) We denote by $\mathrm{Sym}(\mathbb{N})$ the group of all permutations of $\mathbb{N}$. We endow this group with the topology of pointwise convergence, i.e., $\alpha_i \to \alpha$ if for each $n \in \mathbb{N}$ we have that $\alpha_i(n) = \alpha(n)$ for all i large enough. It's not hard to see that Symm($\mathbb{N}$) is a Polish group with this topology. In fact, if we consider a probability measure on $\mathbb{N}$ whose support is all of $\mathbb{N}$, then Symm($\mathbb{N}$) can be identified with $\mathrm{Aut}^*(\mathbb{N}, \mu)$.

5.2 Locally compact groups

When the group is locally compact there exists a **Haar measure** λ, which is a non-zero Radon measure on G satisfying $\lambda(gE) = \lambda(E)$ for all $g \in G$, and Borel subsets $E \subset G$. When a Haar measure λ on G is fixed, we'll often use the notation $\int f(x)\,\mathrm{d}x$ for the integral $\int f(x)\,\mathrm{d}\lambda(x)$.

Any two Haar measures differ by a constant. If $h \in G$, and λ is a Haar measure then $E \mapsto \lambda(Eh)$ is again a Haar measure and hence there is a scalar $\Delta(h) \in (0, \infty)$ such that $\lambda(Eh) = \Delta(h)\lambda(E)$ for all Borel subsets $E \subset G$. The map $\Delta : G \to (0, \infty)$ is the **modular function**, and is easily seen to be a continuous homomorphism. The group G is **unimodular** if $\Delta(g) = 1$ for all $g \in G$, or equivalently if a Haar measure for G is right invariant.

Example 5.2.1. (i) Lebesgue measure is a Haar measure for $\mathbb{R}^n$, and similarly for $\mathbb{T}^n$.

(ii) Counting measure is a Haar measure for any discrete group.

(iii) Consider the group $GL_n(\mathbb{R})$ of invertible matrices. As the determinant of a matrix is given by a polynomial function of the coefficients of the matrix it follows that $GL_n(\mathbb{R})$ is an open dense subset of $\mathbb{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. If $x \in GL_n(\mathbb{R})$ then the transformation induced on $\mathbb{M}_n(\mathbb{R})$ by left multiplication has Jacobian given by $\det(x)^n$, and hence if λ denotes Lebesgue measure on $\mathbb{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, then we obtain a Haar measure for $GL_n(\mathbb{R})$ by

$$\mu(B) = \int_B \frac{1}{|\det(X)|^n}\,\mathrm{d}X.$$

Since the linear transformation induced by right multiplication has the same Jacobian we see that this also gives a right Haar measure, and hence $GL_n(\mathbb{R})$ is unimodular.

(iv) If $G \cong G_1 \times G_2$, and λ_i are Haar measures for G_i, then $\lambda_1 \times \lambda_2$ is a Haar measure for G. For example, consider the group $SL_n(\mathbb{R})$ of matrices with determinant 1. Then $SL_n(\mathbb{R}) \lhd GL_n(\mathbb{R})$, and we also have an embedding of $\mathbb{R}^*$ in $GL_n(\mathbb{R})$ as constant diagonal matrices. The subgroups $SL_n(\mathbb{R})$, and $\mathbb{R}^*$ are both closed and normal, and their intersection is $\{e\}$, hence we have an isomorphism $GL_n(\mathbb{R}) \cong \mathbb{R}^* \times SL_n(\mathbb{R})$. Thus, for suitably chosen Haar measures μ on $GL_n(\mathbb{R})$, and λ on $SL_n(\mathbb{R})$ we have $\mathrm{d}\mu(tx) = \frac{1}{|t|}\mathrm{d}t\, d\lambda(x)$. Note that $SL_n(\mathbb{R})$ is again unimodular.

(v) Consider the group N of upper triangular $n \times n$ matrices with real coefficients and diagonal entries equal to 1. We may identify N with $\mathbb{R}^{n(n-1)/2}$ by means of the homeomorphism $N \ni n \mapsto (n_{ij})_{1 \leq i < j \leq n} \in \mathbb{R}^{n(n-1)/2}$. Under this identification, Lebesgue measure on $\mathbb{R}^{n(n-1)/2}$ is a Haar measure on N. Indeed, if $n, x \in N$, then for $i < j$ we have

$$(nx)_{ij} = n_{ij} + x_{ij} + \sum_{i<k<j} n_{ik} x_{kj}.$$

If we endow the set of pairs (i, j), $1 \leq i < j \leq n$ with the lexicographical order, then it is clear that the Jacobi matrix corresponding to the transformation $x \mapsto nx$ is upper triangular with diagonal entries equal to 1, and hence the Jacobian of this transformation is 1.

The same argument shows that the transformation $n \mapsto nx$ also has Jacobian equal to 1 and hence Lebesgue measure is also right invariant, i.e., N is unimodular.

(vi) Suppose K, and H are locally compact groups with Haar measures $\mathrm{d}k$, and $\mathrm{d}h$ respectively. Suppose also that $\alpha : K \to \mathrm{Aut}(H)$ is a homomorphism which is continuous in the sense that the map $K \times H \ni (k, h) \mapsto \alpha_k(h) \in H$ is jointly continuous. Then for each $k \in K$, the push-forward of $\mathrm{d}h$ under the transformation α_k is again a Haar measure and hence must be of the form $\delta(k)^{-1}\mathrm{d}h$, where $\delta : K \to \mathbb{R}_{>0}$ is a continuous homomorphism.

The **semi-direct product** of K with H is denoted by $K \ltimes H$. Topologically, it is equal to the direct product $K \times H$, however the group law is given by

$$(k_1, h_1)(k_2, h_2) = (k_1 k_2, \alpha_{k_2}^{-1}(h_1) h_2); \qquad k_1, k_2 \in K, \quad h_1, h_2 \in H.$$

A Haar measure for $K \ltimes H$ is given by the product measure $\mathrm{d}k\mathrm{d}h$.

Indeed, if $f \in C_c(G \ltimes H)$, and $k' \in K$, $h' \in H$ then we have

$$\begin{aligned}\int f((k',h')(k,h))\,\mathrm{d}k\mathrm{d}h &= \int f(k'k, \alpha_k^{-1}(h')h)\,\mathrm{d}h\mathrm{d}k \\ &= \int f(k'k,h)\,\mathrm{d}k\mathrm{d}h = \int f(k,h)\,\mathrm{d}k\mathrm{d}h.\end{aligned}$$

The modular function for $K \ltimes H$ is given by $\Delta_{K\ltimes H}(k,h) = \delta(k)\Delta_K(k)\Delta_H(h)$. Indeed, if $f \in C_c(K \ltimes H)$ then

$$\begin{aligned}\int f((k,h)^{-1})\mathrm{d}k\mathrm{d}h &= \int f(k^{-1}, \alpha_k(h^{-1}))\mathrm{d}h\mathrm{d}k \\ &= \int \delta(k) f(k^{-1},h^{-1})\mathrm{d}h\mathrm{d}k \\ &= \int \delta(k)\Delta_K(k)\Delta_H(h) f(k,h)\mathrm{d}k\mathrm{d}h.\end{aligned}$$

A specific case to consider is when $G = A$ is the group of diagonal matrices in $\mathbb{M}_n(\mathbb{R})$ with positive diagonal coefficients, and $H = N$ is the group of upper triangular matrices in $\mathbb{M}_n(\mathbb{R})$ with diagonal entries equal to 1. In this case A acts on N by conjugation and the resulting semi-direct product can be realized as the group B of upper triangular matrices with positive diagonal coefficients.

Both N and A are unimodular, and for each $a = \mathrm{diag}(a_1, a_2, \ldots, a_n) \in A$ we see from the previous example that conjugating N by a multiplies the Haar measure on N by a factor of $\delta(a) = \prod_{1 \le i < j \le n} \frac{a_i}{a_j}$. Hence, the modular operator for B is given by $\Delta_B(g) = \prod_{1 \le i < j \le n} \frac{g_{ii}}{g_{jj}}$.

In the case when $n = 2$ we have the group $G = \{ \left(\begin{smallmatrix} x & y \\ 0 & x^{-1} \end{smallmatrix}\right) \mid x \in \mathbb{R}_+^*, y \in \mathbb{R}\}$, so that $G = \mathbb{R}_+^* \ltimes \mathbb{R}$, $\delta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is given by $\delta(x) = x^2$, and a left Haar measure for G is given by $\frac{1}{x}\mathrm{d}x\,\mathrm{d}y$, while a right Haar measure is given by $\frac{1}{x^3}\mathrm{d}x\,\mathrm{d}y$.

(vii) A metric space (X, d) is proper if each open ball has compact closure. Note that proper metric spaces are also locally compact and separable. For example, if $\mathcal{G}$ is a connected (undirected) graph we may define a metric on the vertices of the graph by defining the distance between two vertices as the minimal number of edges needed to get from one vertex to the other. If each vertex has only finitely many neighbors then we have that $\mathcal{G}$ with its graph metric is a proper metric space.

If (X, d) is a proper metric space then $\mathrm{Isom}(X, d)$ is locally compact. Indeed, if we fix $x_0 \in X$ then we claim that $K = \{\theta \in \mathrm{Isom}(X, d) \mid d(x_0, \theta(x_0)) \leq 1\}$ is compact. Suppose $\{\theta_n\}_n$ is a sequence in K, Since $\overline{B_1(x_0)}$ is compact there must be a subsequence $\{\theta_{n_j}\}$ such that $\theta_{n_j}(x_0)$ converges to some point $x \in \overline{B_1(x_0)}$.

If $y \in X$, we then have that $d(\theta_{n_j}(y), y) \leq 1 + 2d(x, y)$, for each j, and since balls have compact closure it then follows that there is a further subsequence such that $\theta_{n_j}(y)$ converges. By a diagonalization argument if $\{x_i\}_{i \in \mathbb{N}}$ is a countable dense subset of X, then we may find a subsequence $\{\theta_{m_j}\}$ such that $\theta_{m_j}(x_i)$ converges for each $i \in \mathbb{N}$. However, it then follows that $\theta_{m_j}(y)$ converges for each $y \in X$, and again by taking a subsequence we may assume that $\theta_{m_j}^{-1}(y)$ converges for each $y \in X$ as well. The argument in Example 5.1.1 (iv) then shows that this limit $\theta(y) = \lim_{j \to \infty} \theta_{m_j}(y)$ must be an isometry. Hence $\{\theta_m\}$ has a convergence subsequence showing that K is compact.

A subset $E \subset G$ is **locally Borel** if $E \cap B$ is Borel for all Borel sets B such that $\lambda(B) < \infty$. A subset $E \subset G$ is **locally null** if $\lambda(E \cap B) = 0$ for all Borel sets B such that $\lambda(B) < \infty$. A function $f : G \to \mathbb{C}$ is locally measurable if $f^{-1}(E)$ is locally Borel for all Borel subsets $E \subset \mathbb{C}$. A property is said to hold locally almost everywhere if it is satisfied outside a locally null set. We define $L^\infty G$ to be the space of all locally measurable functions which are bounded locally almost everywhere, where we identify two functions if they agree locally almost everywhere.

We have a canonical map from $L^\infty G$ to the dual space of $L^1 G$ given by $\langle f, \xi \rangle = \int \xi(x) f(x) \, \mathrm{d}\lambda(x)$, where $\xi \in L^\infty G$ and $f \in L^1 G$. This map gives an identification between $L^\infty G$ and $(L^1 G)^*$.

Exercise 5.2.2. Let (X, d) be a metric space, suppose $x_0, y_0 \in X$, and $\{\theta_n\}_{n \in \mathbb{N}}$ is a sequence such that $\theta_n(x_0) \to y_0$. If $r > 0$, such that $\overline{B_r(y_0)}$ is compact then show that for every $x \in B_{r/2}(x_0)$ there exists a subsequence $\{n_j\}$ such that $\theta_{n_j}(x)$ converges. Use this to prove the van Dantzig-van der Waerden theorem: If (X, d) is a second countable connected locally compact metric space[2], then $\mathrm{Isom}(X, d)$ is locally compact.

[2]Connected locally compact metric spaces are, in fact, always second countable. See Lemma 3 in Appendix 2 of [**?**]

5.3 The L^1, C^*, and von Neuman algebras of a locally compact group

Let G be a locally compact group with a fixed left Haar measure λ, and let $\Delta : G \to \mathbb{R}_{>0}$ be the modular function on G. If $f : G \to \mathbb{C}$, and $x \in G$ we denote by $L_x(f)$ (resp. $R_x(f)$) the function given by $L_x(f)(y) = f(x^{-1}y)$ (resp. $R_x(f)(y) = f(yx)$, for $y \in G$. Note that L_x defines an isometry on $L^p(G)$, for any $1 \leq p \leq \infty$.

A continuous function $f : G \to \mathbb{C}$ is **left (resp. right) uniformly continuous** if for all $\varepsilon > 0$, there exists $K \subset G$ compact such that $\|L_x(f) - f\|_\infty < \varepsilon$ (resp. $\|R_x(f) - f\|_\infty < \varepsilon$), for all $x \in K$. We let $C_b^{lu}G$ (resp. $C_b^{ru}G$) be the space of all uniformly bounded left (resp. right) uniformly continuous function on G. We set $C_b^uG = C_b^{lu}G \cap C_b^{ru}G$.

We remark that $C_b^{lu}G$, $C_b^{ru}G$, and C_b^uG are all C^*-subalgebras of C_bG. We also have $C_0G \subset C_b^uG$, so that when G is compact these algebras all coincide with C_bG.

For all $1 \leq p < \infty$, and $f \in L^p(G)$ we have $\lim_{x\to e} \|L_x(f) - f\|_p = 0$. This follows easily when $f \in C_cG$, and as C_cG is dense in $L^p(G)$ whenever $1 \leq p < \infty$, it then follows for all functions in $L^p(G)$.

If $f, g : G \to \mathbb{C}$, are measurable then the **convolution** of f with g is given by

$$(f * g)(x) = \int f(y)g(y^{-1}x)\,\mathrm{d}\lambda(y),$$

where this is defined whenever the integral converges absolutely.

If $f \in L^1G$, and $g \in L^pG$ for $1 \leq p \leq \infty$ then from Minkowski's inequality for integrals we have

$$\begin{aligned}\|f * g\|_p &= \left\| \int f(y)L_y(g)\,\mathrm{d}\lambda(y) \right\|_p \\ &\leq \int |f(y)| \|L_y(g)\|_p\,\mathrm{d}\lambda(y) = \|f\|_1 \|g\|_p\end{aligned}$$

so that $f*g \in L^pG$. In particular, if $f, g \in L^1G$, then $\|f*g\|_1 \leq \|f\|_1\|g\|_1$.

For $x \in G$, $f \in L^1G$, $g \in L^pG$ we have $\|L_x(f * g) - f * g\|_p = \|(L_x(f) - f) * g\|_p \leq \|L_x(f) - f\|_1\|g\|_p$. Thus, if $g \in L^\infty G$ then we have that $f * g$ is always left uniformly continuous, since $\lim_{x\to e} \|g - L_x(g)\|_1 = 0$. A similar argument shows that if $f \in L^1G$, and $g \in C_0G$, then $f * g \in C_0G$.

By Fubini's theorem, if $f, g, h \in L^1G$ we have

$$\begin{aligned}((f * g) * h)(x) &= \int (f * g)(y)h(y^{-1}x)\,\mathrm{d}\lambda(y)\\ &= \iint f(z)g(z^{-1}y)h(y^{-1}x)\,\mathrm{d}\lambda(y)\mathrm{d}\lambda(z)\\ &= \iint f(z)g(y)h(y^{-1}z^{-1}x)\,\mathrm{d}\lambda(y)\mathrm{d}\lambda(z) = (f * (g * h))(x).\end{aligned}$$

Thus, L^1G is a Banach algebra. Moreover, we define an involution on L^1G by

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}.$$

Note that $\|f^*\|_1 = \|f\|_1$, $f \mapsto f^*$ is conjugate linear, and if $f, g \in L^1G$ we have

$$\begin{aligned}(f * g)^*(x) &= \Delta(x^{-1})\overline{(f * g)(x^{-1})}\\ &= \int \Delta(x^{-1})\overline{f(y)g(y^{-1}x^{-1})}\,\mathrm{d}\lambda(y)\\ &= \int \Delta(y^{-1})\overline{g(y^{-1})}\Delta(x^{-1}y)\overline{f(x^{-1}y)}\,\mathrm{d}\lambda(y) = (g^* * f^*)(x).\end{aligned}$$

The resulting involutive Banach algebra is the L^1 **group algebra** of G.

If G is discrete and λ is the counting measure on G, then δ_e gives a unit for L^1G. If G is not discrete then the L^1 group algebra will not be a unital algebra. It will however admit an approximate unit. Indeed, if $f_n \in C_cG$ is any net of positive functions such that $\|f_n\|_1 = 1$, and the supports of f_n become arbitrarily small in the sense that for any neighborhood O of the identity we have that f_n is supported in O for large enough n, then for all $g \in L^1G$ we have $\lim_{n\to\infty} \|g * f_n - g\|_1 = \lim_{n\to\infty} \|f_n * g - g\|_1 = 0$. This follows easily from the fact that the maps $y \mapsto L_y(g)$, and $y \mapsto R_y(g)$ are continous for each $g \in L^1G$.

Theorem 5.3.1 (Cohen's factorization theorem)**.** *Let G be a locally compact group, then $C_b^{lu}G = L^1G * L^\infty G$, $C_0G = L^1G * C_0G$, and for $1 \leq p < \infty$ we have $L^pG = L^1G * L^pG$.*

Proof. Since $L^1 * L^pG \subset L^pG$, for $1 \leq p < \infty$, and $L^1 * L^\infty G \subset C_b^{lu}$, by the Cohen-Hewitt factorization theorem it is enough to check that for some left approximate identity $f_n \in L^1G$, we have $\|g - f_n * g\|_p \to 0$ for all $g \in L^pG$, and $\|g - f_n * g\|_\infty \to 0$ for all $g \in C_b^{lu}$, this follows easily from continuity of the action of G on L^pG, for $1 \leq p < \infty$, and on $C_b^{lu}G$. ■

We let $M(G)$ denote the space of all finite complex valued Radon measures on G, which is a Banach space with norm $\|\mu\| = |\mu|(G)$. The Riesz representation theorem allows us to identify $M(G)$ with the space of continuous linear functionals on the space of bounded continuous functions C_bG, by $\mu(f) = \int f \,\mathrm{d}\mu$. Note that the map $f \mapsto f\lambda$ is a Banach space isometry from L^1G into $M(G)$ and that L^1G when viewed in this way as a subspace of $M(G)$ is dense in the weak*-topology.

If $\mu, \nu \in M(G)$ then the **convolution** of μ and ν is the linear functional $\mu * \nu \in M(G)$ which satisfies $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$, and is given by

$$(\mu * \nu)(f) = \iint f(xy) \,\mathrm{d}\mu(x)\mathrm{d}\nu(y).$$

Note that convolution is associative and gives a Banach algebra structure to $M(G)$. Moreover, we may define an involution on $M(G)$ by

$$\mu^*(f) = \int f(x^{-1})\mathrm{d}\overline{\mu}(x).$$

This involution is clearly a conjugate linear isometry on $M(G)$, and also satisfies $(\mu * \nu)^* = \nu^* * \mu^*$. The resulting involutive Banach algebra is the **measure algebra** of G.

We remark that when we view L^1G as a subspace of $M(G)$ then the two definitions of convolution agree. Indeed, if $f, g \in L^1G$ and $h \in C_b(G)$ then

$$\begin{aligned}\iint h(xy)f(x)g(y) \,\mathrm{d}\lambda(x)\mathrm{d}\lambda(y) &= \iint h(y)f(x)g(x^{-1}y) \,\mathrm{d}\lambda(x)\mathrm{d}\lambda(y) \\ &= \int h(y)(f * g)(y) \,\mathrm{d}\lambda(y).\end{aligned}$$

Similarly, we have that the involution on $M(G)$ agrees with that on L^1G when restricted. Indeed, for $f \in L^1$, and $g \in C_b(G)$ we have

$$\int g(x^{-1})\overline{f(x)} \,\mathrm{d}\lambda(x) = \int g(x)\Delta(x^{-1})\overline{f(x^{-1})} \,\mathrm{d}\lambda(x).$$

Also, if δ_x denotes the Dirac mass at $x \in G$, then we also have $\delta_x * \delta_y = \delta_{xy}$, and $\delta_x^* = \delta_{x^{-1}}$. We remark that $M(G)$ is unital, with unit given by δ_e.

5.3.1 Unitary representations

If G is non-trivial, then L^1G will not be a C^*-algebra. There are however C^*-algebras which are closely related to the L^1 group algebra. To define these, we first note that the L^1 group algebra can encode the representation theory of the group G. Specifically, if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G which is measurable in the sense that the **matrix coefficients of the representation** $x \mapsto \langle \pi(x)\xi, \eta \rangle$ are measurable functions for all $\xi, \eta \in \mathcal{H}$, then for any $\mu \in M(G)$, the sesquilinear form defined by

$$(\xi, \eta) \mapsto \int \langle \pi(x)\xi, \eta \rangle \, \mathrm{d}\mu(x)$$

is bounded by $\|\mu\|$ and hence defines a unique bounded operator $\tilde{\pi}(\mu)$ such that

$$\langle \tilde{\pi}(\mu)\xi, \eta \rangle = \int \langle \tilde{\pi}(x)\xi, \eta \rangle \, \mathrm{d}\mu(x),$$

for all $\xi, \eta \in \mathcal{H}$.

It's then easy to see that $\mu \mapsto \tilde{\pi}(\mu)$ gives a continuous $*$-representation of $M(G)$ into $\mathcal{B}(\mathcal{H})$. Restricting to L^1G gives a continuous $*$-representation $\tilde{\pi} : L^1G \to \mathcal{B}(\mathcal{H})$. Moreover, if the representation π is continuous, i.e., if all the matrix coefficients are continuous, then it's easy to see that $\tilde{\pi}$ is **nondegenerate** in the sense that $\xi = 0$ if and only if $\pi(f)\xi = 0$ for all $f \in L^1G$. Indeed, this follows since in this case if $f_n \in C_c(G)_+$ such that $\|f_n\|_1 = 1$, and if $\mathrm{supp}(f)$ is decreasing to $\{e\}$, then $\tilde{\pi}(f_n) \to 1$ in the weak operator topology.

Conversely, suppose $\tilde{\pi} : L^1G \to \mathcal{B}(\mathcal{H})$ is a continuous $*$-representation which is nondegenerate. By Theorem 5.3.1 we have that $\pi(L^1G)\mathcal{H}$ coincides with the closed subspace of continuous vectors. Note that if $\eta \in (\pi(L^1G)\mathcal{H})^\perp$ then for all $\xi \in \mathcal{H}$ and $f \in L^1G$ we have $\langle \pi(f)\eta, \xi \rangle = \langle \eta, \pi(f^*)\xi \rangle = 0$. Since π is nondegenerate we then have $\eta = 0$. Thus, $\pi(L^1G)\mathcal{H} = \mathcal{H}$.

For $x \in G$ we may define the operator $\pi(x) : \mathcal{H} \to \mathcal{H}$ given by $\pi(x)(\tilde{\pi}(f)\xi) = \tilde{\pi}(L_x f)\xi$, for $f \in L^1G$, and $\xi \in \mathcal{H}$. We note that for $f_1, f_2 \in L^1G$, and $\xi_1, \xi_2 \in \mathcal{H}$ we have

$$\langle \tilde{\pi}(L_x f_1)\xi_1, \tilde{\pi}(L_x f_2)\xi_2 \rangle = \langle \tilde{\pi}(f_2^* * f_1)\xi_1, \xi_2 \rangle = \langle \tilde{\pi}(f_1)\xi_1, \tilde{\pi}(f_2)\xi_2 \rangle.$$

Thus, it follows that $\pi(x)$ is a well defined unitary on $\mathcal{H}$. It's then easy to see that $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation whose matrix coefficients are continuous.

Given a representation $\pi : G \to \mathcal{U}(\mathcal{H})$, to ensure that this extends to a continuous nondegenerate $*$-representation of L^1G a much weaker condition than continuity may be used. This then gives rise to an automatic continuity result.

Theorem 5.3.2 (Von Neumann). *Let G be a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ a representation such that $x \mapsto \langle \pi(x)\xi, \eta\rangle$ is measurable for all $\xi, \eta \in \mathcal{H}$. Suppose that either $\mathcal{H}$ is separable, or else the representation satisfies the condition that $\xi = 0$ if and only if $\langle \pi(x)\xi, \xi\rangle = 0$ for almost every $x \in G$. Then $x \mapsto \langle \pi(x)\xi, \eta\rangle$ is continuous for all $\xi, \eta \in \mathcal{H}$.*

Proof. We may assume that G is not discrete, otherwise there is nothing to prove. Note that if $\xi \neq 0$, and $\langle \pi(x)\xi, \xi\rangle = 0$ for almost every $x \in G$, then we also have $\langle \pi(x)\xi, \pi(y)\xi\rangle = \langle \pi(y^{-1}x)\xi, \xi\rangle = 0$ for almost every $x, y \in G$. Hence, these vectors are almost everywhere pairwise orthogonal and since G is not discrete it follows that $\mathcal{H}$ is not separable. This then reduces the case when $\mathcal{H}$ is separable to the case when $\xi = 0$ if and only if $\langle \pi(x)\xi, \xi\rangle = 0$ for almost all $x \in G$.

Suppose this is indeed the case. Then for each non-zero vector $\xi \in \mathcal{H}$ there exists $c > 0$, and a set $F \subset G$ of finite positive measure such that $|\langle \pi(x)\xi, \xi\rangle|^2 \geq c$, for all $x \in F$. Set $f(g) = 1_F(g)\overline{\langle \pi(x)\xi, \xi\rangle} \in L^1G$. Then $\langle \tilde{\pi}(f)\xi, \xi\rangle \geq c\lambda(F) > 0$. Hence the representation $\tilde{\pi} : L^1G \to \mathcal{B}(\mathcal{H})$ is nondegenerate and from the discussion before the theorem we see that the representation π must then be continuous. ∎

Having established the above correspondence between continuous unitary representations of G, and continuous nondegenerate $*$-representations of L^1G, we now consider a norm on L^1G given by $\|f\| = \sup_{\tilde{\pi}} \|\tilde{\pi}(f)\|$, where the supremum is taken over all continuous non-degenerate $*$-representations of L^1G. This clearly gives a norm which satisfies $\|f\| \leq \|f\|_1$. Moreover, this norm satisfies the C^*-identity since each of the norms $\|\tilde{\pi}(f)\|$ does. Thus, we may define the (full) **group C^*-algebra** C^*G to be the completion of L^1G with respect to this norm.

Note that by construction we have that any continuous nondegenerate $*$-representation of L^1G has a unique extension to a continuous $*$-representation of C^*G.

If we consider only the left-regular representation of G on L^2G, then we define the **reduced group C^*-algebra** C^*_rG to be the C^*-algebra generated by L^1G which acts by convolution on L^2G. Note that the identity map on

L^1G extends to a surjection from C^*G onto C^*_rG. We also define the (left) **group von Neumann algebra** LG to be the von Neumann subalgebra of $\mathcal{B}(L^2G)$ generated by the action of L^1G on L^2G given by left convolution.

If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representations. Then a closed subspace $\mathcal{K} \subset \mathcal{H}$ is **invariant** if it is invariant under $\pi(x)$ for each $x \in G$. We denote by $\pi|_{\mathcal{K}}$ the representation obtained by restricting each $\pi(x)$ to $\mathcal{K}$, and we call $\pi|_{\mathcal{K}}$ a **subrepresentation** of π. If the only invariant closed subspaces are $\{0\}$ and $\mathcal{H}$ then π is **irreducible**.

Given two continuous representations $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$, $i = 1, 2$, a bounted operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is **equivariant** (or an **intertwiner**) if $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. The representations π_1 and π_2 are **isomorphic** if there exists a unitary intertwiner, in this case we write $\pi_1 \cong \pi_2$. We say that π_1 is **contained** in π_2 if it is isomorphic to a subrepresentation of π_2.

Lemma 5.3.3 (Schur's Lemma)**.** *Let G be a locally compact group, and $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$, $i = 1, 2$, two continuous irreducible representations with $\mathcal{H}_1 \neq \{0\}$, then the space of intertwiners is either $\{0\}$ or else is one dimensional, and the latter case occurs if and only if π_1 and π_2 are isomorphic.*

Proof. If π_1 and π_2 are isomorphic then there exists a non-zero intertwiner. Conversely, if $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is an intertwiner, then $T^*T \in \mathcal{B}(\mathcal{H}_1)$ satisfies $\pi_1(x)T^*T = T^*T\pi_1(x)$, for all $x \in G$, i.e., $T^*T \in \pi_1(G)'$. We than have that any spectral projection of T^*T is contained in $\pi_1(G)'$, and hence the range of any spectral projection of T^*T is a closed subspace which is invariant under π_1. Since π_1 is irreducible it follows that the only spectral projections of T^*T are either 0, or 1, and hence $T^*T = \alpha \in \mathbb{C}$, for some $\alpha \geq 0$. Since, $T \neq 0$ we have that $\alpha \neq 0$, and hence setting $S = \frac{1}{\sqrt{\alpha}}T$ we have that $S^*S = 1$. Since π_2 is also irreducible, it follows that we must also have $SS^* = 1$, i.e., S is a unitary intertwiner which then shows that π_1 and π_2 are isomorphic.

To show that the space of intertwiners is one dimensional when π_1 and π_2 are isomorphic it suffices to consider the case when $\pi_1 = \pi_2$. We then have that the space of intertwiners coincides with the commutant $\pi_1(G)'$ which is a von Neumann algebra. Since π_1 is irreducible it follows that $\pi_1(G)'$ has no non-trivial projections which then easily implies that $\pi_1(G)' = \mathbb{C}$, which is one dimensional. ∎

Note that it follows from Schur's lemma, and the double commutant theorem, that a continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is irreducible if and only if $\mathcal{B}(\mathcal{H})$ is the von Neumann algebra generated by $\pi(G)$. We also note that it follows easily from the proof of Schur's lemma that two representations π_1 and π_2 have a non-zero intertwiner if and only if they have subrepresentations which are isomorphic.

If $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$, $i \in I$, is a family of continuous unitary representations then we denote by $\oplus_{i\in I}\pi_i : G \to \mathcal{U}(\oplus_{i\in I}\mathcal{H}_i)$ the **direct sum representation** given by $(\oplus_{i\in I}\pi_i)(x) = \oplus_{i\in I}\pi_i(x)$. If I is finite then we denote by $\otimes_{i\in I}\pi_i : G \to \mathcal{U}(\overline{\otimes}_{i\in I}\mathcal{H}_i)$ the **tensor product representation** given by $(\otimes_{i\in I}\pi_i)(x) = \otimes_{i\in I}\pi_i(x)$. It is easy to see that both the direct sum and the tensor product of representations is again continuous. If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representation then the **conjugate** representation is $\overline{\pi} : G \to \mathcal{U}(\overline{\mathcal{H}})$ given by $\overline{\pi}(g) = \overline{\pi(g)}$.

We have two distinguished representations for any locally compact group. The **trivial representation** $1_G : G \to \mathbb{T}$ given by $1_G(x) = 1$, for all $x \in G$. And the **left-regular representation** $\lambda : G \to L^2G$ given by $\lambda_x\xi = L_x(\xi)$ for $x \in G$ and $\xi \in L^2G$. Note that, up to isomorphism, the left-regular representation does not depend on the choice of Haar measure.

Lemma 5.3.4 (Fell's Absorption Principle)**.** *Let G be a locally compact group, then for any continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ we have $\lambda \otimes \pi \cong \lambda \otimes \mathrm{id}$.*

Proof. We may naturally identify the Hilbert space $L^2G\overline{\otimes}\mathcal{H}$ with $L^2(G;\mathcal{H})$ the space of square integrable functions from G to $\mathcal{H}$. Under this identification the representation $\lambda \otimes \pi$ is given by $((\lambda \otimes \pi)(x)\xi)(y) = \pi(x)\xi(x^{-1}y)$, for $\xi \in L^2(G;\mathcal{H})$, and $x, y \in G$. We define the map $U : L^2(G;\mathcal{H}) \to L^2(G;\mathcal{H})$ by $(U\xi)(x) = \pi(x)\xi(x)$, for $\xi \in L^2(G;\mathcal{H})$, and $x \in G$, then U gives a unitary with inverse U^* given by $(U^*\xi)(x) = \pi(x^{-1})\xi(x)$, for $\xi \in L^2(G;\mathcal{H})$, and $x \in G$.

If $\xi \in L^2(G;\mathcal{H})$, and $x, y \in G$, we then have

$$\begin{aligned}(U(\lambda \otimes \mathrm{id})(x)\xi)(y) &= \pi(y)\xi(x^{-1}y)\\ &= \pi(x)\pi(x^{-1}y)\xi(x^{-1}y)\\ &= ((\lambda \otimes \pi)(x)U\xi)(y).\end{aligned}$$

Thus, U gives a unitary intertwiner between $\lambda \otimes \mathrm{id}$ and $\lambda \otimes \pi$. ∎

Exercise 5.3.5. Let G be a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ a continuous unitary representation. Show that the corresponding representation $\pi : M(G) \to \mathcal{B}(\mathcal{H})$ is continuous from $M(G)$ with the weak*-topology, to $\mathcal{B}(\mathcal{H})$ with the weak operator topology. Conclude that $\pi(L^1G)'' = \pi(G)''$.

5.4 Functions of positive type

Let G be a locally compact group with a Haar measure λ. A function $\varphi \in L^\infty G$ is of **positive type** if for all $f \in L^1 G$ we have $\int \varphi(x)(f^* * f)(x)\,\mathrm{d}\lambda(x) \geq 0$.

Proposition 5.4.1. *Suppose $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous representation of G, and $\xi_0 \in \mathcal{H}$, then the function defined by $\varphi(x) = \langle \pi(x)\xi_0, \xi_0\rangle$ is of positive type.*

Proof. If we consider the associated representation of $L^1 G$, then for $f \in L^1 G$ we have

$$\begin{aligned}\int \langle \pi(x)\xi_0, \xi_0\rangle (f^* * f)(x)\,\mathrm{d}\lambda(x) &= \langle \pi(f^* * f)\xi_0, \xi_0\rangle \\ &= \|\pi(f)\xi_0\|^2 \geq 0.\end{aligned}$$

Thus, φ is of positive type. ∎

Corollary 5.4.2. *If $f \in L^2 G$, let $\tilde{f}(x) = \overline{f(x^{-1})}$. Then $f * \tilde{f}$ is of positive type.*

Proof. We have $(f * \tilde{f})(x) = \int f(y)\tilde{f}(y^{-1}x)\,\mathrm{d}y = \langle \overline{\lambda}(x)(f), f\rangle_{L^2 G}$. ∎

We also have the following converse to Proposition 5.4.1.

Theorem 5.4.3 (The GNS-construction)**.** *If $\varphi \in L^\infty G$ is of positive type there exists a continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$, and a vector $\xi_0 \in \mathcal{H}$, such that $\varphi(g) = \langle \pi(g)\xi_0, \xi_0\rangle$, for locally almost every $g \in G$. In particular, every function of positive type agrees with a continuous function locally almost everywhere.*

Proof. This result does not follow directly from the GNS-construction for C^*-algebras, where we used the C^*-algebraic structure (specifically we used that $x^* a^* a x \leq \|a\|^2 x^* x$, for all x and a) to ensure that the representation was bounded. The proof in that case does however easily adapt to this setting where we instead use the group structure to ensure that we get a unitary representation.

Thus, we begin by defining the sesquilinear form on $L^1 G$ given by $\langle f, g\rangle_\varphi = \int \varphi(x)(g^* * f)(x)\,\mathrm{d}\lambda(x)$. This sesquilinear form is non-negative definite and hence satisfies the Cauchy-Schwarz inequality, thus as in the GNS-construction for C^*-algebras we have that

$$N_\varphi = \{f \in L^1G \mid \langle f, f\rangle_\varphi = 0\} = \{f \in L^1G \mid \langle f, g\rangle_\varphi = 0, g \in L^1G\}$$

is a left ideal which closed in L^1G.

We consider $\mathcal{H}_0 = A/N_\varphi$, with the (well defined) positive definite sesquilinear form given by $\langle [f], [g]\rangle = \langle f, g\rangle_\varphi$, where $[f]$ denotes the equivalence class of f in A/N_φ. We then define $\mathcal{H}$ to be the Hilbert space completion of $\mathcal{H}_0$ with respect to this inner-product.

Given $x \in G$, we define an operator $\pi_0(x) : \mathcal{H}_0 \to \mathcal{H}_0$ by $\pi_0(x)[f] = [L_x(f)]$. Note that

$$\begin{aligned}\langle \pi_0(x)[f], \pi_0(x)[g]\rangle &= \int ((L_x(g))^* * (L_x(f)))(y)\varphi(y)\,\mathrm{d}\lambda(y)\\ &= \int (g^* * f)(y)\varphi(y)\,\mathrm{d}\lambda(y) = \langle [f], [g]\rangle.\end{aligned}$$

Hence, $\pi_0(x)$ is well defined and extends to an isometry $\pi(x)$ on $\mathcal{H}$. It is easy to see that we have $\pi(x)\pi(y) = \pi(xy)$, and $\pi(e) = 1$, hence we obtain a unitary representation of G on $\mathcal{H}$. Also note that for all $f, g \in L^1G$ we have

$$\begin{aligned}|\langle \pi(x)[f], [g]\rangle - \langle [f], [g]\rangle| &= |\int (g^* * (L_x(f) - f))(y)\varphi(y)\,\mathrm{d}\lambda(y)|\\ &\leq \|g\|_1 \|L_x(f) - f\|_1 \|\varphi\|_\infty,\end{aligned}$$

and since the action of G on L^1G is continuous it then follows that the representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is also continuous.

Suppose $\{f_i\}_i$ is an approximate identity for L^1G, then $\|[f_i]\|^2 \leq \|f_i^* * f_i\|_1 \|\varphi\|_\infty \leq \|\varphi\|_\infty$, hence $\{[f_i]\}$ describes a bounded net in $\mathcal{H}$, and we have $\langle [f], [f_i]\rangle \to \int f(x)\varphi(x)\,\mathrm{d}\lambda(x)$ for all $f \in L^1G$. It follows that $[f_i]$ converges weakly to a vector $\xi_0 \in \mathcal{H}$ such that $\langle [f], \xi_0\rangle = \int f(x)\varphi(x)\,\mathrm{d}\lambda(x)$ for all $f \in L^1G$.

For all $f \in L^1G$ we then have

$$\begin{aligned}\int f(x)\langle \pi(x)\xi_0, \xi_0\rangle\,\mathrm{d}\lambda(x) &= \lim_{i\to\infty} \int f(x)\langle [L_x(f_i)], \xi_0\rangle\,\mathrm{d}\lambda(x)\\ &= \lim_{i\to\infty} \int\int f(x) f_i(x^{-1}y)\varphi(y)\,\mathrm{d}\lambda(y)\mathrm{d}\lambda(x)\\ &= \lim_{i\to\infty} \int (f * f_i)(y)\varphi(y)\,\mathrm{d}\lambda(y) = \int f(x)\varphi(x)\,\mathrm{d}\lambda(x).\end{aligned}$$

Thus, $\varphi(y) = \langle \pi(x)\xi_0, \xi_0 \rangle$ locally almost everywhere. ■

Corollary 5.4.4. *If $\varphi \in C_bG$ is of positive type then φ is uniformly continuous, and the following statements hold:*

- $\|\varphi\|_\infty = f(e)$*;*
- $\varphi(x^{-1}) = \overline{\varphi(x)}$*, for all $x \in G$;*
- $|\varphi(y^{-1}x) - \varphi(x)|^2 \leq 2\varphi(e)\mathrm{Re}(\varphi(e) - \varphi(y))$*, for all $x, y \in G$.*

Proof. Suppose $\varphi(x) = \langle \pi(x)\xi_0, \xi_0 \rangle$ as in the GNS-construction. For the first statement we have $\varphi(e) \leq \|\varphi\|_\infty = \sup_{x \in G} |\langle \pi(x)\xi_0, \xi_0 \rangle| \leq \|\xi_0\|^2 = \varphi(e)$.

For the second statement we have $\varphi(x^{-1}) = \langle \pi(x^{-1})\xi_0, \xi_0 \rangle = \langle \xi_0, \pi(x)\xi_0 \rangle = \overline{\varphi(x)}$.

For the third statement we have

$$\begin{aligned}
|\varphi(y^{-1}x) - \varphi(x)|^2 &= |\langle \pi(x)\xi_0, \pi(y)\xi_0 - \xi_0 \rangle|^2 \leq \|\xi_0\|^2 \|\pi(y)\xi_0 - \xi_0\|^2 \\
&= 2\|\xi_0\|^2(\|\xi_0\|^2 - \mathrm{Re}(\langle \pi(y)\xi_0, \xi_0 \rangle)) \\
&= 2\varphi(e)\mathrm{Re}(\varphi(e) - \varphi(y)).
\end{aligned}$$

Since φ is continuous at e, this also shows that φ is left uniformly continuous, and right uniform continuity then follows since $\varphi(x^{-1}) = \overline{\varphi(x)}$, for all $x \in G$. ■

We let $\mathcal{P}(G) \subset L^\infty G$ denote the convex cone of positive type functions, $\mathcal{P}_1(G) = \{\varphi \in \mathcal{P}(G) \mid \varphi(e) = 1\}$, and $\mathcal{P}_{\leq 1}(G) = \{\varphi \in \mathcal{P}(G) \mid \varphi(e) \leq 1\}$. We let $\mathcal{E}(\mathcal{P}_1(G))$, and $\mathcal{E}(\mathcal{P}_{\leq 1}(G))$ denote, respectively, the extreme points of these last two convex sets.

Proposition 5.4.5. *Suppose $\varphi \in \mathcal{P}_{\leq 1}(G)$, then the corresponding linear functional on L^1G defined by $\tilde{\varphi}(f) = \int \varphi(x)f(x)\,\mathrm{d}x$ extends to a positive linear functional on C^*G. Moreover, this association gives an affine homeomorphism between $\mathcal{P}_{\leq 1}(G)$ and the set of positive linear functionals on C^*G with norm at most one, where both spaces are endowed with their weak*-topology.*

Proof. Suppose $\varphi \in \mathcal{P}_{\leq 1}(G)$, then by the GNS-construction there exists a continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, and a vector $\xi_0 \in \mathcal{H}$ such that $\varphi(x) = \langle \pi(x)\xi_0, \xi_0 \rangle$ locally almost everywhere. Since the representation is continuous we may define the corresponding representation of L^1G given by $\tilde{\pi}(f) = \int f(x)\pi(x)\,\mathrm{d}x$, and we then have $\tilde{\varphi}(f) = \int \varphi(x)f(x)\,\mathrm{d}x$.

By the universal property of C^*G, $\tilde{\pi}$ extends uniquely to a representation $\tilde{\pi} : C^*G \to \mathcal{B}(\mathcal{H})$, and hence we may extend $\tilde{\varphi}$ to a positive linear functional on C^*G by defining $\tilde{\varphi}(a) = \langle \tilde{\pi}(a)\xi_0, \xi_0\rangle$.

By the GNS-construction for C^*-algebras every positive linear functional on C^*G with norm at most one arrises in this way, and since L^1G is dense in C^*G it is easy to see that this association is then a homeomorphism. We leave it as an exercise to check that this is also an affine map. ■

Corollary 5.4.6. *If $\varphi \in \mathcal{P}_1(G)$, then $\varphi \in \mathcal{E}(\mathcal{P}_1(G))$ if and only if the unitary representation corresponding to φ is irreducible.*

Proof. By the previous proposition this follows from the corresponding result (Proposition 2.3.9) for states on C^*-algebras. ■

Corollary 5.4.7. *If G is an abelian locally compact group, then $\varphi \in \mathcal{E}(\mathcal{P}_1(G))$ if and only if $\varphi : G \to \mathbb{T}$ is a continuous homomorphism.*

Corollary 5.4.8. *The convex hull of $\mathcal{E}(\mathcal{P}_1(G))$ is weak*-dense in $\mathcal{P}_1(G)$.*

Proof. This similarly follows from Proposition 5.4.5, together with the corresponding result (Theorem 2.3.11) for C^*-algebras. ■

Note that since a function φ of positive type is uniformly continuous, it follows that for a left approximate identity $\{f_n\}_n \subset L^1G$ we have that $\|f_n * \varphi - \varphi\|_\infty \to 0$. The next lemma strengthens this approximation by allowing us to consider weak$*$ neighborhoods of φ.

Lemma 5.4.9. *Suppose $\varphi \in \mathcal{P}_1(G)$. Then for all $\varepsilon > 0$, there exists $f \in L^1G$, and a weak$*$ neighborhood O of φ in $\mathcal{P}_1(G)$, such that for all $\psi \in O$ we have*

$$\|f * \psi - \psi\|_\infty < \varepsilon.$$

Proof. Fix $\varphi \in \mathcal{P}_1(G)$, and $\varepsilon > 0$. Since φ is continuous there exists a compact neighborhood V of e such that $|1 - \varphi(x)| < \varepsilon^2/4$, for all $x \in V$.

We set $f = |V|^{-1}1_V \in L^1G$, and set

$$O = \{\psi \in \mathcal{P}_1(G) \mid |V|^{-1} \int_V (\psi - \varphi)(x)\,\mathrm{d}x| < \varepsilon^2/4\}.$$

If $\psi \in O$, and $x \in G$, then we have

$$\begin{aligned}|f * \psi(x) - \psi(x)| &= |V|^{-1}|\int_V \psi(y^{-1}x) - \psi(x)\,\mathrm{d}y| \\ &\leq |V|^{-1}\int_V |\psi(y^{-1}x) - \psi(x)|\,\mathrm{d}y.\end{aligned}$$

By Corollary 5.4.4 we have $|\psi(y^{-1}x) - \psi(x)| \leq \sqrt{2\mathrm{Re}(1 - \psi(y))}$, hence

$$\begin{aligned}|f * \psi(x) - \psi(x)| &\leq |V|^{-1}\int_V \sqrt{2\mathrm{Re}(1 - \psi(y))}\,\mathrm{d}y \\ &\leq |V|^{-1}\sqrt{2}(\int_V \mathrm{Re}(1 - \psi(y))\,\mathrm{d}y)^{1/2}(\int_V \mathrm{d}y)^{1/2} \\ &\leq |V|^{-1/2}\sqrt{2}|\int_V (1 - \psi(y))\,\mathrm{d}y|^{1/2}.\end{aligned}$$

Since $\psi \in O$ we then further have

$$|f * \psi(x) - \psi(x)| < \sqrt{2}(|V|^{-1}\int_V |1 - \varphi(y)|\,\mathrm{d}y + \varepsilon^2/4)^{1/2} < \varepsilon. \qquad \blacksquare$$

Theorem 5.4.10 (Raikov)**.** *The weak*-topology on $\mathcal{P}_1(G)$ agrees with the topology of uniform convergence on compact sets.*

Proof. First, suppose that $\{\varphi_i\}_i$ is a net in $\mathcal{P}_1(G)$ which converges uniformly on compact subsets to $\varphi \in \mathcal{P}_1(G)$. Fix $f \in L^1G$, and $\varepsilon > 0$. Let $Q \subset G$ be a compact subset such that $\int_{G\setminus Q} |f(x)|\,\mathrm{d}x < \varepsilon$. Then for large enough i we have that $\sup_{x\in Q} |\varphi_i(x) - \varphi(x)| < \varepsilon$. Since $\|\varphi_i\|_\infty = \|\varphi\| = 1$ we have

$$\begin{aligned}|\int (\varphi(x) - \varphi_i(x))f(x)\,\mathrm{d}x &\leq \int_{G\setminus Q} |\varphi(x) - \varphi_i(x)||f(x)|\,\mathrm{d}x + \varepsilon\|f\|_1 \\ &\leq 2\varepsilon + \varepsilon\|f\|_1.\end{aligned}$$

Since $\varepsilon > 0$, and $f \in L^1G$ were arbitrary it follows that $\varphi_i \to \varphi$ weak$*$.

Conversely, suppose now that $\varphi_i \to \varphi$ weak$*$. Note that since $\|\varphi_i\| = 1$, if $K \subset L^1G$ is any compact subset then we have that $\int(\varphi(x)-\varphi_i(x))g(x)\,\mathrm{d}x \to 0$, uniformly over all $g \in K$.

If $f \in L^1G$, then as the action of G on L^1G is continuous, for all $Q \subset G$ compact we have that $K = \{L_{x^{-1}}(f) \mid x \in Q\}$ is compact. Moreover, for all

$x \in G$ we have

$$\begin{aligned}(f * \varphi - f * \varphi_i)(x) &= \int f(xy)(\varphi(y^{-1}) - \varphi_i(y^{-1}))\, \mathrm{d}y \\ &= \int L_{x^{-1}}(f)(y)\overline{(\varphi(y) - \varphi_i(y))}\, \mathrm{d}y.\end{aligned}$$

Hence, for all $f \in L^1$ it follows that $(f * \varphi_i) \to (f * \varphi)$ uniformly on compact subsets of G. The result then follows from Lemma 5.4.9. ■

Theorem 5.4.11 (Gelfand-Raikov)**.** *If G is a locally compact group, then the irreducible unitary representations of G separate points of G. That is, if $x, y \in G$, $x \neq y$, then there is an irreducible representation π of G such that $\pi(x) \neq \pi(y)$.*

Proof. Suppose $x, y \in G$, $x, \neq y$. Take $f \in C_c G$, $f \geq 0$, such that $f(x) \neq 0$, and $y\mathrm{supp}(f) \cap \mathrm{supp}(f) = \emptyset$.

If we consider the function of positive type $\varphi = \tilde{f} * f$, then we have $\varphi(x) \neq 0$, while $\varphi(y) = 0$. By the previous theorem and Corollary 5.4.8 there then must exist $\varphi_0 \in \mathcal{E}(\mathcal{P}_1(G))$ such that $\varphi_0(x) \neq 0$, while $\varphi_0(y) = 0$. If we let π_0 be the GNS-representation corresponding to φ_0 with cyclic vector ξ_0, then this is an irreducible representation by Corollary 5.4.6, and we have $\langle(\pi_0(x) - \pi_0(y))\xi_0, \xi_0\rangle = \varphi_0(x) \neq 0$, hence $\pi_0(x) \neq \pi_0(y)$. ■

5.5 The Fourier-Stieltjes, and Fourier algebras

We let $B(G) \subset C_b^u G$ be the set of all matrix coefficients of continuous unitary representations, i.e., $\varphi \in B(G)$ if there exists a continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$, and $\xi, \eta \in \mathcal{H}$, such that $\varphi(x) = \langle \pi(x)\xi, \eta\rangle$, for $x \in G$, we let $\|\varphi\|_{B(G)}$ be the infimum of $\|\xi\|\|\eta\|$ for all possible decompositions.

If $\varphi \in B(G)$, so that $\varphi(x) = \langle \pi(x)\xi, \eta\rangle$, for $x \in G$, then there is a unique continuous linear functional on C^*G given by $\theta_\varphi(f) = \langle \pi(f)\xi, \eta\rangle$. Note that this is well defined since if we also have $\varphi(x) = \langle \pi_0(x)\xi_0, \eta_0\rangle$, for $x \in G$, then for all $f \in L^1 G$ we have $\langle \pi_0(f)\xi_0, \eta_0\rangle = \int f(x)\varphi(x)\, \mathrm{d}x = \langle \pi(f)\xi, \eta\rangle$, and hence by continuity it follows that $\langle \pi(f)\xi, \eta\rangle = \langle \pi_0(f)\xi_0, \eta_0\rangle$ for all $f \in C^*G$.

Theorem 5.5.1 (Eymard)**.** *Let G be a locally compact group, then $B(G)$ is the linear span of $\mathcal{P}(G)$, and is a involutive Banach algebra under pointwise* *multiplication with norm* $\|\cdot\|_{B(G)}$*. Moreover, the map* $\theta : B(G) \to (C^*G)^*$ *defined above is an isometric Banach space isomorphism, which maps* $\mathcal{P}(G)$ *onto the set of positive linear functionals on* C^*G*.*

Proof. That $B(G)$ is the linear span of $\mathcal{P}(G)$ follows from the polarization identity $\langle \pi(x)\xi, \eta\rangle = \frac{1}{4}\sum_{k=0}^{3}\langle \pi(x)(\xi + i^k\eta), \xi + i^k\eta\rangle$.

Clearly, we have $\|\alpha\varphi\|_{B(G)} = |\alpha|\|\varphi\|_{B(G)}$ for $\alpha \in \mathbb{C}$, and $f \in B(G)$. If $\varphi_1, \varphi_2 \in B(G)$, such that $\varphi_i(x) = \langle \pi_i(x)\xi_i, \eta_i\rangle$, $i = 1, 2$, for some continuous representations $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$, then we see that

$$\varphi_1(x) + \varphi_2(x) = \langle (\pi_1(x) \oplus \pi_2(x))(\xi_1 \oplus \xi_2), \eta_1 \oplus \eta_2\rangle, \qquad x \in G,$$

thus we have $\varphi_1 + \varphi_2 \in B(G)$ and $\|\varphi_1 + \varphi_2\|_{B(G)} \leq \|\varphi_1\|_{B(G)} + \|\varphi_2\|_{B(G)}$. Similarly, we have

$$\varphi_1(x)\varphi_2(x) = \langle (\pi_1(x) \otimes \pi_2(x))(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2\rangle, \qquad x \in G.$$

Hence $\varphi_1\varphi_2 \in B(G)$, and $\|\varphi_1\varphi_2\|_{B(G)} \leq \|\varphi_1\|_{B(G)}\|\varphi_2\|_{B(G)}$. We also have $\overline{\varphi_1(x)} = \langle \eta, \pi(x)\xi\rangle = \langle \overline{\pi}(x)\overline{\xi}, \overline{\eta}\rangle$, for $x \in G$, hence $\overline{\varphi_1} \in B(G)$. Thus, we see that $B(G)$ is a normed $*$-algebra, and so to finish the proof it is enough to show that θ is an isometric isomorphism of normed spaces.

Clearly θ is linear. Suppose that $\varphi \in B(G)$, such that $\varphi(x) = \langle \pi(x)\xi, \eta\rangle$, for $x \in G$. Then we have $\|\theta_\varphi\| = \sup_{f \in L^1G, \|f\|_1 = 1} |\langle \pi(f)\xi, \eta\rangle| \leq \|\xi\|\|\eta\|$. Taking the infimum over all such representations gives $\|\theta_\varphi\| \leq \|\varphi\|_{B(G)}$. Conversely, if $\psi \in (C^*G)^*$, then by Theorem 3.11.7 there exists a continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$, and vectors $\xi, \eta \in \mathcal{H}$ such that $\psi(a) = \langle \pi(a)\xi, \eta\rangle$, for all $a \in C^*G$, and such that $\|\psi\| = \|\xi\|\|\eta\|$. If we set $\varphi(x) = \langle \pi(x)\xi, \eta\rangle$, then $\varphi \in B(G)$, and $\theta_\varphi = \psi$. This shows that θ is onto, and we have $\|\theta_\varphi\| = \|\xi\|\|\eta\| \leq \|\varphi\|_{B(G)} \leq \|\theta_\varphi\|$, hence θ is isometric.

The fact that θ maps $\mathcal{P}(G)$ onto the set of positive linear functionals on C^*G follows easily by considering GNS-representations. ■

Corollary 5.5.2. *Let* G *be a locally compact group, and* $\varphi \in B(G)$*, then there exists a continuous unitary representation* $\pi : G \to \mathcal{U}(\mathcal{H})$*, and vectors* $\xi, \eta \in \mathcal{H}$*, such that* $\varphi(x) = \langle \pi(x)\xi, \eta\rangle$*, for* $x \in G$*, and such that* $\|\varphi\|_{B(G)} = \|\xi\|\|\eta\|$*.*

Proof. If we consider the isometric isomorphism $B(G) \cong (C^*G)^*$ defined above, the result then follows from Theorem 3.11.7. ■

The Banach algebra $B(G)$ is the **Fourier-Stieltjes algebra** of G.

We let $A(G) \subset B(G)$ be the subset of $B(G)$ consisting of all functions which are matrix coefficients of a direct sum of the left regular representation, i.e., $\varphi \in A(G)$ if there exist $\xi, \eta \in L^2(G)\overline{\otimes}\ell^2\mathbb{N}$ such that $\varphi(x) = \langle(\lambda \otimes \mathrm{id})(x)\xi, \eta\rangle$, for $x \in G$. There is then a unique normal linear functional on LG given by $\tilde{\theta}_\varphi(f) = \langle(f \otimes 1)\xi, \eta\rangle$. Note that just as in the case for C^*G above, it is not hard to see that $\tilde{\theta}_\varphi$ is well defined.

Theorem 5.5.3 (Eymard)**.** *Let G be a locally compact group, then $A(G)$ is a closed ideal in $B(G)$ which is also closed under involution. Moreover, the map $\tilde{\theta} : A(G) \to (LG)_*$ defined above is an isometric Banach space isomorphism.*

Proof. We clearly have that $A(G)$ is closed under linear combinations. Since the left regular representation λ is isomorphic to its conjugate representation (an explicit unitary intertwiner $U : \overline{L^2G} \to L^2G$ is given by $(U\overline{\xi})(x) = \overline{\xi(x)}$), we have that $A(G)$ is closed under conjugation. Moreover, if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous representation, then by Fell's absorption principle we have $(\lambda \otimes \mathrm{id}) \otimes \pi \cong (\lambda \otimes \mathrm{id})$ and hence it follows that $A(G)$ is an ideal in $B(G)$. To finish the proof it then suffices to show that the map θ defined above is an isometric isomorphism.

If $\psi \in (LG)_*$, then consider the polar decomposition $\psi = v \cdot |\psi|$. By Proposition 3.10.3 there exists a vector $\eta \in L^2G\overline{\otimes}\ell^2\mathbb{N}$ such that $|\psi|(f) = \langle(f \otimes 1)\eta, \eta\rangle$ for all $f \in LG$. Note that $\|\eta\|^2 = |\psi|(1) = \||\psi|\|$. We then have $\psi(f) = \langle(f \otimes 1)(v \otimes 1)\eta, \eta\rangle$, for all $f \in LG$, and setting $\xi = (v \otimes 1)\eta$ gives $\|\psi\| = \||\psi|\| = \|\xi\|^2 \geq \|\xi\|\|\eta\| \geq \|\psi\|$. If we define $\varphi \in A(G)$ by $\varphi(x) = \langle(\lambda \otimes \mathrm{id})(x)\xi, \eta\rangle$ then we see that $\tilde{\theta}_\varphi = \psi$, hence $\tilde{\theta}$ is a bijection.

We have a canonical homomorphism $\lambda_0 : C^*G \to C^*_rG \subset LG$, and hence given $\psi \in (LG)_*$ we may consider the corresponding linear functional $\psi \circ \lambda_0 \in (C^*G)$. It's then easy to check that $\theta^{-1}(\psi \circ \lambda_0) = \tilde{\theta}^{-1}(\psi)$, and since ψ is normal it follows from Kaplansky's density theorem that we have $\|\psi \circ \lambda_0\| = \|\psi\|$. Hence, $\|\tilde{\theta}^{-1}(\psi)\|_{B(G)} = \|\theta^{-1}(\psi \circ \lambda_0)\|_{B(G)} = \|\psi \circ \lambda_0\| = \|\psi\|$. ∎

An easy consequence of the proof of the previous theorem is the following.

Corollary 5.5.4. *Let G be a locally compact group, and $\varphi \in A(G)$. Then there exist $\xi, \eta \in L^2(G)\overline{\otimes}\ell^2\mathbb{N}$ such that $\varphi(x) = \langle(\lambda \otimes \mathrm{id})(x)\xi, \eta\rangle$, for $x \in G$, and such that $\|\xi\|\|\eta\| = \|\varphi\|_{B(G)}$.*

The Banach algebra $A(G)$ is the **Fourier algebra** of G, or the **Wiener algebra** in the case $G = \mathbb{T}$.

Proposition 5.5.5. *Let G be a locally compact group, then $A(G) \cap C_cG$ is dense in both C_0G and $A(G)$ with their respective norms. Also, for each $1 \leq p < \infty$, $A(G) \cap L^pG$ is dense in both L^pG and $A(G)$ with their respective norms.*

Proof. If $\varphi \in A(G)$ such that $\|\varphi\|_{B(G)} = 1$, then there exist $\xi, \eta \in L^2(G)\overline{\otimes}\ell^2\mathbb{N}$ such that $\varphi(x) = \langle(\lambda \otimes \mathrm{id})(x)\xi, \eta\rangle$, for $x \in G$, and such that $\|\xi\| = \|\eta\| = 1$. If $\varepsilon > 0$, then take $\{f_i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}} \subset C_cG$, such that $f_i = g_i = 0$ for all but finitely many terms and setting $\xi_0 = \sum_{i\in\mathbb{N}} f_i \otimes \delta_i$, and $\eta_0 = \sum_{i\in\mathbb{N}} g_i \otimes \delta_i$ we have $\|\xi - \xi_0\|_2, \|\eta - \eta_0\|_2 < \varepsilon$. If we set $\psi(x) = \langle(\lambda \otimes \mathrm{id})(x)\xi_0, \eta_0\rangle$, then we have $\psi \in C_cG \cap A(G)$, and using the triangle inequality we have $\|\varphi - \psi\|_{B(G)} \leq \|\xi\|\|\eta - \eta_0\| + \|\eta_0\|\|\xi - \xi_0\| < \varepsilon + (1 + \varepsilon)\varepsilon$. Since $\varepsilon > 0$ was arbitrary it then follows that $A(G) \cap C_cG$ (and hence also $A(G) \cap L^pG$ for $1 \leq p < \infty$), is dense in $A(G)$.

Note that if $f, g \in C_cG \subset L^2G$, then $f * \tilde{g} \in A(G) \cap C_cG$, and since $\tilde{g}$ is also in C_cG it then follows that $f * g \in A(G) \cap C_cG$ for all $f, g \in C_cG$. Taking f_n to be an approximate identity of compactly supported functions it then follows that $f_n * g \to g$ in C_0G, and hence the uniform closure of $A(G) \cap C_cG$ contains C_cG which is dense in C_0G. Thus, $A(G) \cap C_cG$ is dense in C_0G. It similarly follows that $A(G) \cap L^pG$ is dense in L^pG for all $1 \leq p < \infty$. ■

Proposition 5.5.6. *Let G be a locally compact group, then for $x \in G$, left (resp. right) traslation L_x (resp. R_x) gives an isometry on both $B(G)$ and $A(G)$ and the induced action of G on $B(G)$ is continuous.*

Proof. Suppose $\varphi \in B(G)$ and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representation so that for some $\xi, \eta \in \mathcal{H}$ we have $\varphi(y) = \langle\pi(y)\xi, \eta\rangle$, and such that $\|\varphi\|_{B(G)} = \|\xi\|\|\eta\|$. Then for $x \in G$ we have, $L_x(\varphi)(y) = \varphi(x^{-1}y) = \langle\pi(y)\xi, \pi(x)\eta\rangle$. Thus, $L_x(\varphi) \in B(G)$, and $\|L_x(\varphi)\|_{B(G)} \leq \|\xi\|\|\pi(x)\eta\| = \|\varphi\|_{B(G)}$. By symmetry we also have $\|\varphi\|_{B(G)} = \|L_{x^{-1}}L_x(\varphi)\|_{B(G)} \leq \|L_x(\varphi)\|_{B(G)}$. Thus, left translation induces an action of G on $B(G)$ by isometries. Note that the representation which realizes the matrix coefficient $L_x(\varphi)$ is the same as the respresentation which realizes φ, thus it follows that if $\varphi \in A(G)$, then also $L_x(\varphi) \in A(G)$. To see that this action is continuous, note that if $x_n \to e$, then $\pi(x_n)\eta \to \eta$, hence it follows that $\|L_{x_n}(\varphi) - \varphi\| \leq \|\xi\|\|\pi(x_n)\eta - \eta\| \to 0$.

A similar argument shows that right tranlsation also induces a continuous action of G on $B(G)$ which preserves the subspace $A(G)$. ■

Corollary 5.5.7. *Let G be a locally compact group, then $A(G) \cap L^pG =$*

$B(G) \cap L^pG$ for all $1 \leq p \leq 2$. In particular, we have $A(G) \cap C_c(G) = B(G) \cap C_c(G)$.

Proof. If $\varphi \in B(G)$ then φ is uniformly bounded and so if $\varphi \in L^pG$ for $1 \leq p \leq 2$, then $\varphi \in L^2G$. Thus, it is enough to consider the case when $p = 2$. Since right translation induces a continuous isometric representation of G on $B(G)$, this representation then extends to a continuous representation $R : L^1G \to \mathcal{B}(B(G))$ given by $R(f)(\varphi)(x) = \int \varphi(xy)f(y)\,\mathrm{d}y$. Moreover, if $f_n \in C_cG_+$ is an approximate identity then we have $\|\varphi - R(f_n)(\varphi)\|_{B(G)} \to 0$. If we additionally take f_n such that $f_n(y) = f_n(y^{-1})$, for $y \in G$, then since $\varphi \in L^2G$ we have $A(G) \ni \varphi * \tilde{f}_n = R(f_n)(\varphi) \to \varphi$, and hence $\varphi \in A(G)$. ■

By the previous proposition and Proposition 5.5.5 we see that $A(G)$ can alternately be described as the Banach subalgebra of $B(G)$ which is generated by $C_cG \cap B(G)$.

Lemma 5.5.8. *Let G be a locally compact group, fix $x \in G$, and suppose $\varphi \in A(G)$ such that $\varphi(x) = 0$. Then for each $\varepsilon > 0$, there exists $\psi \in A(G)$ such that ψ vanishes in a neighborhood of x, and such that $\|\varphi - \psi\|_{B(G)} < \varepsilon$.*

Proof. Since the action of G on $A(G)$ given by right translation is continuous, it follows that there exists a compact symmetric neighborhood K of e such that $\left\|\varphi - \varphi * \left(\frac{1}{\lambda(K)}1_K\right)\right\|_{B(G)} < \varepsilon/3$. Moreover, since φ is continuous, we may choose K such that we also have $|\varphi(xy)| < \varepsilon/3$, for all $y \in K$. We may then find $V \subset G$ an open set such that $K \subset V$, $\lambda(V) < 4\lambda(K)$, and such that $|\varphi(xy)| < \varepsilon/3$, for all $y \in V$.

We set $\psi = (1_{G\setminus xV}\varphi) * \left(\frac{1}{\lambda(K)}1_K\right) = (1_{G\setminus xV}\varphi) * \left(\frac{1}{\lambda(K)}\widetilde{1_K}\right) \in A(G)$. For $y \in G$ we have

$$\psi(y) = \frac{1}{\lambda(U)} \int_U \varphi(yz)1_{G\setminus xV}(yz)\,\mathrm{d}z,$$

hence, if $yK \subset xV$ then $\psi(y) = 0$, so that ψ vanishes in a neighborhood of x.

Note that

$$\begin{aligned}
\left\|(1_{xV}\varphi) * \left(\frac{1}{\lambda(K)}\widetilde{1_K}\right)\right\|_{B(G)} &\leq \|1_{xV}\varphi\|_2 \left\|\frac{1}{\lambda(K)}1_K\right\|_2 \\
&\leq \frac{\varepsilon}{3}\lambda(xV)^{1/2}\lambda(K)^{-1/2} \\
&< 2\varepsilon/3.
\end{aligned}$$

Hence,

$$\|\varphi - \psi\|_{B(G)} \leq \left\| \varphi - \varphi * \left(\frac{1}{\lambda(K)} 1_K \right) \right\|_{B(G)} + \left\| (1_{xV}\varphi) * \left(\frac{1}{\lambda(K)} \widetilde{1_K} \right) \right\|_{B(G)}$$
$$< \varepsilon/3 + 2\varepsilon/3 = \varepsilon. \qquad \blacksquare$$

Theorem 5.5.9 (Eymard)**.** *Let G be a locally compact group, and consider the map $\gamma : G \to \sigma(A(G))$ given by $\gamma_x(\varphi) = \varphi(x)$. Then γ is a homeomorphism from G onto $\sigma(A(G))$.*

Proof. Suppose $\chi : A(G) \to \mathbb{C}$ is a continuous homomorphism such that $\chi \neq \gamma_x$, for all $x \in G$. Then for each $x \in G$, there exists $\varphi_x \in A(G)$, such that $\varphi_x(x) = 0$, and $\chi(\varphi_x) = 1$. By the previous lemma we may approximate each φ_x by a function $\tilde{\varphi}_x \in A(G)$ such that we have $\chi(\tilde{\varphi}_x) \neq 0$, and which vanishes in a neighborhood of x.

If $\varphi \in C_c(G) \cap A(G)$, then since $\text{supp}(\varphi)$ is compact, there exist $x_1, \ldots, x_n \in \text{supp}(\varphi)$ such that $\prod_{i=1}^n \tilde{\varphi}_{x_i}$ vanishes on $\text{supp}(\varphi)$. We then have $\varphi \prod_{i=1}^n \tilde{\varphi}_{x_i} = 0$ so that $\chi(\varphi) \prod_{i=1}^n \chi(\tilde{\varphi}_{x_i}) = \chi(\varphi \prod_{i=1}^n \tilde{\varphi}_{x_i}) = 0$. Since $\chi(\tilde{\varphi}_{x_i}) \neq 0$, for $1 \leq i \leq n$, we then have $\chi(\varphi) = 0$. Thus, χ vanishes on $C_c(G) \cap A(G)$ which is dense in $A(G)$, and hence χ is the trivial homomorphism.

We therefore have that γ is surjective. It is then easy to see that it is a homeomorphism. $\blacksquare$

In summary, to each locally compact group G we have associated the following eight Banach algebras: C_0G, $L^\infty G$, L^1G, $M(G)$, C^*G, LG, $A(G)$, and $B(G)$. The latter four algebras are in a sense "dual" to the former four algebras. This is a notion which we'll make more precise in the next section for the case when G is abelian. Both $L^\infty G$, and LG are von Neumann algebras, and both C_0G, and C^*G are C^*-algebras. We have established canonical identifications

$$(C_0G)^* \cong M(G); \qquad (C^*G)^* \cong B(G);$$
$$(L^1G)^* \cong L^\infty G; \qquad (A(G))^* \cong LG.$$

We also have canonical $*$-homomorphisms

$$C_0G \to L^\infty(G); \qquad C^*G \to LG$$
$$L^1G \to M(G); \qquad A(G) \to B(G).$$

We remark that the $*$-homomorphism $C^*G \to LG$ is not injective in general, while the other three $*$-homomorphism are always injective and isometric.

5.6 Pontryagin duality

If G is a locally compact abelian group, a (unitary) **character** on G is a continuous homomorphism $\chi : G \to \mathbb{T}$. We denote the set of all characters by $\hat{G}$, which by Corollary 5.4.7 agrees with $\mathcal{E}(\mathcal{P}_1(G))$, which is canonically identified with $\sigma(C^*G)$, and we endow $\hat{G}$ with the weak*-topology, which by Theorem 5.4.10 is the same as the topology of uniform convergence on compact sets. The set of all characters is also a group under pointwise operations, and these operations are clearly continuous and hence $\hat{G}$ is an abelian locally compact group, which is the **Pontryagin dual group** of G. If $\chi \in \hat{G}$, and $x \in G$, then we'll also use the notation $\langle x, \chi \rangle = \chi(x)$.

We leave it to the reader to verify the following.

Example 5.6.1.

- $\hat{\mathbb{R}} \cong \mathbb{R}$ with the pairing $\langle x, \xi \rangle = e^{2\pi i x \xi}$.
- $\hat{\mathbb{Z}} \cong \mathbb{T}$ with the pairing $\langle n, \lambda \rangle = \lambda^n$.
- $\hat{\mathbb{T}} \cong \mathbb{Z}$ with pairing $\langle \lambda, n \rangle = \lambda^n$.
- $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ with the pairing $\langle j, k \rangle = e^{2\pi jki/n}$.
- If G_1, G_2, are locally compact abelian groups, then $\widehat{G_1 \times G}_2 \cong \hat{G}_1 \times \hat{G}_2$ with pairing $\langle (x, y), (\chi, \omega) \rangle = \langle x, \chi \rangle \langle y, \omega \rangle$.
- If G is a finite abelian group then $\hat{G} \cong G$.

Proposition 5.6.2. *If G is a discrete abelian group then $\hat{G}$ is compact, if G is a compact abelian group then $\hat{G}$ is discrete.*

Proof. If G is discrete then $C^*G \cong C_0\hat{G}$ has a unit, and hence $\hat{G}$ is compact.

If G is compact and $\chi \in \hat{G}$, then for $y \in G$ we have

$$\int \chi(x)\,\mathrm{d}x = \int \chi(yx)\,\mathrm{d}x = \chi(y) \int \chi(x)\,\mathrm{d}x.$$

Thus, if $\chi \neq 1$ then it follows that $\int \chi(x)\,\mathrm{d}x = 0$. This then shows that 1 is isolated in $\hat{G}$, and hence $\hat{G}$ is discrete. ∎

Proposition 5.6.3. *If G is a compact abelian group with Haar measure λ normalized so that $\lambda(G) = 1$ then $\hat{G}$ forms an orthonormal basis for $L^2\hat{G}$.*

Proof. If $\chi_1, \chi_2 \in \hat{G}$, then either $\chi_1 = \chi_2$, in which case we have $\int \chi_1\overline{\chi_2}\,\mathrm{d}\lambda = \int \mathrm{d}\lambda = 1$, or else $\chi_1\overline{\chi_2} \neq 1$ in which case, as in the previous proposition, we have $\int \chi_1\overline{\chi_2}\,\mathrm{d}\lambda = 0$. Thus, $\hat{G}$ forms an orthonormal set.

The span of $\hat{G}$ forms a self-adjoint unital subalgebra of CG which separates points by the Gelfand-Raikov theorem. Thus, by the Stone-Weierstrauss theorem we have that the span of $\hat{G}$ is dense in CG, and hence also L^2G. ■

The Gelfand transform gives an isomorphism $C^*G \cong C_0(\hat{G})$, which on the dense subspace $L^1G \subset C^*G$ is given by

$$\Gamma(f)(\chi) = \int f(x)\chi(x)\,\mathrm{d}\lambda(x).$$

We introduce the closely related **Fourier transform** $\mathcal{F} : L^1G \to C_0(\hat{G})$ given by $\mathcal{F}(f)(\chi) = \hat{f}(\chi) = \int f(x)\overline{\chi(x)}\,\mathrm{d}\lambda(x)$. Note that since $\chi \mapsto \overline{\chi}$ is a homeomorphism on $\hat{G}$ it follows that $\mathcal{F}$ also extends continuously to an isomorphism from C^*G onto $C_0(\hat{G})$.

We also extend the Fourier transform to the measure algebra $M(G)$. If $\mu \in M(G)$ then its **Fourier-Stieltjes transform** is a continuous function on $\hat{G}$ given by

$$\hat{\mu}(\chi) = \int \overline{\chi(x)}\,\mathrm{d}\mu(x), \qquad \chi \in \hat{G}.$$

If $\mu, \nu \in M(G)$ then we have

$$(\mu * \nu)\hat{}(\chi) = \iint \overline{\chi(xy)}\,\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \iint \overline{\chi(x)\chi(y)}\,\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \hat{\mu}(\chi)\hat{\nu}(\chi).$$

We also have

$$\widehat{\mu^*}(\chi) = \int \overline{\chi(x^{-1})}\,\mathrm{d}\overline{\mu}(x) = \overline{\int \overline{\chi(x)}\,\mathrm{d}\mu(x)} = \overline{\hat{\mu}(\chi)},$$

and

$$\widehat{x_*\mu}(\chi) = \int \overline{\chi(x^{-1}y)}\,\mathrm{d}\mu(y) = \chi(x^{-1})\hat{\mu}(\chi).$$

In particular, restricting to $L^1(G)$ we have

$$(L_x f)\hat{}(\chi) = \chi(x)\hat{f}(\chi); \quad (\eta f)\hat{}(\chi) = (L_\eta \hat{f})(\chi); \quad (f * g)\hat{}(\chi) = f(\chi)g(\chi), \tag{5.1}$$

for all $f, g \in L^1G$, $x \in G$, and $\eta, \chi \in \hat{G}$.

Similarly, for each $\mu \in M(\hat{G})$ we define a continuous function on G by

$$\check{\mu}(x) = \int \chi(x)\,\mathrm{d}\mu(\chi), \qquad x \in G.$$

We again have

$$(\mu * \nu)\check{} = \check{\mu}\check{\nu}; \qquad \check{\mu^*} = \overline{\check{\mu}}; \qquad (\chi_* \mu)\check{} = \chi\check{\mu}, \tag{5.2}$$

for $\mu, \nu \in M(\hat{G})$, and $\chi \in \hat{G}$.

Note that if $\mu \in M(G)$, and $\nu \in M(\hat{G})$ then we have

$$\begin{aligned}\int \check{\nu}(x)\,\mathrm{d}\mu(x) &= \iint \chi(x)\,\mathrm{d}\nu(\chi)\mathrm{d}\mu(x)\\ &= \int \chi(x)\,\mathrm{d}\mu(x)\mathrm{d}\nu(\chi)\\ &= \int \hat{\mu}(\chi^{-1})\,\mathrm{d}\nu(\chi).\end{aligned} \tag{5.3}$$

Theorem 5.6.4. *The map $M(\hat{G}) \ni \mu \mapsto \check{\mu} \in C_bG$ gives an isometric $*$-Banach algebra isomorphism from $M(\hat{G})$ onto $B(G)$. Moreover, the restriction to $M(\hat{G})_+$ maps onto $\mathcal{P}(G)$.*

Proof. If $\mu \in M(\hat{G})$, consider the polar decomposition $\mu = v|\mu|$, where $v : \hat{G} \to \mathbb{T}$ is a Borel function. We have a continuous representation $\pi : G \to \mathcal{U}(L^2(\hat{G}, |\mu|))$ given by $\pi(x)(\chi) = \chi(x)$, for $\chi \in \hat{G}$. If we consider $\xi, \eta \in L^2(\hat{G}, |\mu|)$ given by $\xi = v$, and $\eta = 1$, then we have

$$\langle \pi(x)\xi, \eta\rangle = \int \chi(x)v(x)\,\mathrm{d}|\mu|(\chi) = \int \chi(x)\,\mathrm{d}\mu(\chi) = \check{\mu}.$$

Thus, we see that $\check{\mu} \in B(G)$, and if we let $\theta : B(G) \to (C^*G)^*$ be defined as before Theorem 5.5.1 then for $f \in L^1G$ we have

$$\theta_{\check{\mu}}(f) = \langle \pi(f)\xi, \eta\rangle = \int f(x)\langle \pi(x)\xi, \eta\rangle\,\mathrm{d}x = \int\int f(x)\chi(x)\,\mathrm{d}\chi\mathrm{d}x.$$

As noted above, we have a canonical homeomorphism $\iota : \hat{G} \to \sigma(C^*G)$ such that for $\chi \in \hat{G}$, and $f \in L^1G \subset C^*G$, we have $\iota_\chi(f) = \int f(x)\chi(x)\,\mathrm{d}x$.

The Gelfand transform then gives a $*$-isomorphism $\Gamma : C^*G \to C_0(\hat{G})$, such that for $f \in L^1G$ we have $\Gamma(f)(\chi) = \iota_\chi(f) = \int f(x)\chi(x)\,\mathrm{d}x$. The dual map then gives an isometric isomorphism $\Gamma^* : (C_0(\hat{G}))^* \to (C^*G)^*$, given by $\Gamma^*(\varphi) = \varphi \circ \Gamma$.

Using the identification $(C_0(\hat{G}))^* \cong M(\hat{G})$ given by the Riesz representation theorem, and using the identification $\theta^{-1} : (C^*G)^* \to B(G)$ given in Theorem 5.5.1, we then obtain an isometric Banach space isomorphism $\theta^{-1} \circ \Gamma^* : M(\hat{G}) \to B(G)$.

If $\mu \in M(\hat{G})$, and $f \in L^1(G)$, we have

$$\Gamma^*(\mu)(f) = \int \Gamma(f)(\chi)\,\mathrm{d}\mu(\chi) = \int\int f(x)\chi(x)\,\mathrm{d}x\mathrm{d}\mu(\chi) = \theta_{\check{\mu}}(f).$$

Hence, we have $\Gamma^*(\mu) = \theta_{\check{\mu}}$, and thus the map $M(\hat{G}) \ni \mu \mapsto \check{\mu}$ agrees with the isometric Banach space isomorphism $\theta^{-1} \circ \Gamma$. From (5.2) we see that this map is also a $*$-homomorphism.

A measure $\mu \in M(\hat{G})$ is a positive measure if and only if $\Gamma^*(\mu)$ is a positive state, which by Theorem 5.5.1 is if and only if $\check{\mu} = \theta^{-1} \circ \Gamma^*(\mu) \in \mathcal{P}(G)$. ■

By considering functions of positive type on G, the previous theorem gives Bochner's Theorem, which was first proved by Hergoltz in the case $G = \mathbb{Z}$, Bachner in the case $G = \mathbb{R}$, and Weil in general.

Corollary 5.6.5 (Bochner's Theorem)**.** *Let G be a locally compact abelian group. If $\varphi \in \mathcal{P}(G)$ then there exists a unique positive measure $\mu \in M(\hat{G})$ such that $\varphi(x) = \check{\mu}(x) = \int \chi(x)\,\mathrm{d}\mu(\chi)$, for all $x \in G$.*

Lemma 5.6.6. *Let G be a locally compact abelian group, then $\mathcal{F}(L^1G)$ is dense in $C_0\hat{G}$.*

Proof. Since the Fourier transform is a $*$-homomorphism from L^1G into $C_0(G)$, the result follows from the Stone-Weierstrauss Theorem once we show that $\mathcal{F}(L^1G)$ separates points, and vanishes nowhere.

Let $\chi_1, \chi_2 \in \hat{G}$ be given, with $\chi_1 \neq \chi_2$. Then there exists $x_0 \in G$ such that $\chi_1(x_0) \neq \chi_2(x_0)$. Set $c_0 = |\chi_1(x_0) - \chi_2(x_0)| \leq 2$. Since χ_1, and χ_2 are continuous, there exists a neighborhood U of x_0 such that $|f(x) - \chi_i(x_0)| < c_0/2$, for all $x \in U$, and $i = 1, 2$. If we take $f \in C_cG_+$, such that f is supported on U, and such that $\|f\|_1 = 1$, then for $i = 1, 2$, we have

$|\hat{f}(\chi_i) - \chi_i(x_0)| \leq \int f(x)|\chi_i(x) - \chi_i(x_0)| \, \mathrm{d}x < c_0/2 \leq 1$. Hence, $\hat{f}(\chi_i) \neq 0$, for $i = 1, 2$, and

$$|\hat{f}(\chi_1) - \hat{f}(\chi_2)| \geq |\chi_1(x_0) - \chi_2(x_0)| - |\hat{f}(\chi_1) - \chi_1(x_0)| - |\hat{f}(\chi_2) - \chi_2(x_0)| > 0.$$

■

Theorem 5.6.7 (Fourier Inversion Theorem)**.** *If $f \in L^1(G) \cap A(G)$, then $\hat{f} \in L^1\hat{G}$. Moreover, there exists a unique Haar measure $\hat{\lambda}$ on $\hat{G}$ such that $(f\hat{\ })\check{\ } = f$, for all $f \in L^1(G) \cap A(G)$.*

Proof. If $f \in L^1G \cap A(G)$, then by Theorem 5.6.4 there exists a unique measure, $\mu_f \in M(\hat{G})$ such that $f = \check{\mu_f}$. If we also have $h \in L^1G$, then from Equation (5.3) we have

$$(h * f)(e) = \int h(x^{-1})f(x) \, \mathrm{d}x = \int \hat{h} \, \mathrm{d}\mu_f.$$

Hence, if $f, g \in L^1G \cap A(G)$, and $h \in L^1G$, we have

$$\begin{aligned} \int \hat{h}\hat{f} \, \mathrm{d}\mu_g &= \int \widehat{h * f} \, \mathrm{d}\mu_g = ((h * f) * g)(e) \\ &= ((h * g) * f)(e) = \int \hat{h}\hat{g} \, \mathrm{d}\mu_f. \end{aligned}$$

By Lemma 5.6.6 we have that $\mathcal{F}(L^1G)$ is dense in $C_0\hat{G}$ and hence it follows that $\hat{f}\mathrm{d}\mu_g = \hat{g}\mathrm{d}\mu_f$.

We now define a positive linear functional Υ on $C_c\hat{G}$. Since C_cG is dense in L^1G, and since $\mathcal{F}(L^1G)$ is dense in $C_0\hat{G}$, it follows that for each $\chi \in \hat{G}$ there exists $f \in C_cG$ such that $\hat{f}(\chi) \neq 0$. Thus, for any nonempty compact set $K \subset \hat{G}$, there exist $f_1, \ldots, f_n \in C_cG$ such that if we set $f = \sum_{i=1}^n f_i * \tilde{f}_i$, then $f \in L^1G \cap A(G)$, and $\hat{f} = \sum_{i=1}^n |\hat{f}_i|^2 > 0$, on K.

If $\psi \in C_c\hat{G}$, such that ψ is supported on K, then we set $\Upsilon(\psi) = \int \frac{\psi}{\hat{f}} \, \mathrm{d}\mu_f$. Since $\hat{f} \geq 0$, and μ_f is positive we have that $\Upsilon(\psi) \geq 0$, whenever $\psi \geq 0$. If we also have $g \in L^1G \cap A(G)$ such that $\hat{g}$ does not vanish on the support of ψ, then we have

$$\int \frac{\psi}{\hat{f}\hat{g}}\hat{f} \, \mathrm{d}\mu_g = \int \frac{\psi}{\hat{f}\hat{g}}\hat{g} \, \mathrm{d}\mu_f,$$

hence, Υ extends to a well defined positive linear functional on $C_c\hat{G}$. Since μ_f is non-trivial there exists some $\psi \in C_c\hat{G}$ with support in K such that

$\int \psi \, \mathrm{d}\mu_f \neq 0$. We then have $\Upsilon(\psi \hat{f}) = \int \psi \, \mathrm{d}\mu_f \neq 0$, and so Υ is a non-zero linear functional.

If $\psi \in C_c\hat{G}$, and $\chi \in \hat{G}$, take $f \in L^1G \cap A(G)$ such that $\hat{f}$ does not vanish on the supports of ψ or $L_\chi(\psi)$. If we set $g = \chi^{-1}f$, then from Equations (5.1) and (5.2) we have $\hat{g} = L_{\chi^{-1}}(\hat{f})$, and $\mu_g = {\chi^{-1}}_* \mu_f$. Hence,

$$\Upsilon(L_\chi(\psi)) = \int \frac{L_\chi(\psi)}{\hat{f}} \, \mathrm{d}\mu_f = \int \frac{\psi}{\hat{g}} \, \mathrm{d}\mu_g = \Upsilon(\psi).$$

Thus Υ is translation invariant and so for some Haar measure $\hat{\lambda}$ on $\hat{G}$ we have $\Upsilon(\psi) = \int \psi \, \mathrm{d}\hat{\lambda}$, for all $\psi \in C_c\hat{G}$.

If $f \in L^1G \cap A(G)$, then for all $\psi \in C_c\hat{G}$ we have

$$\int \psi \hat{f} \, \mathrm{d}\hat{\lambda} = \Upsilon(\psi \hat{f}) = \int \psi \, \mathrm{d}\mu_f.$$

Thus, we conclude that $\mu_f = \hat{f} \, \mathrm{d}\hat{\lambda}$, which shows that $\hat{f} \in L^1\hat{G}$, and $(f\hat{\ })\check{\ } = f$. ■

Theorem 5.6.8 (Plancherel's Theorem)**.** *Let G be a locally compact abelian group. The Fourier transform when restricted to $L^1G \cap L^2G$ is an isometry (with respect to the L^2-norms) onto a dense subspace of $L^2\hat{G}$, with its dual Haar measure. Hence, it has a unique extension to a unitary from L^2G to $L^2\hat{G}$.*

Proof. If $f \in L^1G \cap L^2G$, then by the Fourier inversion theorem we have

$$\begin{aligned} \int |f(x)|^2 \, \mathrm{d}\lambda(x) &= \int f(x)\tilde{f}(x^{-1}) \, \mathrm{d}\lambda(x) = (f * \tilde{f})(e) \\ &= ((f * \tilde{f})\hat{\ })\check{\ }(e) = \int (f * \tilde{f})\hat{\ }(\chi) \, \mathrm{d}\hat{\lambda}(\chi) \\ &= \int |\hat{f}(\chi)|^2 \, \mathrm{d}\hat{\lambda}(\chi). \end{aligned}$$

Hence, the Fourier transform is isometric. Suppose $\psi \in L^2\hat{G}$ such that $\int \hat{f}\overline{\psi} \, \mathrm{d}\hat{\lambda} = 0$ for all $f \in L^1G \cap L^2G$. Then, for each $f \in L^1G \cap L^2G$, and $x \in G$ we have that $\hat{f}\overline{\psi} \in L^1\hat{G}$, and

$$(\hat{f}\overline{\psi})\check{\ }(x) = \int \hat{f}(\chi)\overline{\psi(\chi)}\chi(x) \, \mathrm{d}\hat{\lambda}(\chi) = \int L_x\hat{(f)}\overline{\psi} \, \mathrm{d}\hat{\lambda} = 0.$$

Thus, by Theorem 5.6.4 it follows that $\widehat{f\overline{\psi}} = 0$ almost everywhere, for each $f \in L^1G \cap L^2G$. By Lemma 5.6.6 we have that $\mathcal{F}(L^1G \cap L^2G)$ is dense in $C_0(\hat{G})$, and hence we must have that $\psi = 0$ almost everywhere. Thus, $\mathcal{F}(L^1G \cap L^2G)$ is dense in L^2G. ■

The extension of the Fourier transform to a unitary from L^2G to $L^2\hat{G}$ is also denoted by $\mathcal{F}$, and for $f \in \hat{f}$ we again write $\hat{f} = \mathcal{F}(f)$.

Corollary 5.6.9 (Parseval's Formula)**.** *If G is a locally compact abelian group, then for all $f, g \in L^2G$ we have*

$$\int f\overline{g}\,\mathrm{d}\lambda = \int \hat{f}\overline{\hat{g}}\,\mathrm{d}\hat{\lambda}.$$

Theorem 5.6.10. *The map $L^1\hat{G} \ni \psi \mapsto \check{\psi} \in B(G)$ gives an isometric $*$-Banach algebra isomorphism from $L^1\hat{G}$ onto $A(G)$.*

Proof. We already know from Theorem 5.6.4, that this is an isometric $*$-Banach algebra homomorphism into $B(G)$, thus we only need to show that $A(G)$ is the range of $L^1\hat{G}$ under this homomorphism.

By the Fourier Inversion Theorem we have $\mathcal{F}(L^1G \cap A(G)) \subset L^1\hat{G}$. If $\psi_1, \psi_2 \in L^2\hat{G}$ with $\|\psi_i\|_2 \leq 1$, and if $\varepsilon > 0$, then from Plancherel's Theorem there exist $f_i \in L^1G \cap C_cG$ such that $\|\psi_i - \hat{f}_i\|_2 < \varepsilon$, hence

$$\begin{aligned}\left\|\psi_1\overline{\psi_2} - \widehat{f_1 * \tilde{f}_2}\right\|_1 &= \left\|\psi_1\overline{\psi_2} - \hat{f}_1\overline{\hat{f}_2}\right\|_1 \\ &\leq \|\psi_1\|_2 \left\|\psi_2 - \hat{f}_2\right\|_2 + \left\|\hat{f}_2\right\|_2 \left\|\psi_1 - \hat{f}_1\right\|_2 \leq 2\varepsilon + \varepsilon^2.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary it follows that $\mathcal{F}(L^1G \cap A(G))$ is dense in $L^1\hat{G}$. Since $L^1G \cap A(G)$ is dense in $A(G)$ it then follows from the Fourier Inversion Theorem, that the range of $L^1\hat{G}$ is precisely $A(G)$. ■

Theorem 5.6.11 (Pontryagin duality)**.** *Let G be a locally compact abelian group. The natural embedding $G \to \hat{\hat{G}}$ given by $x \mapsto \chi_x$ where $\chi_x(\varphi) = \varphi(x)$, gives a homeomorphism $G \cong \hat{\hat{G}}$.*

Proof. Clearly the map $x \mapsto \chi_x$ is continuous. Suppose that $x \in G$, and $\{x_i\}_i \subset G$ is a net such that $\chi_{x_i} \to \chi_x$. Then for each function $f \in C_cG$ we have $f(x) = \chi_x(f) = \lim_{i\to\infty} \chi_{x_i}(f) = f(x_i)$. By considering f to have support in an arbitrary neighborhood of x it then follows easily that $x_i \to x$. Thus, the map $x \mapsto \chi_x$ is a homeomorphism onto its range and so it is enough to show that it is surjective.

If $\gamma \in \hat{\hat{G}}$, then this induces a character on $L^1\hat{G}$ given by $\gamma(\psi) = \int \psi(\chi)\gamma(\chi)\,\mathrm{d}\chi$. By the previous theorem this then gives a character η on $A(G)$ given by $\check{\psi} \mapsto \gamma(\psi)$, for all $\psi \in L^1\hat{G}$. By Theorem 5.5.9 there then exist $x \in G$ such that $\eta(f) = f(x)$ for all $f \in A(G)$. Thus, for each $\psi \in L^1\hat{G}$ we have

$$\int \psi(\chi)\chi(x)\,\mathrm{d}\chi = \check{\psi}(x) = \gamma(\psi) = \int \psi(\chi)\gamma(\chi)\,\mathrm{d}\chi.$$

It then follows that $\gamma(\chi) = \chi(x)$, for almost every $\chi \in \hat{G}$, and as both functions are continuous we then have that $\gamma = \chi_x$. ■

Corollary 5.6.12. *If G is a locally compact abelian group, then G is discrete if and only if $\hat{G}$ is compact, and G is compact if and only if $\hat{G}$ is discrete.*

Proposition 5.6.13. *If G is a discrete abelian group, then λ is counting measure if and only if $\hat{\lambda}(\hat{G}) = 1$.*

Proof. If G is discrete and λ is counting measure then $\hat{\delta_e} = 1$, and so $\check{1} = \delta_e$. Hence, $1 = \delta_e(e) = \check{1}(e) = \int \mathrm{d}\hat{\lambda} = \hat{\lambda}(\hat{G})$.

Similarly, if G is compact and $\lambda(G) = 1$, then $\hat{1} = \delta_e$, and so $\check{\delta_e} = 1$. Hence, $\hat{\lambda}(\{e\}) = \int \delta_e(\chi)\,\mathrm{d}\hat{\lambda}(\chi) = \check{\delta_e}(e) = 1$. The result then follows by duality. ■

Proposition 5.6.14. *Under the identification $\hat{\mathbb{R}} \cong \mathbb{R}$ given by the paring $\langle x, \xi \rangle = e^{2\pi i x\xi}$, $x, \xi \in \mathbb{R}$, Lebesgue measure is self dual.*

Proof. Consider $\phi(x) = e^{-\pi x^2} \in L^1\mathbb{R}$. Then we have $\hat{\phi}(\xi) = \int e^{-\pi x^2 - 2\pi i x\xi}\,\mathrm{d}x$. Differentiating under the integral and applying integration by parts gives $\hat{\phi}'(\xi) = -2\pi\xi\hat{\phi}(\xi)$, and $\hat{\phi}(0) = \int \phi(x)\,\mathrm{d}x = 1$. Solving this differential equation shows $\hat{\phi} = \phi$ (and hence $\phi \in A(\mathbb{R})$. Since ϕ is even we also have $\phi(x) = \int e^{-\pi\xi^2 + 2\pi i x\xi}\,\mathrm{d}\xi = (\hat{\phi})\check{}(x)$. ■

Using the identification between G and $\hat{\hat{G}}$ given by Pontryagin duality, we see that for $\mu \in M(G)$, and $\chi \in \hat{G}$ we have $\check{\mu}(\chi) = \int \chi(x)\,\mathrm{d}\mu(x) = \hat{\mu}(\chi^{-1})$. The following corollary is then immediate.

Corollary 5.6.15. *The Fourier transform gives an isometric* $*$*-Banach algebra isomorphism between* $M(G)$*, and* $B(\hat{G})$*, such that* L^1G *is mapped onto* $A(\hat{G})$*.*

Theorem 5.6.16. *Let* G *be a locally compact abelian group, and consider the unitary* $\mathcal{F} : L^2G \to L^2\hat{G}$ *given by Plancherel's theorem. If* $f \in L^1G$*, then* $\mathcal{F}\lambda(f) = M_{\hat{f}}\mathcal{F}$*. Consequently, conjugation by* $\mathcal{F}$ *implements a normal* $*$*-isomorphism* $LG \cong L^\infty\hat{G}$*.*

Proof. If $f \in L^1G$, and $\xi \in L^1G \cap L^2G$, then

$$\mathcal{F}\lambda(f)\xi = \mathcal{F}(f * \xi) = \hat{f}\hat{\xi} = M_{\hat{f}}\mathcal{F}\xi.$$

Since $L^1G \cap L^2G$ is dense in L^2G we then have $\mathcal{F}\lambda(f) = M_{\hat{f}}\mathcal{F}$.

We have that $\lambda(L^1G)$ is dense in the strong operator topology in LG, and by Lemma 5.6.6 we have that $A(\hat{G})$ is dense in the strong operator topology in $L^\infty\hat{G}$. Thus, conjugation by $\mathcal{F}$ implements a normal $*$-isomorphism from LG onto $L^\infty\hat{G}$. ∎

In summary, If G is a locally compact abelian group, and $\hat{G}$ is its dual group, then through the Fourier and Fourier-Stieltjes transforms we have established the following isometric $*$-isomorphisms of involutive Banach algebras:

$$C^*G \cong C_0\hat{G}; \qquad LG \cong L^\infty\hat{G}; \qquad A(G) \cong L^1\hat{G}; \qquad B(G) \cong M(\hat{G});$$

5.6.1 Subgroups and quotients

If G is a locally compact abelian group and $H < G$ is a closed subgroup, then the **annihilator** of H is $\Lambda = \{\chi \in \hat{G} \mid \chi(x) = 1, x \in H\}$. It's easy to see that Λ is a closed subgroup of $\hat{G}$.

Lemma 5.6.17. *Using the notation above,* H *is also the annihilator of* Λ*.*

Proof. We clearly have that H is contained in the annihilator of Λ. Conversely, if $x \in G \setminus H$, then there exists a continuous character χ on G/H such that $\chi(xH) \neq 1$. If we then view χ as a continuous character on G, then we have that $\chi(x) \neq 1$, and $\chi \in \Lambda$, hence x is not in the annihilator of Λ. ∎

Theorem 5.6.18. *Using the notation above, we have $\widehat{G/H} \cong \Lambda$ with pairing $\langle xH, \chi \rangle = \chi(x)$. We also have $\hat{G}/\Lambda \cong \hat{H}$ with pairing $\langle x, \chi\Lambda \rangle = \chi(x)$, for $x \in H$, and $\chi \in \hat{G}$.*

Proof. The pairing of Λ with G/H given above induces a homomorphism $\pi : \Lambda \to \widehat{G/H}$. This homomorphism is clearly injective, and if we have a continuous character χ on G/H, then we may view this as a continuous character on G which annihilates H, thus π is also surjective. The quotient map from G to G/H is a continuous open map, and hence if $K \subset G/H$ is compact there then exists a compact subset $C \subset G$ such that $K = CH$. It then follows that if $\chi_i \in \Lambda$ such that $\chi_i \to 1$ uniformly on C, then we have that $\pi(\chi_i) \to 1$ uniformly in K. Since K was an arbitrary compact subset it then follows that π is a continuous map. It is even easier to see that π^{-1} is also continuous, hence π is a homeomorphism. ■

If G and H are locally compact abelian groups and $\pi : G \to H$ is a continuous homomorphism, then we obtain a continuous homomorphism $\hat{\pi} : \hat{H} \to \hat{G}$ given by $\langle x, \hat{\pi}(\chi) \rangle = \langle \pi(x), \chi \rangle$, for $x \in G$, and $\chi \in \hat{H}$. We leave it to the reader to check that π is injective if and only if $\hat{\pi}$ is surjective (and vice versa by duality). We also have $\pi_1 \hat{\circ}\, \pi_2 = \hat{\pi}_2 \circ \hat{\pi}_1$. In particular, we obtain an anti-isomorphism of automorphism groups $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\hat{G})$. As an example, consider $\hat{\mathbb{R}^n} \cong \mathbb{R}^n$ with pairing $\langle x, \xi \rangle = e^{2\pi i x \cdot \xi}$. Then, we have that $\mathrm{Aut}(\mathbb{R}^n)$ can be identified with $GL_n(\mathbb{R})$ acting on $\mathbb{R}^n$ by left matrix multiplication. If $A \in GL_n(\mathbb{R})$, then the dual automorphism is given by $\langle x, \hat{A}\xi \rangle = \langle Ax, \xi \rangle$, for $x, \xi \in \mathbb{R}^n$, i.e., $\hat{A}$ is nothing but left multiplication by the transpose A^t.

5.6.2 Restricted products

Let $\{G_i\}_{i \in I}$ be a family of locally compact groups, $S \subset I$ a subset such that $I \setminus S$ is finite, and for each $i \in S$, let $K_i < G_i$ be an open compact subgroup of G_i, then the **restricted product** of $\{G_i\}_{i \in I}$ with respect to $\{K_i\}_{i \in S}$ is the subgroup $\prod'_{i \in I} G_i$ of $\prod_{i \in I} G_i$ consisting of all elements $(x_i)_{i \in I}$ such that $x_i \in K_i$ for all but finitely many $i \in S$. We define a topology on $\prod'_{i \in I} G_i$ by defining a basis of open sets to be those of the form $\prod_{i \in I} A_i$ such that $A_i \subset G_i$ is open for $i \in I$, and $A_i = K_i$ for all but finitely many $i \in S$. It's then not hard to see that $\prod'_{i \in I} G_i$ is a locally compact group and $(\prod_{i \in I \setminus S} G_i) \times (\prod_{i \in S} K_i)$ is an open subgroup.

Suppose now that G_i is abelian for each $i \in I$, and let Λ_i denote the annihilator of K_i for each $i \in S$. As K_i is compact for $i \in S$, we have that $\hat{K}_i \cong \hat{G}_i/\Lambda_i$ is discrete and hence Λ_i is open for $i \in S$. Similarly, since K_i is open for $i \in S$, it also follows that Λ_i is compact for $i \in S$. Thus, we may consider the restricted product $\prod'_{i\in I} \hat{G}_i$ with respect to $\{\Lambda_i\}_{i\in S}$.

If $(x_i)_{i\in I} \in \prod'_{i\in I} G_i$, and $(\chi_i)_{i\in I} \in \prod'_{i\in I} \hat{G}_i$, then as Λ_i is the annihilator of K_i it then follows that $\langle x_i, \chi_i\rangle = 1$ for all but finitely many $i \in I$. Thus, we obtain a pairing of $\prod'_{i\in I} G_i$ and $\prod'_{i\in I} \hat{G}_i$ given by $\langle (x_i)_{i\in I}, (\chi_i)_{i\in I}\rangle = \prod_{i\in I}\langle x_i, \chi_i\rangle$. This paring then gives an isomorphism $(\prod'_{i\in I} G_i)\hat{} \cong \prod'_{i\in I} \hat{G}_i$.

Exercise 5.6.19. Suppose G is a locally compact abelian group, $K < G$ is a compact open subgroup, and $\Lambda < \hat{G}$ is the annihilator of K. If λ is the Haar measure on G which satisfies $\lambda(K) = 1$, then show that the dual Haar measrue $\hat{\lambda}$ satisfies $\hat{\lambda}(\Lambda) = 1$.

5.6.3 Stone's theorem

Just as Bochner's theorem associates functions of positive type on G to measures on $\hat{G}$, we may associate representations of G to spectral measures on $\hat{G}$.

Theorem 5.6.20. *Let G be a locally compact abelian group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a continuous unitary representation. Then there exists a unique spectral measure E on $\hat{G}$ relative to $\mathcal{H}$ such that for all $f \in L^1G$ we have $\pi(f) = \int \hat{f}\, \mathrm{d}E$. Moreover, we also have $\pi(\mu) = \int \hat{\mu}\, \mathrm{d}E$ for all $\mu \in M(G)$.*

Proof. Suppose $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representation. Then this representation is associated to a non-degenerate $*$-representation $\pi : C^*G \to \mathcal{B}(\mathcal{H})$. The Fourier transform extends to an isomorphism $\mathcal{F} : C^*G \to C_0\hat{G}$ and so we obtain a non-degenerate representation $\pi\circ\mathcal{F}^{-1} : C_0\hat{G} \to \mathcal{B}(\mathcal{H})$ By the spectral theorem there then exists a unique spectral measure E on $\sigma(\hat{G})$ such that for $\psi \in C_0(\hat{G})$ we have $\pi \circ \mathcal{F}^{-1}(\psi) = \int \psi\, \mathrm{d}E$. In particular, for $f \in L^1G$ we have $\pi(f) = \int \hat{f}\, \mathrm{d}E$. Since L^1G is dense in C^*G this shows that E must also be unique.

If $f_i \in L^1G$, such that $f_i \to \mu \in M(G)$ weak*, then its not hard to check that $\int \hat{f}_i\, \mathrm{d}E \to \int \hat{\mu}\, \mathrm{d}E$ in the weak operator topology. Thus, we also have $\pi(\mu) = \int \hat{\mu}\, \mathrm{d}E$ for all $\mu \in M(G)$. ■

The previous theorem gives us an alternative proof of Stone's Theorem which avoids some of the technicalities which arise in dealing with unbounded operators.

Corollary 5.6.21 (Stone)**.** *There is a bijective correspondence between self-adjoint operators $A \in \mathcal{C}(\mathcal{H})$ and continuous unitary representations $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ given by $U_t = \exp(itA)$.*

Proof. If $A \in \mathcal{C}(\mathcal{H})$ is self-adjoint then from the spectral theorem we have that $t \mapsto \exp(itA)$ defines a continuous unitary representation. Moreover, if $B \in \mathcal{C}(\mathcal{H})$ is also self-adjoint such that $\exp(itA) = \exp(itB)$ for all $t \in \mathbb{R}$, then differentiating with respect to t and evaluating at 0 gives $A = B$.

Conversely, if $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representation, then by the previous theorem there exists a spectral measure E on $\hat{\mathbb{R}} \cong \mathbb{R}$ such that $U(\mu) = \int \hat{\mu}\, \mathrm{d}E$ for all $\mu \in M(\mathbb{R})$. If we let $A = \int s\, \mathrm{d}E(s)$, then A is a self-adjoint operator in $\mathcal{C}(\mathcal{H})$, and for all $\mu \in M(\mathbb{R})$ we have $\hat{\mu}(A) = \int \hat{\mu}\, \mathrm{d}E = U(\mu)$. If we take $\mu = \delta_t$, the Dirac measure at t, then we have $e^{itA} = U_t$, for all $t \in \mathbb{R}$. ∎

5.7 The Peter-Weyl Theorem

Recall that if G is a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$, $\rho : G \to \mathcal{U}(\mathcal{K})$ are representations, then we may naturally identify the Hilbert space $\mathcal{H}\overline{\otimes}\overline{\mathcal{K}}$ with the space of Hilbert-Schmidt operators from $\mathcal{K}$ to $\mathcal{H}$ as in Section 3.2. Under this identification we have that the representation $\pi \otimes \overline{\pi}$ acts on $\mathrm{HS}(\mathcal{K}, \mathcal{H})$ as $(\pi \otimes \overline{\rho})(x)\Xi = \pi(x)\Xi\rho(x^{-1})$.

Lemma 5.7.1. *Let G be a compact group with Haar measure λ normalized so that $\lambda(G) = 1$. Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a continuous unitary representation. Then $\int \pi(x)\, \mathrm{d}x$ is the orthogonal projection onto the space of G-invariant vectors $\mathcal{H}^G$.*

Proof. Set $P = \int \pi(x)\, \mathrm{d}x$. If $y \in G$, then $\pi(y)P = \int \pi(yx)\, \mathrm{d}x = P$. Hence, we see that the range of P is contained in $\mathcal{H}^G$. This also shows that $P^2 = \int \pi(y) \left(\int \pi(x)\, \mathrm{d}x \right) \mathrm{d}y = P$. Since, G is compact, it is unimodular, hence $P^* = \int \pi(x^{-1})\, \mathrm{d}x = P$. Thus, P is a projection onto a closed subspace of $\mathcal{H}^G$. If $\xi \in \mathcal{H}^G$, then $P\xi = \int \pi(x)\xi\, \mathrm{d}x = \xi$, hence P is the projection onto $\mathcal{H}^G$. ∎

Corollary 5.7.2. *Let G be a compact group withHaar measure λ normalized so that $\lambda(G) = 1$. Suppose $\pi : G \to \mathcal{U}(\mathcal{H})$ and $\rho : G \to \mathcal{U}(\mathcal{K})$ are finite dimensional irreducible representations, such that $\pi \ncong \rho$. Then, $\int (\pi \otimes \overline{\rho})(x)\,\mathrm{d}x = 0$, and for each $\Xi \in \mathrm{HS}(\mathcal{H})$ we have $\int \pi(x)\Xi\pi(x^{-1})\,\mathrm{d}x = \frac{1}{\dim(\mathcal{H})}\mathrm{Tr}(\Xi)$.*

Proof. From the previous lemma we have that $\int (\pi \otimes \overline{\rho})(x)\,\mathrm{d}x$ is the projection onto $(\mathcal{H}\overline{\otimes}\overline{\mathcal{K}})^G = (\mathrm{HS}(\mathcal{K}, \mathcal{H}))^G$. If $\pi \ncong \rho$ then by Schur's lemma this space is trivial and hence $\int (\pi \otimes \overline{\rho})(x)\,\mathrm{d}x = 0$.

For the case when $\pi = \rho$ then we again have by Schur's lemma that $\int (\pi \otimes \overline{\pi})(x)\,\mathrm{d}x$ is the projection onto $\frac{1}{\dim(\mathcal{H})^{1/2}} I \in \mathrm{HS}(\mathcal{H})$. Thus, if $\Xi \in \mathrm{HS}(\mathcal{H})$ we have

$$\int \pi(x)\Xi\pi(x^{-1})\,\mathrm{d}x = \frac{1}{\dim(\mathcal{H})^{1/2}}\langle \Xi, I\rangle_2 = \frac{1}{\dim(\mathcal{H})^{1/2}}\mathrm{Tr}(\Xi).$$

∎

Proposition 5.7.3. *Let G be a compact group, then every continuous unitary representation decomposes as a direct sum of finite dimensional irreducible representations.*

Proof. Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a continuous representation and let $\{\mathcal{H}_\alpha\}_{\alpha \in I}$ be a maximal family of pairwise orthogonal finite dimensional irreducible subrepresentations. We set $\mathcal{K} = \mathcal{H} \ominus (\oplus_{\alpha \in I}\mathcal{H}_\alpha)$. Then $\mathcal{K}$ is invariant under the action of G and $\mathcal{K}$ has no finite dimensional irreducible subrepresentation.

By a dimension argument we see that any finite dimensional representation of G has non-trivial irreducible subrepresentations and hence it follows that $\mathcal{K}$ has no finite dimensional subrepresentation other than $\{0\}$. Thus, $\mathcal{B}(\mathcal{K})$ has no finite rank intertwiner, and hence by the spectral theorem $\mathcal{B}(\mathcal{K})$ has no Hilbert-Schmidt intertwiner, i.e., $\pi|_{\mathcal{K}} \otimes \overline{\pi}|_{\overline{\mathcal{K}}}$ has no invariant vectors.

If we set $P = \int \pi|_{\mathcal{K}}(x) \otimes \overline{\pi}|_{\overline{\mathcal{K}}}(x)\,\mathrm{d}x$, then by Lemma 5.7.1, P is the projection onto the space of invariant vectors in $\mathcal{K}\overline{\otimes}\overline{\mathcal{K}}$, and so $P = 0$. If $\xi \in \mathcal{K}$ we then have

$$\begin{aligned} 0 &= \left\langle \int (\pi(x) \otimes \overline{\pi}(x))\xi \otimes \overline{\xi}, \xi \otimes \overline{\xi} \right\rangle \mathrm{d}x \\ &= \int |\langle \pi(x)\xi, \xi\rangle|^2\,\mathrm{d}x, \end{aligned}$$

hence $\xi = 0$. Therefore $\mathcal{K} = \{0\}$ and $\mathcal{H} = \oplus_{\alpha \in I}\mathcal{H}_\alpha$. ∎

Let G be a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ a representation. We denote by $\mathcal{E}_\pi \subset B(G)$ the set of all conjugate matrix coefficients of the representation $\pi \otimes \mathrm{id}$ on $\mathcal{H}\overline{\otimes}\overline{\mathcal{H}}$. I.e., $\mathcal{E}_\pi$ is the set of functions $x \mapsto \mathrm{Tr}(\pi(x^{-1})A)$, for $A \in L^1(\mathcal{B}(\mathcal{H}))$. Note that $\mathcal{E}_\pi$ is a linear subspace of $B(G)$.

If G is a locally compact group we let $\hat{G}$ be the set of equivalence classes of irreducible representations. Note that when G is abelian then irreducible representations correspond to unitary characters and hence this notation should not cause any confusion.

If G is a compact group, and $[\pi] \in \hat{G}$, then $\mathcal{E}_\pi \subset B(G) \subset L^2G$, and $\mathrm{HS}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))$ since $\mathcal{H}$ is finite dimensional by Proposition 5.7.3. We may therefore define the map $\mathcal{F}_\pi^* : \mathrm{HS}(\mathcal{H}) \to L^2G$ by

$$\mathcal{F}_\pi^*(\Xi)(x) = \dim(\mathcal{H})^{1/2}\mathrm{Tr}(\pi(x^{-1})\Xi).$$

Proposition 5.7.4. *Let G be a compact group, and $[\pi_1], [\pi_2] \in \hat{G}$, $[\pi_1] \neq [\pi_2]$, then $\mathcal{E}_{\pi_1}$ and $\mathcal{E}_{\pi_2}$ are orthogonal subspaces of L^2G.*

Proof. Since $\mathcal{E}_{\pi_i}$ is spanned by functions of the type $x \mapsto \langle \xi_i, \pi_i(x)\eta_i\rangle$, for $\xi_i, \eta_i \in \mathcal{H}$, it is enough to show that these functions are orthogonal for $i = 1, 2$. From Corollary 5.7.2 we have

$$\int \langle \xi_1, \pi_1(x)\eta_1\rangle \overline{\langle \xi_2, \pi_2(x)\eta_2\rangle}\, \mathrm{d}x = \left\langle \xi_1 \otimes \overline{\xi_2}, \left(\int (\pi_1 \otimes \overline{\pi_2})(x)\, \mathrm{d}x \right) (\eta_1 \otimes \overline{\eta_2}) \right\rangle = 0.$$

∎

If G is a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous representation, then we denote by $\pi \odot \overline{\pi} : G \times G \to \mathcal{U}(\mathrm{HS}(\mathcal{H}))$ the representation given by $(\pi \odot \overline{\pi})(x, y)\Xi = \pi(x)\Xi\pi(y^{-1})$. We also denote by $\lambda \cdot \rho : G \times G \to \mathcal{U}(L^2G)$ the representation given by $(\lambda \cdot \rho)(x, y) = \lambda(x)\rho(y)$.

Theorem 5.7.5 (Peter-Weyl)**.** *Let G be a compact group with Haar measure λ normalized so that $\lambda(G) = 1$, then $L^2G = \oplus_{\pi \in \hat{G}}\mathcal{E}_\pi$, and for each $[\pi] \in \hat{G}$ the map $\mathcal{F}_\pi^* : \mathrm{HS}(\mathcal{H}) \to \mathcal{E}_\pi$ is a unitary intertwiner between the $G \times G$ representations $\pi \odot \overline{\pi}$, and $\lambda \cdot \rho$.*

Proof. From Proposition 5.7.3 we have that $B(G)$ is spanned by the matrix coefficients of irreducible representations, thus $\sum_{\pi \in \hat{G}} \mathcal{E}_\pi$ is dense in $B(G)$ which is then dense in L^2G. By Proposition 5.7.4 we then have that $L^2G = \oplus_{\pi \in \hat{G}}\mathcal{E}_\pi$.

Suppose $\pi \in \hat{G}$. For $\Xi \in \mathrm{HS}(\mathcal{H})$, and $x, y, z \in G$, we have

$$\begin{aligned}((\lambda \cdot \rho)(x,y)\mathcal{F}_\pi^*\Xi)(z) &= (\mathcal{F}_\pi^*\Xi)(x^{-1}zy) \\ &= \dim(\mathcal{H})^{1/2}\mathrm{Tr}(\pi(y^{-1}z^{-1}x)\Xi) \\ &= \dim(\mathcal{H})^{1/2}\mathrm{Tr}(\pi(z^{-1})(\pi(x)\Xi\pi(y^{-1}))) \\ &= (\mathcal{F}_\pi^*(\pi \odot \overline{\pi})(x,y)\Xi)(z).\end{aligned}$$

Thus, $\mathcal{F}_\pi^*$ is an intertwiner. Since $\mathcal{F}_\pi^*$ is clearly surjective it remains to show $\langle \mathcal{F}_\pi^*\Xi_1, \mathcal{F}_\pi^*\Xi_2\rangle = \langle \Xi_1, \Xi_2\rangle$, for all $\Xi_1, \Xi_2 \in \mathrm{HS}(\mathcal{H})$. For this it suffices to consider the case when $\Xi_i = \xi_i \otimes \overline{\eta_i}$ is a rank one operator for $i = 1, 2$. Using Corollary 5.7.2 we then have

$$\begin{aligned}\langle \mathcal{F}_\pi^*(\xi_1 \otimes \overline{\eta_1}), \mathcal{F}_\pi^*(\xi_2 \otimes \overline{\eta_2})\rangle &= \dim(\mathcal{H}) \int \langle \pi(x^{-1})\xi_1, \eta_1\rangle \overline{\langle \pi(x^{-1})\xi_2, \eta_2\rangle}\, \mathrm{d}x \\ &= \dim(\mathcal{H}) \left\langle \left(\int (\pi \otimes \overline{\pi})(x)\, \mathrm{d}x \right) (\xi_1 \otimes \overline{\xi_2}), \eta_1 \otimes \overline{\eta_2} \right\rangle \\ &= \langle \xi_1, \xi_2\rangle \overline{\langle \eta_1, \eta_2\rangle} \\ &= \langle \xi_1 \otimes \overline{\eta_1}, \xi_2 \otimes \overline{\eta_2}.\rangle \qquad \blacksquare\end{aligned}$$

5.8 The Stone-von Neumann theorem

Fix $n \geq 0$, and consider the **Heisenberg group**

$$H_n = \left\{ M(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \mid a^t, b, \in \mathbb{R}^n, c \in \mathbb{R} \right\}.$$

It's easy to check that we have $\mathcal{Z}(H_n) = \{M(0,0,c) \mid c \in \mathbb{R}\} \cong \mathbb{R}$, and $H_n/\mathcal{Z}(H_n) \cong \mathbb{R}^{2n}$.

For each $h \in \mathbb{R} \setminus \{0\}$, the **Schrödinger representation** $U_h : H_n \to \mathcal{U}(L^2(\mathbb{R}^n))$ is given by

$$\left(U_h \begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \xi \right)(x) = e^{2\pi i(x \cdot b - hc)}\xi(x - ha).$$

It is easy to see that for each $h \in \mathbb{R} \setminus \{0\}$, U_h defines a continuous unitary representation. If we restrict U_h to the center, then this gives a character, and these characters are pairwise distinct for $h \in \mathbb{R} \setminus \{0\}$, thus we see that the representations $\{U_h\}_h$ are pairwise inequivalent.

We also have that U_h is irreducible, for each $h \in \mathbb{R} \setminus \{0\}$. To see this, notice that considering the matrices $M(0,b,0)$, we have that the von Neumann algebra generated by $U_h(H_n)$ contains $L^\infty(\mathbb{R}^n) \subset \mathcal{B}(L^2(\mathbb{R}^n))$ which is a maximal abelian subalgebra. Thus, any bounded operator which commutes with $U_h(H_n)$ must be in $L^\infty(\mathbb{R}^n)$. However, if $f \in L^\infty(\mathbb{R}^n)$, $a \in \mathbb{R}^n$, and $u = U_h(M(a,0,0))$, then as an operators in $\mathcal{B}(L^2(\mathbb{R}^n))$, we $(ufu^*)(x) = f(x-ha)$. If f were invariant under the representation U_h, then for every $a \in \mathbb{R}^n$ we would have $f(x) = f(x-ha)$ for almost every $x \in \mathbb{R}^n$, this is not possible unless f is essentially constant and hence we see that $U_h(H_n)' = \mathbb{C}$, showing that U_h is irreducible.

If $F, G \in L^1\mathbb{R}^{2n}$ we define the twisted convolution $F \natural G$ by

$$(F \natural G)(a,b) = \iint F(a-x, b-y) G(x,y) e^{-2\pi i (a-x)\cdot y} \, \mathrm{d}x \mathrm{d}y.$$

We also define

$$F^\sharp(a,b) = \overline{F(-a,-b)} e^{-2\pi i a \cdot b},$$

and for $x, y \in \mathbb{R}^n$ we define

$$(L_{x,y} \natural F)(a,b) = e^{-2\pi i (x\cdot(b-y))} F(a-x, b-y);$$

$$(F \natural L_{x,y})(a,b) = e^{2\pi i ((a-x)\cdot y)} F(a-x, b-y).$$

If $F \in L^1\mathbb{R}^{2n}$, and $\pi : H_n \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then we define $\pi(F)$ to be the operator $\pi(F) = \iint F(x,y) \pi(M(x,y,0)) \, \mathrm{d}x \mathrm{d}y$.

Lemma 5.8.1. *If $\pi : H_n \to \mathcal{U}(\mathcal{H})$ is a continuous representation such that $\pi(M(0,0,c)) = e^{-2\pi i c}$, then for all $F, G \in L^1\mathbb{R}^{2n}$, and $x, y \in \mathbb{R}^n$ we have*

(i) $\pi(F \natural G) = \pi(F)\pi(G)$;

(ii) $\pi(F^\sharp) = \pi(F)^*$;

(iii) $\pi(L_{x,y} \natural F) = \pi(M(x,y,0))\pi(F)$.

(iv) $\pi(F \natural L_{x,y}) = \pi(F)\pi(M(x,y,0))$.

Proof. For (i) we have

$$\begin{aligned}\pi(F)\pi(G) &= \iiiint F(a,b)G(x,y)\pi(M(a,b,0))\pi(M(x,y,0))\,\mathrm{d}a\mathrm{d}b\mathrm{d}x\mathrm{d}y \\ &= \iiiint F(a,b)G(x,y)e^{-2\pi i a\cdot y}\pi(M(a+x,b+y,0))\,\mathrm{d}a\mathrm{d}b\mathrm{d}x\mathrm{d}y \\ &= \iiiint F(a-x,b-y)G(x,y)e^{-2\pi i(a-x)\cdot y}\pi(M(a,b,0))\,\mathrm{d}a\mathrm{d}b\mathrm{d}x\mathrm{d}y \\ &= \iint (F\natural G)(a,b)\pi(M(a,b,0))\,\mathrm{d}a\mathrm{d}b = \pi(F\natural G);\end{aligned}$$

For (ii) we have

$$\begin{aligned}\pi(F)^* &= \iint \overline{F(x,y)}\pi(M(x,y,0))^*\,\mathrm{d}x\mathrm{d}y \\ &= \iint \overline{F(x,y)}e^{-2\pi i(x\cdot y)}\pi(M(-x,-y,0))\,\mathrm{d}x\mathrm{d}y \\ &= \iint \overline{F(-x,-y)}e^{-2\pi i(x\cdot y)}\pi(M(x,y,0))\,\mathrm{d}x\mathrm{d}y = \pi(F^\sharp);\end{aligned}$$

For (iii) (and similarly for (iv)) we have

$$\begin{aligned}\pi(x,y,0)\pi(F) &= \iint F(a,b)e^{-2\pi i x\cdot b}\pi(M(a+x,b+y,0))\,\mathrm{d}a\mathrm{d}b \\ &= \iint F(a-x,b-y)e^{-2\pi i(x\cdot(b-y))}\pi(M(a,b,0))\,\mathrm{d}a\mathrm{d}b = \pi(L_{x,y}\natural F).\end{aligned}$$

∎

If $F \in L^1R^{2n}$ we denote by $V(F)$ the function

$$V(F)(a,b) = \int e^{2\pi i x\cdot b}F(x-a,x)\,\mathrm{d}x.$$

Then V is a composition of the measure preserving change of variables $(a,x) \mapsto (x-a,x)$, and the inverse Fourier transform with respect to the second variable. Since both these transformations are unitaries on $L^2\mathbb{R}^{2n}$ it follows that V has a unique extension to a unitary operator (which we again denote by V) on $L^2\mathbb{R}^{2n}$.

If $f, g \in L^2\mathbb{R}^n$, then we set $V_{f,g} = V(f \otimes g) \in L^2\mathbb{R}^{2n}$, where $f \otimes g \in L^2\mathbb{R}^{2n}$ is given by $(f \otimes g)(x, y) = f(x)g(y)$, for $x, y \in \mathbb{R}^n$, i.e.,

$$V_{f,g}(a,b) = \int e^{2\pi i x\cdot b} f(x-a)g(x)\,\mathrm{d}x = \langle U_1(M(a,b,0))f, \overline{g}\rangle.$$

for almost every $a, b \in \mathbb{R}^n$. Since, V is a unitary we have $\langle V_{f_1,g_1}, V_{f_2,g_2}\rangle = \langle f_1, f_2\rangle\langle g_1, g_2\rangle$, for $f_1, f_2, g_1, g_2 \in L^2\mathbb{R}^n$.

Lemma 5.8.2. *If $\pi : H_n \to \mathcal{U}(\mathcal{H})$ is a continuous representation such that $\pi(M(0,0,c)) = e^{-2\pi i c}$, and if $F \in L^1\mathbb{R}^{2n}$, such that F is not almost everywhere 0, then $\pi(F) \neq 0$.*

Proof. If $F \in L^1\mathbb{R}^{2n}$ such that $\pi(F) = 0$, then from Lemma 5.8.1, for all $a, b \in \mathbb{R}^n$, and $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned} 0 &= \langle \pi(M(x,y,0))\pi(F)\pi(M(-x,-y,0))\xi, \eta\rangle \\ &= \langle \pi(L_{x,y}\natural(F\natural L_{-x,-y}))\xi, \eta\rangle \\ &= e^{2\pi i x\cdot y}\int e^{-2\pi i(x\cdot b + a\cdot y)}F(a,b)\langle \pi(M(a,b,0))\xi, \eta\rangle\,\mathrm{d}a\mathrm{d}b. \end{aligned}$$

From the Fourier inversion theorem we then have that for all $\xi, \eta \in \mathcal{H}$ and almost every $x, y \in \mathbb{R}^n$, $F(x,y)\langle \pi(M(x,y,0))\xi, \eta\rangle = 0$. From this it follows easily that $F(x,y) = 0$ almost everywhere. ∎

Lemma 5.8.3. *Fix $\phi(x) = e^{-\pi\|x\|^2/2}$, and set*

$$\Phi(a,b) = \overline{V_{\phi,\phi}(a,b)} = \int e^{-2\pi i x\cdot b}\phi(x-a)\phi(x)\,\mathrm{d}x.$$

then we have:

(i) $\Phi(a,b) = e^{-\pi\|a\|^2/4}e^{i\pi a\cdot b}e^{-\pi\|b\|^2} \in L^1\mathbb{R}^{2n}$;

(ii) $\Phi^\sharp = \Phi$.

(iii) $\Phi\natural\Phi = \Phi$;

(iv) $\Phi\natural(L_{x,y}\natural\Phi)(a,b) = \langle U_1(M(a,b,0))\phi, \phi\rangle\Phi(a,b)$;

Proof. For (i) we have

$$\begin{aligned}\Phi(a,b) &= \int e^{-2\pi i x\cdot b} e^{-\pi\|x-a\|^2/2} e^{-\pi\|x\|^2/2}\,\mathrm{d}x \\ &= e^{-\pi\|a\|^2/4}\int e^{-2\pi i x\cdot b} e^{-\pi\|x+\frac{a}{2}\|^2}\,\mathrm{d}x \\ &= e^{-\pi\|a\|^2/4} e^{\pi i a\cdot b}\int e^{-2\pi i x\cdot b} e^{-\pi\|x\|^2}\,\mathrm{d}x \\ &= e^{-\pi\|a\|^2/4} e^{i\pi a\cdot b} e^{-\pi\|b\|^2}.\end{aligned}$$

Where the last equality follows as in the proof of Proposition 5.6.14.

Note that if $f, g \in L^2\mathbb{R}^n$, then we have

$$\begin{aligned}\langle U_1(\Phi)f, g\rangle &= \int \Phi(a,b)\langle U_1(a,b)f, g\rangle\,\mathrm{d}a\mathrm{d}b \\ &= \int \overline{V_{\phi,\phi}}(a,b)V_{f,\overline{g}}(a,b)\,\mathrm{d}a\mathrm{d}b \\ &= \langle f,\phi\rangle\langle\overline{g},\phi\rangle = \langle f,\phi\rangle\langle\phi, g\rangle.\end{aligned}$$

Thus, $U_1(\Phi)$ is the rank one projection onto $\phi \in L^2\mathbb{R}^n$. From Lemma 5.8.1 we therefore have that $U_1(\Phi^\sharp) = U_1(\Phi)^* = U_1(\Phi)$, $U_1(\Phi\natural\Phi) = U_1(\Phi)^2 = U_1(\Phi)$, and if $x, y \in \mathbb{R}^n$, then

$$\begin{aligned}U_1(\Phi\natural(L_{x,y}\natural\Phi)) &= U_1(\Phi)U_1(x,y,0)U_1(\Phi) \\ &= \langle U_1(x,y,0)\phi,\phi\rangle U_1(\Phi).\end{aligned}$$

From Lemma 5.8.2 we have that $U_1(F) = 0$ only if $F = 0$ almost everywhere, and hence (ii), (iii), and (iv) follow directly. ∎

Theorem 5.8.4 (Stone-von Neumann). *If $\pi : H_n \to \mathcal{U}(\mathcal{H})$ is a unitary representation such that $\pi(M(0,0,c)) = e^{2\pi i h c}$ for some $h \in \mathbb{R}\setminus\{0\}$ then $\mathcal{H} = \oplus\mathcal{H}_\alpha$ where $\mathcal{H}_\alpha$ are pairwise orthogonal invariant subspaces such that $\pi|_{\mathcal{H}_\alpha}$ is unitarily conjugate to the Schrödinger representation U_h, for each α. In particular, if π is an irreducible representation of H_n on $L^2(\mathbb{R}^n)$, then π is unitarily conjugate to U_h for some $h \in \mathbb{R}\setminus\{0\}$.*

Proof. We'll consider only the case when $h = 1$, the other cases will follow similarly. Suppose $\pi : H_n \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representation, such that $\pi(M(0,0,c)) = e^{2\pi i c}$, for each $c \in \mathbb{R}$.

Fix $\phi(x) = e^{-\pi\|x\|^2/2}$ and let $\Phi(a,b) = \overline{V_{\phi,\phi}(a,b)} \in L^1\mathbb{R}^{2n}$ be as in Lemma 5.8.3. If we set $P = \pi(\Phi)$, then by Lemmas 5.8.1 and 5.8.3 we have that P is a projection. Moreover, by Lemma 5.8.2 we have $P \neq 0$.

If $x, y \in \mathbb{R}^n$, then from Lemma 5.8.3 we have

$$P\pi(M(x,y,0))P = \langle U_1(M(x,y,0))\phi, \phi\rangle P,$$

for all $x, y \in \mathbb{R}^n$.

Let $\{\xi_\alpha\}_{\alpha\in I}$ be an orthonormal basis for $P\mathcal{H}$, and for each $\alpha \in I$, let $\mathcal{H}_\alpha \subset \mathcal{H}$ denote the cyclic subrepresentation generated by ξ_α. If $\alpha, \beta \in I$, then for all $x, y, z \in \mathbb{R}^n$ we have

$$\begin{aligned}\langle \pi(M(x,y,z))\xi_\alpha, \xi_\beta\rangle &= e^{2\pi i z}\langle P\pi(M(x,y,0))P\xi_\alpha, \xi_\beta\rangle \\ &= e^{2\pi i z}\langle U_1(M(x,y,0))\phi, \phi\rangle\langle \xi_\alpha, \xi_\beta\rangle \\ &= \delta_{\alpha,\beta}\langle U_1(M(x,y,z))\phi, \phi\rangle.\end{aligned}$$

Thus, we see that if $\alpha \neq \beta$ then $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ are orthogonal, while for $\alpha = \beta$ we see that the the function of positive type given by ξ_α agrees with that given by ϕ. By the uniqueness of the GNS-representation there then exists a unitary intertwiner $U : L^2\mathbb{R}^n \to \mathcal{H}_\alpha$ such that $U\phi = \xi_\alpha$. ■

A locally compact group G is **type I** if every continuous representation of G which generates a factor, generates a type I factor. Clearly, abelian groups are type I. It also follows from the Peter-Weyl theorem that compact groups are type I.

Corollary 5.8.5. *The Heisenberg group H_n is type I.*

Proof. Fix $\pi : H_n \to \mathcal{U}(\mathcal{H})$, a continuous unitary representation such that $\pi(H_n)''$ is a factor. Then $\pi(\mathcal{Z}(H_n)) \subset \mathcal{Z}(\pi(H_n)'') = \mathbb{C}$, and hence there exists $h \in H$ such that $\pi(0,0,c) = e^{2\pi i h c}$ for all $c \in \mathbb{R}$. If $h = 0$, then π factors through $H_n/\mathcal{Z}(H_n)$ and hence $\pi(H_n)'' \cong \mathbb{C}$.

Otherwise, by the Stone-von Neumann theorem, $\mathcal{H} = \oplus\mathcal{H}_\alpha$ where $\mathcal{H}_\alpha$ are pairwise orthogonal invariant subspaces such that $\pi|_{\mathcal{H}_\alpha}$ is unitarily conjugate to the Schrödinger representation U_h, for each α. Equivalently, there exists a non-empty set Λ such that π is unitarily equivalent to the representation $U_h \otimes \mathrm{id}$ on $L^2(\mathbb{R}^n)\overline{\otimes}\ell^2\Lambda$. Hence, $\pi(H_n)'' \cong U_h(H_n)'' \cong \mathcal{B}(L^2(\mathbb{R}^n))$. ■

Corollary 5.8.6. *If $\pi : H_n \to \mathcal{U}(\mathcal{H})$ is a continuous irreducible representation, then either*

- $\dim(\mathcal{H}) = \infty$ *in which case* π *is conjugate to* U_h *for some* $h \in \mathbb{R} \setminus \{0\}$, *or*
- $\dim(\mathcal{H}) = 1$ *in which case there exists* $(x_0, \lambda) \in \mathbb{R}^{2n}$ *such that* $\pi(a, b, c) = e^{2\pi i(a \cdot x_0 + \lambda b \cdot \lambda)}$, *for all* $a, b \in \mathbb{R}^n$, $c \in \mathbb{R}$.

If G is a locally compact group, then a **covariant pair of representations** of G and C_0G on a Hilbert space $\mathcal{H}$ consists of a a continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, and a non-degenerate $*$-representation $\alpha : C_0G \to \mathcal{B}(\mathcal{H})$ such that for all $f \in C_0G$, and $x \in G$ we have $\alpha(L_x(f)) = \pi(x)\alpha(f)\pi(x^{-1})$. The **standard representation** for the pair G and C_0G is the pair (λ, M), where $\lambda : G \to \mathcal{U}(L^2G)$ is the left-regular representation and $M : C_0G \to \mathcal{B}(L^2G)$ is the representation given by pointwise multiplication.

If G is abelian then the by Theorem 5.6.20, there exists a continuous representation $\rho : \hat{G} \to \mathcal{U}(\mathcal{H})$ such that $\rho(f) = \alpha(\hat{f})$ for all $f \in L^1\hat{G}$. The covariance condition then gives $\pi(x)\rho(\chi)\pi(x^{-1}) = \langle x, \chi\rangle\rho(\chi)$, for all $x \in G$, $\chi \in \hat{G}$. In the case when $G = \mathbb{R}^n$, then G is self-dual with the pairing given by $\langle x, \xi\rangle = e^{2\pi i x \cdot \xi}$. Thus, we see that a pair of representations π and ρ above correspond uniquely to a representation $\tilde{\pi} : H_n \to \mathcal{U}(\mathcal{H})$ given by $\tilde{\pi}(M(x, y, z)) = e^{2\pi i z}\rho(y)\pi(x)$. We thus have the following alternate version of the Stone-von Neumann theorem.

Theorem 5.8.7 (Alternate form of the Stone-von Neumann theorem)**.** *Any covariant pair of representations of* $\mathbb{R}^n$ *and* $C_0\mathbb{R}^n$ *on a Hilbert space* $\mathcal{H}$ *is a multiple of the standard representation on* $L^2\mathbb{R}^n$.

Chapter 6

Group representations and approximation properties

6.1 Ergodicity and weak mixing

Let G be a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ a continuous unitary representation. A vector $\xi \in \mathcal{H}$ is an **invariant vector** if $\pi_x \xi = \xi$ for all $x \in G$. If the representation has no non-zero invariant vectors then it is **ergodic**.

Proposition 6.1.1. *Let G be a group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c > 0$ such that $\mathrm{Re}(\langle \pi_x \xi, \xi \rangle) \geq c\|\xi\|^2$ for all $x \in G$, then π contains an invariant vector ξ_0 such that $\mathrm{Re}(\langle \xi_0, \xi \rangle) \geq c\|\xi\|^2$.*

Proof. Let K be the closed convex hull of the orbit $\pi(G)\xi$. Then K is G-invariant and $\mathrm{Re}(\langle \eta, \xi \rangle) \geq c\|\xi\|^2$ for every $\eta \in K$. Let $\xi_0 \in K$ be the unique element of minimal norm, then since G acts isometrically we have that for each $x \in G$, $\pi_x \xi_0$ is the unique element of minimal norm for $\pi_x K = K$, and hence $\pi_x \xi_0 = \xi_0$ for each $x \in G$. Since $\xi_0 \in K$ we have that $\mathrm{Re}(\langle \xi_0, \xi \rangle) \geq c\|\xi\|^2$. ∎

Corollary 6.1.2. *Let G be a group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c < \sqrt{2}$ such that $\|\pi_x \xi - \xi\| \leq c\|\xi\|$ for all $x \in G$, then π contains a non-zero invariant vector.*

Proof. For each $x \in G$ we have

$$2\mathrm{Re}(\langle \pi_x \xi, \xi \rangle) = 2\|\xi\|^2 - \|\pi_x \xi - \xi\|^2 \geq (2 - c^2)\|\xi\|^2.$$

Hence, we may apply Proposition 6.1.1. ■

6.1.1 Mixing representations

Let G be a locally compact group, a continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is **weak mixing** if for each finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ there exists $x \in \Gamma$ such that

$$|\langle \pi_x \xi, \xi \rangle| < \varepsilon,$$

for all $\xi \in \mathcal{F}$.

The representation π is (**strong**) **mixing** if G is not compact, and for each $\xi \in \mathcal{H}$, we have

$$\lim_{x \to \infty} |\langle \pi_x \xi, \xi \rangle| = 0.$$

Note that mixing implies weak mixing, which in turn implies ergodicity. It is also easy to see that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is mixing (resp. weak mixing) then so is $\pi^{\oplus\infty}$, and if π is mixing then so is $\pi \otimes \rho$ for any representation ρ.

Lemma 6.1.3. *Let G be a locally compact group, a continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is weak mixing if and only if for each finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ there exists $x \in G$ such that*

$$|\langle \pi_x \xi, \eta \rangle| < \varepsilon,$$

for all $\xi, \eta \in \mathcal{F}$.

The representation π is mixing if G is not compact and for each $\xi, \eta \in \mathcal{H}$, we have

$$\lim_{x \to \infty} |\langle \pi_x \xi, \eta \rangle| = 0.$$

Proof. This follows from the polarization identity:

$$\langle \pi_x \xi, \eta \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \pi_x(\xi + i^k \eta), (\xi + i^k \eta) \rangle.$$ ■

Theorem 6.1.4 (Dye)**.** *Let G be a locally compact group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a continuous unitary representation. The following are equivalent:*

(*i*) *π is weak mixing.*

(*ii*) *$\pi \otimes \overline{\pi}$ is ergodic.*

(iii) $\pi \otimes \rho$ *is ergodic for all continuous unitary representation* $\rho : G \to \mathcal{U}(\mathcal{K})$.

(iv) π *does not contain a finite dimensional sub-representation.*

Proof. For (i) $\implies$ (iv), if π is weak mixing then for $\mathcal{L} \subset \mathcal{H}$ any non-trivial finite dimensional subspace with orthonormal basis $\mathcal{F} \subset \mathcal{H}$, there exists $x \in G$ such that $|\langle \pi_x \xi, \eta \rangle| < 1/\sqrt{\dim(\mathcal{L})}$, for all $\xi, \eta \in \mathcal{F}$. Hence, if $\xi \in \mathcal{F}$ then

$$\|[\mathcal{L}](\pi_x \xi)\|^2 = \sum_{\eta \in \mathcal{F}} |\langle \pi_x \xi, \eta \rangle|^2 < 1 = \|\xi\|^2$$

showing that $\mathcal{L}$ is not an invariant subspace.

To show (iv) $\implies$ (iii) suppose $\rho : G \to \mathcal{U}(\mathcal{K})$ is a continuous unitary representation such that $\pi \otimes \rho$ not ergodic. Identifying $\mathcal{H} \overline{\otimes} \mathcal{K}$ with the space of Hilbert-Schmidt operators $\mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$ we then have that there exists $T \in \mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$, non-zero, such that $\pi_x T \overline{\rho}_{x^{-1}} = T$, for all $x \in G$. Then $TT^* \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ is positive, non-zero, compact, and $\pi_x TT^* \pi_{x^{-1}} = TT^*$, for all $x \in G$. By taking the range of a non-trivial spectral projection of TT^* we then obtain a finite dimensional invariant subspace of π.

(iii) $\implies$ (ii) is obvious. To show (ii) $\implies$ (i) suppose $\pi \otimes \overline{\pi}$ is ergodic. If $\mathcal{F} \subset \mathcal{H}$ is finite, and $\varepsilon > 0$, then setting $\zeta = \sum_{\xi \in \mathcal{F}} \xi \otimes \overline{\xi}$ it then follows from Proposition 6.1.1 that there exists $x \in G$ such that

$$\sum_{\xi, \eta \in \mathcal{F}} |\langle \pi_x \xi, \eta \rangle|^2 = \sum_{\xi, \eta \in \mathcal{F}} \langle \pi_x \xi, \eta \rangle \langle \overline{\pi}_x \overline{\xi}, \overline{\eta} \rangle = \mathrm{Re}(\langle (\pi \overline{\otimes} \overline{\pi})(x) \zeta, \zeta \rangle) < \varepsilon.$$

Thus, π is weak mixing. ■

Corollary 6.1.5. *Let* G *be a locally compact group and let* $\pi : G \to \mathcal{U}(\mathcal{H})$ *be a continuous unitary representation. Then* π *is weak mixing if and only if* $\pi \otimes \overline{\pi}$ *is weak mixing, if and only if* $\pi \overline{\otimes} \rho$ *is weak mixing for all continuous unitary representations* ρ.

Corollary 6.1.6. *Let* G *be a locally compact group and let* $\pi : G \to \mathcal{U}(\mathcal{H})$ *be a weak mixing continuous unitary representation. If* $H < G$ *is a finite index closed subgroup then* $\pi_{|H}$ *is also weak mixing.*

Proof. Let $D \subset G$ be a set of coset representatives for H. If $\pi_{|H}$ is not weak mixing, then by Theorem 6.1.4 there is a finite dimensional subspace $\mathcal{L} \subset \mathcal{H}$ which is H-invariant. We then have that $\sum_{x \in D} \pi_x \mathcal{L} \subset \mathcal{H}$ is finite dimensional and G-invariant. Hence, again by Theorem 6.1.4, π is not weak mixing. ■

6.2 Almost invariant vectors

Let G be a locally compact group. A continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$ contains **almost invariant vectors** if for each compact subset $K \subset G$, and $\varepsilon > 0$, there exists $\xi \in \mathcal{H}$, such that

$$\|\pi_k \xi - \xi\| < \varepsilon \|\xi\|, \text{ for all } k \in K.$$

If $\mu \in \mathrm{Prob}(G)$, then for a continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$ the μ-**gradient** operator $\nabla_\mu : \mathcal{H} \to L^2(G, \mu; \mathcal{H})$ is given by

$$(\nabla_\mu \xi)(x) = \xi - \pi_x \xi.$$

Note that we have $\|\nabla_\mu\| \leq \sqrt{2}$. The μ-**divergence** operator $\mathrm{div}_\mu : L^2(G, \mu; \mathcal{H}) \to \mathcal{H}$ is given by

$$\mathrm{div}_\mu(\eta) = \int (\eta(x) - \pi_{x^{-1}} \eta(x)) \, \mathrm{d}\mu(x).$$

If $\xi \in \mathcal{H}$, and $\eta \in L^2(G, \mu; \mathcal{H})$ then we have

$$\begin{aligned} \langle \nabla_\mu \xi, \eta \rangle &= \int \langle \xi - \pi_x \xi, \eta(x) \rangle \, \mathrm{d}\mu(x) \\ &= \int \langle \xi, \eta(x) - \pi_{x^{-1}} \eta(x) \rangle \, \mathrm{d}\mu(x) \\ &= \langle \xi, \mathrm{div}_\mu \eta \rangle, \end{aligned}$$

hence $\mathrm{div}_\mu = \nabla_\mu^*$. The μ-**Laplacian** is defined to be $\Delta_\mu = \mathrm{div}_\mu \nabla_\mu$, which we can compute directly as

$$\Delta_\mu = \int (2 - \pi_x - \pi_{x^{-1}}) \, \mathrm{d}\mu(x) = (2 - \pi(\mu) - \pi(\mu^*)).$$

Note that if μ is **symmetric**, i.e., $\mu^* = \mu$, then we have $\Delta_\mu = 2(1 - \pi(\mu))$.

Proposition 6.2.1 (Kesten)**.** *Let G be a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ a continuous unitary representation. Then the following conditions are equivalent:*

(i) *π contains almost invariant vectors;*

(ii) *For every $\mu \in \mathrm{Prob}(G)$ we have $0 \in \sigma(\Delta_\mu)$;*

(*iii*) *For every* $\mu \in \mathrm{Prob}(G)$ *we have* $\|\pi(\mu)\| = 1$.

Moreover, if G *is second countable then above conditions are also equivalent to*

(*iv*) *There exists* $\mu \in \mathrm{Prob}(G)$ *which is absolutely continuous with respect to Haar measure, such that* $e \in \mathrm{supp}(\mu)$, $\langle \mathrm{supp}(\mu) \rangle = G$, *and* $\|\pi(\mu)\| = 1$.

(*iiv*) *There exists* $\mu \in \mathrm{Prob}(G)$ *which is absolutely continuous with respect to Haar measure, such that* $\mathrm{supp}(\mu) = G$, *and* $\|\pi(\mu)\| = 1$.

Proof. For (i) $\implies$ (ii), suppose that π has almost invariant vectors. If $\mu \in \mathrm{Prob}(G)$ and $\varepsilon > 0$ then let $K \subset G$ be a compact subset such that $\mu(K) > 1 - \varepsilon$. Since π contains almost invariant vectors there exists $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$, and $\|\xi - \pi_k \xi\| < \varepsilon$ for all $k \in K$. We then have

$$\|\nabla_\mu \xi\|^2 = \int \|\xi_i - \pi_k \xi_i\|^2 \, \mathrm{d}\mu(k) < 2\varepsilon + \varepsilon(1 - \varepsilon).$$

As $\varepsilon > 0$ was arbitrary this shows that $0 \in \sigma(\Delta_\mu)$.

(ii) $\implies$ (iii). By considering the spectral radius we see that $0 \in \sigma(\Delta_\mu)$ if and only if $\|\pi(\mu + \mu^*)\| = 2$. In this case, the triangle inequality gives $\|\pi(\mu)\| = 1$.

We next show (iii) $\implies$ (i). For this fix $K \subset G$ compact, and take $f \subset L^1 G_+$ such that $\int f(x) \, \mathrm{d}x = 1$, and $K \cup \{e\} \subset \mathrm{supp}(f)$. Then $\|\pi(f)\| = 1$ and hence $\|\pi(f^* * f)\| = 1$. Moreover, we have $K \subset \mathrm{supp}(f^* * f)$, and so replacing f with $f^* * f$ we will also assume that $\pi(f)$ is a positive operator. Since $\|\pi(f)\| = 1$ and $\pi(f) \geq 0$, we have $1 \in \sigma(\pi(f))$ and so $0 \in \sigma(1 - \pi(f))$. Hence, there exists a sequence $\{\xi_i\} \subset \mathcal{H}$ such that $\|\xi_i\| = 1$, and $\|\xi_i - \pi(f)\xi_i\| \to 0$.

Since we have

$$\langle \pi(f)\xi_i, \xi_i \rangle = \int f(x) \langle \pi_x \xi_i, \xi_i \rangle \, \mathrm{d}x,$$

and since $|\langle \pi(x)\xi_i, \xi_i \rangle| \leq 1$ we then have the functions $\varphi_i(x) = \langle \pi(x)\xi_i, \xi_i \rangle$ converge to 1 in measure on $\mathrm{supp}(f)$.

If $\tilde{\varphi}_i(x) = \langle \pi(x)\pi(f)\xi_i, \pi(f)\xi_i \rangle$, then we have $\|\tilde{\varphi}_i - \varphi_i\|_\infty \to 0$, and hence $\tilde{\varphi}_i \to 1$ in measure on $\mathrm{supp}(f)$.

Next observe that the sequence $\{\tilde{\varphi}_i\}$ is equicontinuous since for all $x \in G$ we have $\|\pi(x)\pi(f)\xi_i - \pi(f)\xi_i\| \leq \|\delta_x * f - f\|_1$ and the action of G on $L^1 G$ is continuous. Since $\tilde{\varphi}_i$ converges to 1 in measure, and since it is an equicontinuous family it then follows that $\tilde{\varphi}_i$ converges to 1 uniformly on $K \subset \mathrm{supp}(f)$. Thus we have

$$\max_{k \in K} \|\pi(f)\xi_i - \pi(k)\pi(f)\xi_i\|^2 = \max_{k \in K} 2(\|\pi(f)\xi_i\|^2 - \mathrm{Re}\tilde{\varphi}_i(k)). \to 0$$

Since K was an arbitrary compact subset this shows (iii) $\implies$ (i).

For (i) $\implies$ (ii) just note that if $f \in L^1 G_+$ such that $f\, \mathrm{d}x$ satisfies the hypotheses in (i), then we may take a sequence $\{a_n\}_{n \in \mathbb{N}}$ such that $0 < a_n < 1$, and $\sum_{n \in \mathbb{N}} a_n = 1$, and if we set $g = \sum_{n \in \mathbb{N}} a_n (f^* f)^n$, then just as in the first part of (iii) $\implies$ (i) above we see that g satisfies the hypotheses in (ii).

Note that in the proof of (iii) $\implies$ (i) the dependence of f on K was only to ensure that $K \subset \mathrm{supp}(f^* f)$. Hence if $\mathrm{supp}(f^* f) = G$ then the above argument shows that $\pi(f^* f)\xi_i$ is a sequence of almost invariant vectors. This shows (ii) $\implies$ (i) for the case when G is second countable. ■

Corollary 6.2.2. *Let G be a locally compact group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a continuous unitary representation. Then π contains almost invariant vectors if and only if $\pi^{\oplus n}$ contains almost invariant vector, where $n \geq 1$ is any cardinal number.*

Proof. If $\mu \in \mathrm{Prob}(G)$ then $\|\pi(\mu)\| = \|\pi^{\oplus n}(\mu)\|$ and hence the corollary follows from Proposition 6.2.1 (iii). ■

6.3 Amenability

A (left) **invariant mean** m on a locally compact group G is a finitely additive Borel probability measure on G, which is absolutely continuous with respect to Haar measure, and which is invariant under the action of left multiplication, i.e., $m : \mathrm{Borel}(G) \to [0, 1]$ such that $m(G) = 1$, $m(E) = 0$ if $\lambda(E) = 0$, where λ is the Haar measure on G, and if $A_1, \ldots, A_n \in \mathrm{Borel}(G)$ are disjoint then $m(\cup_{j=1}^n A_n) = \sum_{j=1}^n m(A_n)$, and if $A \in \mathrm{Borel}(G)$, then $m(xA) = m(A)$ for all $x \in G$. If G possesses an invariant mean then G is **amenable**. We can similarly define right invariant means, and in fact if m is a left invariant mean then $m^*(A) = m(A^{-1})$ defines a right invariant mean. Amenable groups were first introduced by von Neumann in his investigations of the Banach-Tarski paradox.

Given a right invariant mean m on G it is possible to define an integral over G just as in the case if m were a countably additive measure. We therefore obtain a state $\phi_m \in (L^\infty G)^*$ by the formula $\phi_m(f) = \int f \, dm$, and this state is left invariant, i.e., $\phi_m(L_x(f)) = \phi_m(f)$ for all $x \in G$, $f \in L^\infty G$. Conversely, if $\phi \in (L^\infty G)^*$ is a left invariant state, then restricting ϕ to characteristic functions defines a right invariant mean.

Example 6.3.1. Let $\mathbb{F}_2$ be the free group on two generators a, and b. Let A^+ be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a, let A^- be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a^{-1}, let B^+, and B^- be defined analogously, and consider $C = \{e, b, b^2, \ldots\}$. Then we have that

$$\begin{aligned}\mathbb{F}_2 = A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C) \\ = A^+ \sqcup aA^- \\ = b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C).\end{aligned}$$

If m were a left-invariant mean on $\mathbb{F}_2$ then we would have

$$\begin{aligned}m(\mathbb{F}_2) = m(A^+) + m(A^-) + m(B^+ \setminus C) + m(B^- \cup C) \\ = m(A^+) + m(aA^-) + m(b^{-1}(B^+ \setminus C)) + m(B^- \cup C) \\ = m(A^+ \sqcup aA^-) + m(b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C)) = 2m(\mathbb{F}_2).\end{aligned}$$

Hence, $\mathbb{F}_2$ is non-amenable.

An **approximately invariant mean** on G is a net $f_i \in L^1(G)_+$ such that $\int f_i = 1$, and $\|L_x(f_i) - f_i\|_1 \to 0$, uniformly on compact subsets of G.

A **Følner net** is a net of non-null finite measure Borel subsets $F_i \subset G$ such that $\lambda(F_i \Delta x F_i)/\lambda(F_i) \to 0$, uniformly on compact subsets of G. Note that we do not require that $\Gamma = \cup_i F_i$, nor do we require that F_i are increasing, however, if G is not compact then it is easy to see that any Følner net $\{F_i\}_i$ must satisfy $\lambda(F_i) \to \infty$.

Theorem 6.3.2. *Let G be a locally compact group, then the following conditions are equivalent.*

(*i*) *G is amenable.*

(*ii*) *$C_{\mathrm{b}}G$ admits a left invariant state.*

(*iii*) *$C_{\mathrm{b}}^{\mathrm{lu}}(G)$ admits a left invariant state.*

(*iv*) *$L^\infty G$ has an L^1G-invariant state.*

(*v*) *G has an approximate invariant mean.*

(*vi*) *G has a Følner net.*

(*vii*) *The left regular representation $\lambda : G \to \mathcal{U}(L^2G)$ has almost invariant vectors.*

(*viii*) *For any $\mu \in \mathrm{Prob}(G)$ we have $0 \in \sigma(\lambda(\Delta_\mu))$.*

(*ix*) *Any $\mu \in \mathrm{Prob}(G)$ satisfies $\|\lambda(\mu)\| = 1$.*

(*x*) *The representation $\lambda : G \to \mathcal{U}(L^2G)$ has almost invariant vectors when G is viewed as a discrete group.*

(*xi*) *There exists a state $\varphi \in (\mathcal{B}(L^2G))^*$ such that $\varphi(\lambda(x)T) = \varphi(T\lambda(x))$ for all $x \in G$, $T \in \mathcal{B}(L^2G)$.*

(*xii*) *The (discontinuous) action of G on its Stone-Čech compactification βG which is induced by left-multiplication admits an invariant Radon probability measure.*

(*xiii*) *Any continuous action $G \curvearrowright K$ on a compact metric space K admits an invariant Radon probability measure.*

Proof. First note that (i) $\implies$ (ii), and (ii) $\implies$ (iii) are obvious.

To see (iii) $\implies$ (iv) suppose m is a left invariant state on $C_{\mathrm{b}}^{\mathrm{lu}}(G)$. Note that since G acts continuously on $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ it follows that for all $f \in L^1G$, and $g \in C_{\mathrm{b}}^{\mathrm{lu}}(G)$ we have that the integral $f * g = \int f(y)\delta_y * g \, \mathrm{d}\lambda(y)$ converges in norm. Hence we have $m(f * g) = m(g)$, for all $f \in L^1(G)_{1,+} = \{f \in L^1G \mid f \geq 0; \int f = 1\}$ and $g \in C_{\mathrm{b}}^{\mathrm{lu}}(G)$.

Fix $f_0 \in L^1(G)_{1,+}$ and define a state on $L^\infty G$ by $\tilde{m}(g) = m(f_0 * g)$ (recall that $L^1G * L^\infty G \subset C_{\mathrm{b}}^{\mathrm{lu}}(G)$). If $\{f_n\}_{n\in\mathbb{N}} \subset L^1(G)_{1,+}$ is an approximate identity then $m(f_0 * g) = \lim_{n\to\infty} m(f_0 * f_n * g) = \lim_{n\to\infty} m(f_n * g)$, and thus $\tilde{m}$ is independent of f_0. Thus, for $f \in L^1(G)_{1,+}$ and $g \in L^\infty G$ we have $\tilde{m}(f * g) = m(f_0 * f * g) = \tilde{m}(g)$.

We show (iv) $\implies$ (v) using the method of Day: Since $L^\infty G = (L^1G)^*$, the unit ball in L^1G is weak*-dense in the unit ball of $(L^\infty G)^* = (L^1G)^{**}$. It follows that $L^1(G)_{1,+}$ is weak*-dense in the state space of $L^\infty G$.

Let $S \subset L^1G_{+,1}$, be finite and let $K \subset \prod_{f \in S} L^1G$ be the weak-closure of the set $\{S \ni g \mapsto (g * f - f) \mid f \in L^1(G)_{1,+}\}$. Since G has an L^1G-invariant state on $L^\infty G$, and since $L^1(G)_{1,+}$ is weak*-dense in the state space of $L^\infty G$, we have that $0 \in K$. However, K is convex and so by the Hahn-Banach separation theorem the weak-closure coincides with the norm closure. Thus, there exists a net $\{f_i\} \subset L^1(G)_{1,+}$ such that for all $g \in L^1G_{+,1}$ we have

$$\|g * f_i - f_i\|_1 \to 0.$$

If $S \subset L^1G_{+,1}$ is compact then the above convergence may be taken uniformly for $g \in S$. Indeed, if $\varepsilon > 0$ then let $S_0 \subset S$ be finite such that $\inf_{g_0 \in S_0} \|g - g_0\|_1 < \varepsilon$, for all $g \in S$. Then there exists f_i such that $\|g_0 * f_i - f_i\|_1 < \varepsilon$ for all $g_0 \in S_0$, and we then have $\|g * f_i - f_i\|_1 < 2\varepsilon$ for all $g \in S$.

If $K \subset G$ is compact and $g_0 \in L^1G_{+,1}$, then since the action of G on L^1G is continuous it follows that $\{L_k(g_0) \mid k \in K\} \subset L^1G_{+,1}$ is compact, and hence we have

$$\lim_{i \to \infty} \sup_{k \in K} \|L_k(g_0 * f_i) - (g_0 * f_i)\|_1 \leq \lim_{i \to \infty} \sup_{k \in K} \|L_k(g_0) * f_i - f_i\|_1 + \|f_i - g_0 * f_i\|_1 = 0.$$

We show (v) $\Longrightarrow$ (vi) using the method of Namioka: Denote by E_r the characteristic function on the set (r, ∞). If $f \in L^1(G)_{1,+}$ then we have

$$\begin{aligned} \|L_x(f) - f\|_1 &= \int |f(x^{-1}y) - f(y)| \, \mathrm{d}\lambda(y) \\ &= \iint_{\mathbb{R}_{\geq 0}} |E_r(f(x^{-1}y)) - E_r(f(y))| \, \mathrm{d}r \, \mathrm{d}\lambda(y) \\ &= \iint_{\mathbb{R}_{\geq 0}} |E_r(f(x^{-1}y)) - E_r(f(y))| \, \mathrm{d}\lambda(y) \, \mathrm{d}r \\ &= \int_{\mathbb{R}_{\geq 0}} \|E_r(L_x(f)) - E_r(f)\|_1 \, \mathrm{d}r. \end{aligned}$$

By hypothesis, if $\varepsilon > 0$ and $K \subset G$ is compact with $\lambda(K) > 0$ then there exists $f \in L^1(G)_{1,+}$ such that $\|L_x(f) - f\|_1 < \varepsilon / \lambda(K)$ for all $x \in K$, and hence for each $x \in K$ we have

$$\iint_{\mathbb{R}_{\geq 0}} \|E_r(L_x(f)) - E_r(f)\|_1 \, dr \, \mathrm{d}\lambda(x) < \varepsilon = \varepsilon \int_{\mathbb{R}_{\geq 0}} \|E_r(f)\|_1 \, dr.$$

Hence, if we denote by $F_r \subset \Gamma$ the (finite measure) support of $E_r(f)$, then for some $r > 0$ we must have

$$\int \lambda(xF_r \Delta F_r)\, \mathrm{d}\lambda(x) = \int \|E_r(x_* f) - E_r(f)\|_1 \, \mathrm{d}\lambda(x) < \varepsilon \|E_r(f)\|_1 = \varepsilon \lambda(F_r).$$

Since K and $\varepsilon > 0$ were arbitrary, it then follows that there exists a net of Borel subsets $\{F_i\}$ with $\lambda(F_i) > 0$, such that the positive type functions $\varphi_i(x) = \frac{1}{\lambda(F_i)} \langle \lambda(x) 1_{F_i}, 1_{F_i} \rangle = \frac{\lambda(xF_i \cap F_i)}{\lambda(F_i)}$ converge in measure to 1. By Raikov's theorem it then follows that φ_i converge uniformly on compact subsets of G and (vi) then follows.

For (vi) $\implies$ (vii) just notice that if $F_i \subset \Gamma$ is a Følner net, then $\frac{1}{\lambda(F_i)^{1/2}} 1_{F_i} \in L^2 G$ is a net of almost invariant vectors.

(vii) $\iff$ (viii) $\iff$ (ix) follows from Proposition 6.2.1. We also clearly have (vii) $\implies$ (x).

For (x) $\implies$ (xi) let $\xi_i \in L^2 G$ be a net of almost invariant vectors for G as a discrete group. We define states φ_i on $\mathcal{B}(L^2 G)$ by $\varphi_i(T) = \langle T\xi_i, \xi_i \rangle$. By weak* compactness of the state space, we may take a subnet and assume that this converges in the weak* topology to $\varphi \in \mathcal{B}(L^2 G)^*$. We then have that for all $T \in \mathcal{B}(L^2 G)$ and $x \in G$,

$$\begin{aligned} |\varphi(\lambda(x)T - T\lambda(x))| &= \lim_i |\langle (\lambda(x)T - T\lambda(x))\xi_i, \xi_i \rangle| \\ &= \lim_i |\langle T\xi_i, \lambda(x^{-1})\xi_i \rangle - \langle T\lambda(x)\xi_i, \xi_i \rangle| \\ &\leq \lim_i \|T\| (\|\lambda(x^{-1})\xi_i - \xi_i\| + \|\lambda(x)\xi_i - \xi_i\|) = 0. \end{aligned}$$

For (xi) $\implies$ (i), we consider the usual embedding $M : L^\infty G \to \mathcal{B}(L^2 G)$ given by point-wise multiplication. For $f \in L^\infty G$ and $x \in G$ we have $\lambda(x) M_f \lambda(x^{-1}) = M_{L_x(f)}$. Thus, if $\varphi \in \mathcal{B}(L^2 G)^*$ is a state which is invariant under the conjugation by $\lambda(x)$, then restricting this state to $L^\infty G$ gives a state on $L^\infty G$ which is G-invariant.

(ii) $\iff$ (xii), follows from the G-equivariant identification $C_b G \cong C(\beta G)$, together with the Riesz representation theorem.

For (xii) $\implies$ (xiii), suppose G acts continuously on a compact Hausdorff space K, and fix a point $x_0 \in K$. Then the map $f(g) = gx_0$ on G extends uniquely to a continuous map $\beta f : \beta G \to K$, moreover since f is G-equivariant, so is βf. If μ is an invariant Radon probability measure for the action on βG then we obtain the invariant Radon probability measure $f_* \mu$ on K.

For (xiii) $\implies$ (iii), fix $F \subset C_b^{\text{lu}}G$ any finite subset. Then since G acts continuously on $C_b^{\text{lu}}G$ it follows that the G-invariant unital C^*-subalgebra $A \subset C_b^{\text{lu}}G$ generated by F is separable, and hence $\sigma(A)$ is a compact metrizable space and the natural action of G on $\sigma(A)$ is continuous. By hypothesis there then exists a G-invariant probabiliy measure on $\sigma(A)$ which corresponds to a G-invariant state φ_F on A. By the Hahn-Banach theorem we may extend φ_F to a (possibly no longer G-invariant) state $\tilde{\varphi}_F$ on $C_b^{\text{lu}}G$. Considering the finite subsets of $C_b^{\text{lu}}G$ as a partially ordered set by inclusion we then have a net of states $\{\tilde{\varphi}_F\}$ on $C_b^{\text{lu}}G$, and we may let φ_∞ be a cluster point of this set. It is then easy to see that φ_∞ is G-invariant. ∎

The previous theorem is the combined work of many mathematicians, including von Neumann, Følner, Day, Namioka, Hulanicki, Reiter, and Kesten.

Example 6.3.3. Any compact group is amenable, and from part (vi) of Theorem 6.3.2 we see that any group which is locally amenable (each compactly generated subgroup is amenable) is also amenable. The group $\mathbb{Z}^n$ is amenable (consider the Følner sequence $F_k = \{1, \ldots, k\}^n$ for example). From this it then follows easily that all discrete abelian groups are amenable. Moreover, from part (xiii) we see that if a locally compact group is amenable as a discrete group then it is also amenable as a locally compact group, thus all abelian locally compact groups are amenable.

Closed subgroups of amenable groups are also amenable (hence any locally compact group containing $\mathbb{F}_2$ as a closed subgroup is non-amenable). This follows from the fact that if $H < G$ is a closed subgroup, then there exists a Borel set $\Sigma \subset G$ such that the map $H \times \Sigma \ni (h, \sigma) \mapsto h\sigma \in G$ gives a Borel bijection. This then gives an H-equivariant homomorphism θ from $L^\infty H \to L^\infty G$, given by $\theta(f)(h\sigma) = f(h)$. Restricting a G-invariant state to the image of $L^\infty H$ then gives an H-invariant state on $L^\infty H$.

If G is amenable and $H \lhd G$ is a closed subgroup then G/H is again amenable. Indeed, we may view $L^\infty(G/H)$ as the space of right H-invariant functions in $L^\infty G$, and hence we may restrict a G-invariant mean to $L^\infty(G/H)$.

From part (xiii) in Theorem 6.3.2 it follows that if $H \lhd G$ is closed, such that H and G/H are amenable then G is also amenable. Indeed, if $G \curvearrowright K$ is a continuous action on a compact Hausdorff space, then if we consider $\text{Prob}(K)^H$ the set of H-invariant probability measures, then $\text{Prob}(K)^H$ is a non-empty compact set on which G/H acts continuously. Thus there is a G/H-invariant probability measure $\tilde{\mu} \in \text{Prob}(\text{Prob}(K)^H)$ and if we consider the barycenter $\mu = \int \nu \, d\tilde{\mu}(\nu)$, then μ is a G invariant probability measure on K.

It then follows that all solvable groups amenable.

6.4 Lattices

Let G be a locally compact group. A **lattice** in G is a discrete subgroup $\Gamma < G$ such that G/Γ has a G-invariant probability measure.

6.4.1 An example: $SL_n(\mathbb{Z}) < SL_n(\mathbb{R})$

Fix $n \in \mathbb{N}$, and set $G = SL_n(\mathbb{R})$. We also set $K = SO(n) < G$, A the abelian subgroup of G consisting of diagonal matrices in G with positive diagonal coefficients, and N the subgroup of upper triangular matrices in G with diagonal entries equal to one. We also set $\Gamma = SL_n(\mathbb{Z})$. We denote by $\{e_i\}_{1 \leq i \leq n}$, the standard basis vectors for $\mathbb{R}^n$. For $1 \leq i, j \leq n$, $i \neq j$, we denote by $E_{ij} \in \Gamma$ the elementary matrix which has diagonal entries and the ijth entry equal to 1, and all other entries equal to 0.

Proposition 6.4.1. *For every $v \in \mathbb{Z}^n \setminus \{0\}$, there exists $\gamma \in SL_n(\mathbb{Z})$ such that $\gamma v \in \mathbb{N}e_1$.*

Proof. The proposition is trivially true for $n = 1$. For $n = 2$, suppose $v_0 \in \mathbb{Z}^2 \setminus \{0\}$, multiplying by $-I$ if necessary we may obtain a vector v_1 with at least 1 positive entry, multiplying by an appropriate power of E_{12}, or E_{21} we obtain a vector $v_2 = (\alpha_1, \alpha_2)$ with both entries non-negative. We now try to minimize these entries as follows: If $0 < \alpha_1 \leq \alpha_2$ we multiply by E_{21}^{-1}, and if $0 < \alpha_2 < \alpha_1$ we multiply by E_{12}^{-1}. Repeating this procedure we eventually obtain a vector v_3 with one positive entry and the other entry equal to 0. Thus, either $v_3 \in \mathbb{N}e_1$, or $v_3 \in \mathbb{N}e_2$. In the latter case, multiplying by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ gives a vector $v_4 \in \mathbb{N}e_1$.

If $n > 2$ then for each $1 \leq i < j \leq n$, we may realize $SL_2(\mathbb{Z})$ as the subgroup of $SL_n(\mathbb{Z})$ which fixes e_k, for $k \neq i, j$. By considering the embeddings corresponding to $(1, j)$ as j decreases from n to 2, then for $v \in \mathbb{Z}^n \setminus \{0\}$ we may inductibely find elements $\gamma_j \in SL_n(\mathbb{Z})$ such that $\gamma_2 \gamma_3 \cdots \gamma_n v \in \mathbb{N}e_1$. ■

Theorem 6.4.2 (Iwasawa decomposition for $SL_n(\mathbb{R})$)**.** *The product map $K \times A \times N \to G$ is a homeomorphism.*

Proof. Fix $g \in G$, and suppose that g has column vectors $x_1, \ldots, x_n$. Using the Gram-Schmidt process we inductively construct unit orthogonal vectors $\tilde{y}_1, \ldots, \tilde{y}_n$ by setting $y_i = x_i - \sum_{1 \leq j < i} \langle \tilde{y}_j, x_j \rangle \tilde{y}_j$, and then $\tilde{y}_i = y_i / \|y_i\|$, for $1 \leq i \leq n$.

We may then consider the orthogonal transformation $k \in O(n)$, such that $ke_i = \tilde{y}_i$. It is then easy to check that $k^{-1}g$ is an upper triangular matrix with diagonal entries equal to $\|y_i\| > 0$. Note that we have $k \in SO(n)$, since $\det(k^{-1}) = \det(k^{-1}g) = \prod_{i=1}^n \|y_i\| \neq -1$.

If $a = \mathrm{diag}(\|y_1\|, \|y_2\|, \ldots, \|y_n\|)$ then we have $g = kan$ where $n \in N$. This association is clearly continuous and it is easy to see that it is an inverse to the product map $K \times A \times N \to G$. Hence, this must be a homeomorphism. ■

Given the decomposition above, it is then natural to see how Haar measures for K, A, and N relate to Haar measures for G.

Theorem 6.4.3. *Suppose* $\mathrm{d}k$*,* $\mathrm{d}a$*, and* $\mathrm{d}n$*, are Haar measure for* K*,* A*, and* N *respectively. Then, with respect to the Iwasawa decomposition, a Haar measure for* G *is given by* $\mathrm{d}g = \delta(a)\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n$*, where* $\delta(a) = \prod_{1 \leq i < j \leq n} \frac{a_i}{a_j}$*, for* $a = \mathrm{diag}(a_1, a_2, \ldots, a_n) \in A$*.*

Proof. From Example 5.2.1, we see that a right Haar measure for $B = AN$ is given by $\delta(a)\,\mathrm{d}a\mathrm{d}n$. Hence a right Haar measure for $K \times B$ is given by $\delta(a)\,\mathrm{d}k\mathrm{d}a\mathrm{d}n$.

We denote by $\mathrm{d}x$ a measure on $K \times B$ which is obtained from a Haar measure on G through the homeomorphism $K \times B \ni (k, b) \mapsto kb \in G$. Since G is unimodular it follows that $\mathrm{d}x$ is right invariant under the actions of B, and left invariant under the action of K. Hence $\psi_* \mathrm{d}x$ is a right invariant measure on $K \times B$, where $\psi : K \times B \to K \times B$ is given by $\psi(k, b) = (k^{-1}, b)$, and hence must be a scalar multiple of $\delta(a)\,\mathrm{d}k\mathrm{d}a\mathrm{d}n$. But K is unimodular, and hence it follows that $\mathrm{d}x = \psi_* \mathrm{d}x$ is a scalar multiple of $\delta(a)\,\mathrm{d}k\mathrm{d}a\mathrm{d}n$. ■

For $t > 0$ we let A_t be the subset of A given by diagonal matrices a such that $a_{ii}/a_{(i+1)(i+1)} \leq t$, for all $1 \leq i \leq n-1$. For $u > 0$ we let N_u be the subset of N consisting of those matrixes (n_{ij}) such that $|n_{ij}| \leq u$, for all $1 \leq i < j \leq n$. Note that N_u is compact. A **Segal set** in $SL_n\mathbb{R}$ is a set of the form $\Sigma_{t,u} = KA_tN_u$.

Lemma 6.4.4. *We have* $N = N_{1/2}(N \cap \Gamma)$*.*

Proof. We will prove this by induction on n. Note that for $n \in \{1, 2\}$ this is easy. If $n > 2$, and $u \in N$, then we have $u = \left(\begin{smallmatrix} 1 & * \\ 0 & u_0 \end{smallmatrix}\right)$, where $u_0 \in SL_{n-1}(\mathbb{R})$ is upper triangular with diagonal entries equal to one. Thus, by induction there exists $\gamma_0 \in SL_{n-1}(\mathbb{Z})$ upper triangular with diagonal entries equal to one, such that the non-diagonal entries in $u_0\gamma_0$ have magnitude at most 1/2. We may then write $u \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & x \\ 0 & u_0\gamma_0 \end{smallmatrix}\right)$.

If we take $y \in \mathbb{Z}^{n-1}$, such that $y + x$ has entires with magnitude at most 1/2, then we have $u \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma_0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & y \\ 0 & I \end{smallmatrix}\right) \in N_{1/2}$. ■

Lemma 6.4.5. *If $g \in AN$ such that $\|ge_1\| \leq \|gv\|$, for all $v \in \mathbb{Z}^n \setminus \{0\}$, then $g_{11}/g_{22} \leq 2/\sqrt{3}$.*

Proof. Suppose $g \in AN$ is as above. Note that since N stabilizes e_1, if $\gamma \in \Gamma \cap N$, then $g\gamma$ again satisfies $\|g\gamma e_1\| \leq \|g\gamma v\|$, for all $v \in \mathbb{Z}^n \setminus \{0\}$. Also, g and $g\gamma$ have the same diagonal entries if $\gamma \in \Gamma \cap N$. Thus, from the previous lemma it is enough to consider the case $g = an$, with $a \in A$, and $n \in N_{1/2}$.

In this case we have $ge_1 = a_{11}e_1$, and $ge_2 = a_{11}n_{12}e_1 + a_{22}e_2$, with $|n_{12}| \leq 1/2$. Hence,

$$a_{11}^2 = \|ge_1\|^2 \leq \|ge_2\|^2 = a_{11}^2|n_{12}|^2 + a_{22}^2 \leq a_{11}^2/4 + a_{22}^2.$$

Hence, $\frac{3}{4}g_{11}^2 = \frac{3}{4}a_{11}^2 \leq a_{22}^2 = g_{22}^2$. ■

Theorem 6.4.6. *For $t \geq 2/\sqrt{3}$, and $u \geq 1/2$ we have $G = \Sigma_{t,u}\Gamma$.*

Proof. By Lemma 6.4.4 it is enough to show $G = KA_{2/\sqrt{3}}N\Gamma$, which we will do by induction, with the case $n = 1$ being trivial.

Assume therefore that $n > 1$, and this holds for $n - 1$. We fix $g \in G$. Since $g(\mathbb{Z}^n \setminus \{0\})$ is discrete, there exists $v_0 \in \mathbb{Z}^n \setminus \{0\}$ which achieves the minimum of $\{\|gv\| \mid v \in \mathbb{Z}^n \setminus \{0\}\}$. Note that we cannot have $v_0 = \alpha v$, for some $v \in \mathbb{Z}^n \setminus \{0\}$, and $\alpha \in \mathbb{Z}$, unless $\alpha = \pm 1$. Hence, by Proposition 6.4.1 there exists $\gamma \in \Gamma$ such that $\gamma e_1 = v_0$.

We consider the Iwasawa decomposition $g\gamma = kan$, and write $an = \left(\begin{smallmatrix} \lambda & * \\ 0 & \lambda^{-1}h_0 \end{smallmatrix}\right)$, where $h_0 \in SL_{n-1}(\mathbb{R})$. By the induction hypothesis there then exists $k_0 \in SO(n-1)$, and $\gamma_0 \in SL_{n-1}(\mathbb{Z})$ such that $k_0^{-1}h_0\gamma_0$ is upper triangular and whose positive diagonal entries $\{a_{i,i}\}_{i=1,n-1}$ satisfy $a_{i,i}/a_{i+1,i+1} \leq 2/\sqrt{3}$, for $1 \leq i \leq n-2$.

Thus, if we consider $\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & k_0^{-1} \end{pmatrix} k^{-1}g\gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma_0 \end{smallmatrix}\right)$, then we see that $\tilde{g} \in AN$, and the diagonal entries of $\tilde{g}$ satisfy $\tilde{g}_{i,i}/\tilde{g}_{i+1,i+1} \leq 2/\sqrt{3}$ for all $2 \leq i \leq n-1$.

Thus, to finish the proof it suffices to show that we also have $\tilde{g}_{1,1}/\tilde{g}_{2,2} \leq 2/\sqrt{3}$. However, this follows from Lemma 6.4.5 since for all $v \in \mathbb{Z}^n \setminus \{0\}$ we have

$$\begin{aligned} \|\tilde{g}e_1\| &= \|g\gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma_0 \end{smallmatrix}\right) e_1\| = \|g\gamma e_1\| = \|gv_0\| \\ &\leq \|g\gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma_0 \end{smallmatrix}\right) v\| = \|\tilde{g}v\|. \end{aligned}$$

■

Theorem 6.4.7. *$SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$.*

Proof. By the previous theorem it suffices to show that $\Sigma_{t,u}$ has finite Haar measure for $t = 2/\sqrt{3}$, and $u = 1/2$. By Theorem 6.4.3 a Haar measure for G is given by $\mathrm{d}g = \delta(a)\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n$, where $\delta(a) = \prod_{1\leq i<j\leq n} \frac{a_i}{a_j}$ for $a = \mathrm{diag}(a_1, \ldots, a_n)$. Hence, $\int_{\Sigma_{t,u}} \mathrm{d}g = (\int_K \mathrm{d}k)(\int_{A_t} \delta(a)\,\mathrm{d}a)(\int_{N_u} \mathrm{d}u)$. Note that since K and N_u are compact, this integral is finite if and only if the integral $\int_{A_t} \delta(a)\,\mathrm{d}a$ is finite.

Consider the isomorphism $\mathbb{R}^{n-1} \to A$ given by

$$(x_1, x_2, \ldots, x_{n-1}) \mapsto \lambda \mathrm{diag}(e^{x_1+x_2+\cdots+x_{n-1}}, e^{x_2+\cdots+x_{n-1}}, \ldots, e^{x_{n-1}}, 1),$$

where $\lambda^n = \prod_{i=1}^{n-1} e^{-ix_i}$. We then have an explicit Haar measure for A given by the push forward of Lebesgue measure on $\mathbb{R}^{n-1}$. Moreover the preimage of A_t under this map is given by $E = (-\infty, \log(2/\sqrt{3})]^{n-1}$. Thus, we may compute directly

$$\int_{A_t} \delta(a)\,\mathrm{d}a = \int_E \prod_{1\leq i\leq j\leq n-1} e^{x_i+\cdots+x_j}\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1} = \prod_{i=1}^{n-1} \int_{-\infty}^{\log(2/\sqrt{3})} e^{b_i x}\,\mathrm{d}x,$$

where b_i are positive integers. Hence, $\int_{A_t} \delta(a)\,\mathrm{d}a < \infty$. ■

6.5 The Howe-Moore property for $SL_n(\mathbb{R})$

A locally compact group G has the **Howe-Moore property** if every continuous representation without invariant vectors is mixing.

Fix $n \in \mathbb{N}$, and set $G = SL_n(\mathbb{R})$, and $K = SO(n) < G$, as above. We let A_+ denote the set of matrices $g \in G$ such that g is a diagonal matrix whose diagonal entries are positive and non-increasing, and we let A be the subgroup generated by A_+. Note that if $g \in G$, then if we consider the polar decomposition of g we may write $g = k_0h$, where h is positive-definite and $k_0 \in K$. Since positive-definite matrices can be diagonalized there then exists $k_1 \in K$ so that $h = k_2^{-1} a k_2$, for $a \in A_+$. If we set $k_1 = k_0 k_2^{-1}$, then we have $g = k_1 a k_2$, with $k_1, k_2 \in K$, and $a \in A_+$. Hence, we have established the **Cartan decomposition** $G = K A_+ K$.

Theorem 6.5.1 (Howe-Moore)**.** *$SL_n(\mathbb{R})$ has the Howe-Moore property for $n \geq 2$.*

Proof. We first consider the case $G = SL_2(\mathbb{R})$. Suppose $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous representation which is not mixing, then we will show that there exists a non-zero G-invariant vector. Since the representation is not mixing, there exists a sequence $\pi(g_n)$ such that $g_n \to \infty$, and $\pi(g_n)$ does not converge to 0 in the weak operator topology. By taking a subsequence we may assume that $\pi(g_n)$ converges weakly to a non-zero operator $S \in \mathcal{B}(\mathcal{H})$. Using the Cartan decomposition we may write $g_n = k_n a_n k_n'$ where $k_n, k_n' \in K$, and $a_n \in A_+$. Since K is compact we have $a_n \to \infty$, and we may take another subsequence so that $\pi(k_n)$ and $\pi(k_n')$ converge in the strong operator topology to unitaries v and w respectively. If we set $T = v^* S w^* \neq 0$ then we have that $\pi(a_n)$ converges in the weak operator topology to T.

Write $a_n = \begin{pmatrix} r_n & 0 \\ 0 & r_n^{-1} \end{pmatrix}$, where $r_n \to \infty$, and consider the subgroup $N \subset G$ consisting of upper triangular matrices with entries 1 on the diagonal. Note that the conjugation action of $A = \langle A_+ \rangle$ on N is given by

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1 & r^2 s \\ 0 & 1 \end{pmatrix},$$

thus, for $x \in N$ we have $a_n^{-1} x a_n \to e \in G$. Hence $\pi(a_n^{-1} x a_n) \to 1$ in the strong operator topology, and so $\pi(x a_n) = \pi(a_n)\pi(a_n^{-1} x a_n) \to T$ in the weak operator topology. But we also have that $\pi(x a_n) \to \pi(x)T$ in the weak operator topology, and so we conclude that $\pi(x)T = T$ for all $x \in N$, and hence $\pi(x)TT^* = TT^*$ for all $x \in N$. Note that $TT^* \neq 0$ since $\|TT^*\| = \|T\|^2 \neq 0$. Replacing a_n with a_n^{-1} then shows that $\pi(y)T^*T = T^*T$ for all $y \in N^t$, where N^t is the transpose of N consisting of lower triangular matrices with 1's down the diagonal.

Since T and T^* are both weak limits of unitaries from A, and since A is abelian, we have $TT^* = T^*T$, and since N and N^t generate $SL_2(\mathbb{R})$ we then have that $\pi(g)TT^* = TT^*$ for all $g \in SL_2(\mathbb{R})$, thus any non-zero vector in the range of TT^* gives a non-zero invariant vector for $SL_2(\mathbb{R})$, completely the proof for $G = SL_2(\mathbb{R})$.

For the case when $G = SL_m(\mathbb{R})$, with $m > 2$ we first note that again if π is not mixing then there exists a sequence $a_n \in A_+$ such that $\pi(a_n) \to T \neq 0$

in the weak operator topology. Where we again have that the upper left entry of a_n is tending to ∞, and that the lower right diagonal entry is tending to 0.

For $i \neq j$, let $N_{i,j} \subset SL_m(\mathbb{R})$ denote the subgroup consisting of matrices with diagonal entries equal to 1, and all other entries zero except possibly the (i,j)-th entry, then exactly as above we conclude that any non-zero vector in the range of TT^* is fixed by the copy of $SL_2(\mathbb{R})$ generated by $N_{1,m}$ and $N_{m,1}$, and in particular, is fixed by the subgroup $A_{1,m}$ consisting of those diagonal matrices with positive entries which are 1 except for possibly the first or mth diagonal entries.

We let $\mathcal{K}$ denote the set of $A_{1,m}$-invariant vectors, then to finish the proof it is enough to show that $\mathcal{K}$ is G-invariant. Indeed, if this is the case then $A_{1,m}$ is contained in the kernel of the representation restricted to $\mathcal{K}$ and since G is simple this must then be the trivial representation.

To see that $\mathcal{K}$ is G-invariant note that $N_{i,j}$ commutes with $A_{1,m}$ whenever $\{i,j\} \cap \{1,m\} = \emptyset$, in which case $N_{i,j}$ leaves $\mathcal{K}$ invariant. On the other hand, if $\{i,j\} \cap \{1,m\} \neq \emptyset$ then $A_{1,m}$ acts on $B_{i,j}$ by conjugation, and this action is isomorphic to the action of A on N described above for $SL_2(\mathbb{R})$. Thus, as above we must have that any vector which is fixed by $A_{1,m}$ is also fixed by $B_{i,j}$ and in particular we have that $B_{i,j}$ leaves $\mathcal{K}$ invariant in this case as well.

Since G is generated by $B_{i,j}$, for $1 \leq i,j \leq m$ this then shows that $\mathcal{K}$ is indeed G-invariant. ■

We remark that the proof above also works equally well for $SL_m(K)$ where K is any non-discrete local field.

6.6 Property (T)

Let G be a locally compact group, and $H < G$ a closed subgroup. The pair (G,H) has **relative property (T)** if every representation of G which has almost invariant vectors, has a non-zero H-invariant vector. G has **property (T)** if the pair (G,G) has relative property (T).

Suppose that G and A are locally compact groups such that A is abelian. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism, and let $\hat{\alpha} : G \to \mathrm{Aut}(\hat{A})$ denote the dual homomorphism given by $\hat{\alpha}_x(\chi) = \chi \circ \alpha_{x^{-1}}$, for $x \in G$, and $\chi \in \hat{A}$. Note that for $f \in L^1 H$, and $x \in G$ we have $\widehat{f \circ \alpha_x}(\chi) =$

$\int f \circ \alpha_x(y)\overline{\chi}(y)\,\mathrm{d}y = \int f(y)\overline{\hat{\alpha}_x(\chi)(y)}\frac{\mathrm{d}\alpha_{x^{-1}}y}{\mathrm{d}y}\,\mathrm{d}y = \left(\widehat{f\frac{\mathrm{d}\alpha_{x^{-1}}y}{\mathrm{d}y}}\right) \circ \hat{\alpha}_x(\chi).$

Lemma 6.6.1. *Let G and A be as above. Suppose $\{\varphi_i\}_{i \in I} \subset \mathcal{P}_1(G \ltimes A)$ is a net of positive type functions, and let $\nu_i \in \mathrm{Prob}(\hat{A})$ denote the net of probability measures which corresponds by Bochner's theorem to the functions φ_i restricted to A. If $\varphi_i \to 1$ uniformly on compact subsets of G then $\|\hat{\alpha}_x\nu_i - \nu_i\| \to 0$ uniformly on compact subsets of G.*

Proof. We let $\pi_i : G \ltimes A \to \mathcal{U}(\mathcal{H}_i)$ be the cyclic representations associated to φ_i with cyclic vector $\xi_i \in \mathcal{H}_i$. For $f \in L^1H$ we have

$$\begin{aligned}\int \hat{f}\,\mathrm{d}\hat{\alpha}_x\nu_i &= \int \hat{f} \circ \hat{\alpha}_x\,\mathrm{d}\nu_i\\ &= \int f \circ \alpha_x(y)\langle \pi_i(y)\xi_i, \xi_i\rangle\,\mathrm{d}\alpha_x(y)\\ &= \int f(y)\langle \pi_i(y)\pi(x^{-1})\xi_i, \pi(x^{-1})\xi_i\rangle\,\mathrm{d}y\\ &= \langle \pi_i(f)\pi_i(x^{-1})\xi_i, \pi(x^{-1})\xi_i\rangle.\end{aligned}$$

Thus, we have $|\int \hat{f}\,\mathrm{d}\hat{\alpha}_x\nu_i - \int \hat{f}\,\mathrm{d}\nu_i| \leq 2\|\pi_i(f)\|\|\xi_i - \pi_i(x)\xi_i\| = 2\|\hat{f}\|_\infty\|\xi_i - \pi(x)\xi_i\|$.

Since $\mathcal{F}(L^1H)$ is dense in $C_0\hat{A}$ it then follows that $\|\hat{\alpha}_x\nu_i - \nu_i\| \leq 2\|\xi_i - \pi_i(x)\xi_i\| \to 0$ uniformly on compact subsets of G. ■

Lemma 6.6.2. *Let G and A be second countable locally compact groups such that A is abelian. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism. Then the following conditions are equivalent:*

(*i*) *$(G \ltimes A, A)$ does not have relative property (T).*

(*ii*) *There exists a net $\{\nu_i\}_{i \in I} \subset \mathrm{Prob}(\hat{A})$, such that $\nu_i(\{e\}) = 0$, $\nu_i \to \delta_{\{e\}}$ weak*, and $\|\hat{\alpha}_y\nu_i - \nu_i\| \to 0$ uniformly on compact subsets of G.*

Proof. For (i) $\implies$ (ii), suppose that $(G \ltimes A, A)$ does not have relative property (T). Thus, there exists a continuous representation $\pi : G \ltimes A \to \mathcal{U}(\mathcal{H})$ without invariant vectors, and a net of unit vectors $\{\xi_i\}_i \subset \mathcal{H}$ such that $\|\pi(x)\xi_i - \xi_i\| \to 0$ uniformly on compact subsets of $G \ltimes H$.

We let $\varphi_i : G \ltimes A \to \mathbb{C}$ denote the function of positive type given by $\varphi_i(x) = \langle \pi(x)\xi_i, \xi_i\rangle$, and we let $\nu_i \in \mathrm{Prob}(\hat{A})$ denote the probability measure corresponding to φ_i restricted to A, given by Bochner's theorem. Then as $\varphi_i \to 1$ uniformly on compact sets of A we have $\nu_i \to \delta_{\{e\}}$ weak*, and since $\varphi_i \to 1$ uniformly on compact subsets of G by the previous lemma we have that $\|\hat{\alpha}_x \nu_i - \nu_i\| \to 0$ uniformly on compact subsets of G.

Conversely, for (ii) $\implies$ (i), let $\{\nu_i\}_{i \in I} \subset \mathrm{Prob}(A)$ be the net given by (ii). Fix $\mu_0 \in \mathrm{Prob}(G)$ in the same measure class as Haar measure, and set $\tilde{\nu}_i = \hat{\alpha}(\mu_0) * \nu_i = \int \hat{\alpha}_x \nu_i \, \mathrm{d}\mu_0(x)$. Then we again have that $\tilde{\nu}_i \to \delta_{\{e\}}$ weak*, and $\|\hat{\alpha}_x \tilde{\nu}_i - \tilde{\nu}_i\| \to 0$ uniformly on compact subsets, and moreover we have that $\tilde{\nu}_i$ is quasi-invariant for the G action on $\hat{A}$.

We define $\pi_i : G \ltimes H \to \mathcal{U}(L^2(\hat{A}, \tilde{\nu}_i))$ by $\pi_i(xh) = U_x h$ where U is the Koopman representation corresponding to the action of G on $(\hat{A}, \tilde{\nu}_i)$. Then π_i gives a unitary representation, and as $\tilde{\nu}_i \to \delta_{\{e\}}$ weak* we see that $\{\xi_i\}_{i \in I}$ forms a net of almost invariant vectors for H. Moreover, for $x \in G$ we have

$$\begin{aligned} |\langle \pi_i(x)\xi_i - \xi_i, \xi_i \rangle| &\leq \int \left| \left(\frac{\mathrm{d}\hat{\alpha}_x \nu_i}{\mathrm{d}\nu_i} \right)^{1/2} - 1 \right| \mathrm{d}\nu_i \\ &\leq \left\| \frac{\mathrm{d}\hat{\alpha}_x \nu_i}{\mathrm{d}\nu_i} - 1 \right\|_1^{1/2} \\ &\leq \|\hat{\alpha}_x \nu_i - \nu_i\|^{1/2}. \end{aligned}$$

Hence, $\{\xi_i\}_{i \in I}$ also forms a net of almost invariant vectors for G. We let $\mathcal{K}_i \subset L^2(\hat{A}, \tilde{\nu}_i)$ denote the space of A-invariant vectors. Then as $A \lhd G \ltimes A$ is normal it follows that $\mathcal{K}_i$ is $A \lhd G$-invariant and hence so is $\mathcal{K}_i^\perp$. If $(G \ltimes A, A)$ had relative property (T) then since $\{P_{\mathcal{K}_i}^\perp \xi_i\}_{i \in I}$ is almost invariant for $G \ltimes A$ in $\oplus_{i \in I} \pi_i$ it then follows that we must have $P_{\mathcal{K}_i}^\perp \xi_i \to 0$. Hence, it follows that $\tilde{\nu}_i(\{e\}) \to 1$. However, $\tilde{\nu}_i(\{e\}) = \tilde{\nu}_i(\{e\}) - \nu_i(\{e\}) \to 0$ and we would then have a contradiction. ■

Corollary 6.6.3. *Let G and A be second countable locally compact groups such that A is abelian. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism. If $(G \ltimes A, A)$ does not have relative property (T) then there exists a state $\varphi \in B_\infty(\hat{A})^*$ such that $\varphi(1_{\{e\}}) = 0$, $\varphi(1_O) = 1$ for every neighborhood O of e, and $\varphi(f \circ \hat{\alpha}_x) = \varphi(f)$ for all $f \in B_\infty(\hat{A})$, and $x \in G$.*

Proof. Suppose $(G \ltimes A, A)$ does not have relative property (T), and let $\{\nu_i\}_{i \in I} \subset \mathrm{Prob}(\hat{A})$ be as in the previous lemma. Then each ν_i gives a state on $B_\infty(\hat{A})$. If we let φ be a weak*-cluster point of $\{\nu_i\}_{i \in I}$ then we have $\varphi(1_{\{e\}}) = 0$ since $\nu_i(\{e\}) = 0$ for all $i \in I$. We also have $\varphi(1_O) = 1$ for every neighborhood O of e since $\nu_i \to \delta_{\{e\}}$ weak*. And we have $\varphi(f \circ \hat{\alpha}_x) = \varphi(f)$ for all $f \in B_\infty(\hat{A})$, and $x \in G$, since $\|\hat{\alpha}_x \nu_i - \nu_i\| \to 0$ for all $x \in G$. ■

We consider the natural action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$ given by matrix multiplication, and let $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the semi-direct product.

Theorem 6.6.4. *The pair* $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ *has relative property (T).*

Proof. Under the identification $\hat{\mathbb{R}}^2 = \mathbb{R}^2$ given by the pairing $\langle a, \xi \rangle = e^{\pi i a \cdot \xi}$ we have that the dual action of $SL_2(\mathbb{R})$ is given by matrix multiplication with the inverse transpose.

Suppose that $\varphi \in B_\infty(\mathbb{R}^2)^*$ is a $SL_2(\mathbb{R})$-invariant state such that $\varphi(1_O) = 1$ for any neighborhood O of $(0,0)$.

We set

$$A = \{(x,y) \in \mathbb{R}^2 \mid x > 0, -x < y \leq x\};$$

$$B = \{(x,y) \in \mathbb{R}^2 \mid y > 0, -y \leq x < y\};$$

$$C = \{(x,y) \in \mathbb{R}^2 \mid x < 0, x \leq y < -x\};$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid y < 0, y < x \leq -y\}.$$

A simple calculation shows that for $k \geq 0$ the sets $A_k = \left(\begin{smallmatrix} 1 & 0 \\ 2^k & 1 \end{smallmatrix}\right) A$ are pairwise disjoint. Thus, we must have that $\varphi(1_A) = 0$. A similar argument also shows that $\varphi(1_B) = \varphi(1_C) = \varphi(1_D) = 0$. Hence we conclude that $\varphi(1_{\{(0,0)\}}) = 1 - \varphi(1_{A \cup B \cup C \cup D}) = 1$. By the previous corollary it then follows that $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has relative property (T). ■

Theorem 6.6.5 ([?])**.** $SL_m(\mathbb{R})$ *has property (T) for* $m \geq 3$.

Proof. We consider the group $SL_2(\mathbb{R}) < SL_m(\mathbb{R})$ embedded as matrices in the upper left corner. We also consider the group $\mathbb{R}^2 < SL_m(\mathbb{R})$ embedded as those matrices with 1's on the diagonal, and all other entries zero except possibly the $(1,n)$th, and $(2,n)$th entries. Note that the embedding of $SL_2(\mathbb{R})$ normalizes the embedding of $\mathbb{R}^2$, and these groups generate a copy of $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$.

If $\pi : SL_m(\mathbb{R})$ is a representation which has almost invariant vectors, then by Theorem 6.6.4 we have that the copy of $\mathbb{R}^2$ has a non-zero invariant vector. By the Howe-Moore property it then follows that π has an $SL_m(\mathbb{R})$-invariant vector. ■

Suppose that G is a locally compact group and $\Gamma < G$ is a lattice. If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation, we denote by $L^2(G, \mathcal{H})^\Gamma$ the set of measurable function $f : G \to \mathcal{H}$ which satisfy $\pi(\gamma^{-1})f(g\gamma) = f(g)$ for all $g \in G$, and $\gamma \in \Gamma$, and $\int_{G/\Gamma} \|f(g)\|^2 \mathrm{d}g < \infty$, where we identify two functions if they agree almost everywhere. We define an inner-product on $L^2(G, \mathcal{H})^\Gamma$ by $\langle f_1, f_2 \rangle = \int_{G/\Gamma} f_1(g)\overline{f_2(g)}\, \mathrm{d}g$. With this inner-product it is not hard to see that $L^2(G, \mathcal{H})^\Gamma$ forms a Hilbert space.

The **induced representation** $\tilde{\pi} : G \to \mathcal{U}(L^2(G, \mathcal{H})^\Gamma)$ is given by $(\tilde{\pi}(x)f)(y) = f(x^{-1}y)$. It is easy to see that $\tilde{\pi}$ gives a continuous unitary representation of G.

Theorem 6.6.6. *[?] Let G be a second countable locally compact group, and $\Gamma < G$ a lattice, if G has property (T) then Γ has property (T).*

Proof. We fix a Borel fundamental domain Σ for Γ so that the map $\Sigma \times \Gamma \ni (\sigma, \gamma) \to \sigma\gamma \in G$ is a Borel isomorphism, and we choose a Haar measure on G so that Σ has measure 1. We let $\alpha : G \times \Sigma \to \Gamma$ be defined so that $\alpha(g, \sigma)$ is the unique element in Γ which satisfies $g\sigma\alpha(g, \sigma) \in \Sigma$.

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a representation with almost invariant unit vectors $\{\xi_n\}_{n\in\mathbb{N}}$. We consider the vectors $\tilde{\xi}_n \in L^2(G, \mathcal{H})^\Gamma$ given by $\tilde{\xi}_n(\sigma\gamma) = \pi(\gamma)\xi_n$, for $\sigma \in \Sigma$, $\gamma \in \Gamma$. Then for $g \in G$ we have

$$\|\tilde{\pi}(g)\tilde{\xi}_n - \tilde{\xi}_n\|^2 = \int_{G/\Gamma} \|\tilde{\xi}_n(g^{-1}x) - \tilde{\xi}_n(x)\|^2 \,\mathrm{d}x = \int_\Sigma \|\xi_n - \pi(\alpha(g^{-1}, \sigma))\xi_n\|^2 \,\mathrm{d}\sigma \to 0.$$

Thus, $\{\tilde{\xi}_n\}_{n\in\mathbb{N}}$ forms a sequence of almost invariant vectors for $\tilde{\pi}$. Since G has property (T), it follows that there exists a non-zero invariant vector $\tilde{\xi}_0 \in L^2(G, \mathcal{H})^\Gamma$, i.e., $\tilde{\xi}_0(gx) = \tilde{\xi}_0(x)$ for almost all $g, x \in G$. It then follows that $\tilde{\xi}_0$ is essentially constant. We let $\xi_0 \neq 0$ denote the essential range of $\tilde{\xi}_0$. Since $\tilde{\xi}_0 \in L^2(G, \mathcal{H})^\Gamma$ we have that $\pi(\gamma)\xi_0 = \xi_0$ for all $\gamma \in \Gamma$. Thus, $\xi_0 \in \mathcal{H}$ is a non-zero invariant vector and hence Γ has property (T). ∎

We remark that the converse of the previous theorem is also true.

Corollary 6.6.7. *$SL_m(\mathbb{Z})$ has property (T) for $m \geq 3$.*

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