Algorithms For Minimization Without Derivatives describes and analyzes some practical methods for finding approximate zeros and minima of functions.

Contents include the use of successive interpolation for finding simple zeros of a function and its derivatives; an algorithm with guaranteed convergence for finding a zero of a function; an algorithm with guaranteed convergence for finding a minimum of a function of one variable; global minimization given an upper bound on the second derivative; and a new algorithm for minimizing a function of several variables without calculating derivatives.

Among the Features

- Provides reliable FORTRAN and ALGOL computer programs.
- Offers many numerical examples.
- Presents methods that are easily used on computers. These methods require no derivatives, and only function values need to be computed.
- Discusses in detail a method for finding global minima or maxima of functions of one or more variables.

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- Contains a complete analysis of the rate of convergence of some commonly used methods for finding zeros and minima of functions.
- Proves convergence for most of the algorithms, and provides error bounds that allow for the effect of rounding errors.
- Includes a comprehensive and up-to-date bibliography.

Richard P. Brent is on the Research Staff of the IBM Thomas J. Watson Research Center at Yorktown Heights, New York. He received the Ph.D. from Stanford University. Dr. Brent is a member of the Society for Industrial and Applied Mathematics and the Association for Computing Machinery.

Algorithms for Minimization Without Derivatives

Richard P. Brent

The ever-growing relevance of computers to our daily lives increases the importance of developing algorithms suitable for computer use. This outstanding text for graduate students and research workers proposes improvements to existing algorithms, extends their related mathematical theories, and offers details on new algorithms for approximating local and global minima. None of the algorithms discussed requires an evaluation of derivatives; all depend entirely on sequential function evaluation, a highly practical scenario in the frequent event of difficult-to-evaluate derivatives.

Topics include the use of successive interpolation for finding simple zeros of a function and its derivatives; an algorithm with guaranteed convergence for finding a minimum of a function of one variation; global minimization given an upper bound on the second derivative; and a new algorithm for minimizing a function of several variables without calculating derivatives.

Many numerical examples appear here, along with a complete analysis of the rate of convergence for most of the algorithms and error bounds that allow for the effect of rounding errors.

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PREFACE

The problem of finding numerical approximations to the zeros and extrema of functions, using hand computation, has a long history. Recently considerable progress has been made in the development of algorithms suitable for use on a digital computer. In this book we suggest improvements to some of these algorithms, extend the mathematical theory behind them, and describe some new algorithms for approximating local and global minima. The unifying thread is that all the algorithms considered depend entirely on sequential function evaluations: no evaluations of derivatives are required. Such algorithms are very useful if derivatives are difficult to evaluate, which is often true in practical problems.

An earlier version of this book appeared as Stanford University Report CS-71-198, *Algorithms for finding zeros and extrema of functions without calculating derivatives*, now out of print. This expanded version is published in the hope that it will interest graduate students and research workers in numerical analysis, computer science, and operations research.

I am greatly indebted to Professors G. E. Forsythe and G. H. Golub for their advice and encouragement during my stay at Stanford. Thanks are due to them and to Professors J. G. Herriot, F. W. Dorr, and C. B. Moler, both for their careful reading of various drafts and for many helpful suggestions. Dr. T. J. Rivlin suggested how to find bounds on polynomials (Chapter 6), and Dr. J. H. Wilkinson introduced me to Dekker's algorithm (Chapter 4). Parts of Chapter 4 appeared in Brent (1971d), and are included in this book by kind permission of the Editor of *The Computer Journal*. Thanks go to

Professor F. Dorr and Dr. I. Sobel for their help in testing some of the algorithms; to Michael Malcolm, Michael Saunders, and Alan George for many interesting discussions; and to Phyllis Winkler for her nearly perfect typing. I am also grateful for the influence of my teachers V. Grenness, H. Smith, Dr. D. Faulkner, Dr. E. Strzelecki, Professors G. Preston, J. Miller, Z. Janko, R. Floyd, D. Knuth, G. Polya, and M. Schiffer.

Deepest thanks go to Erin Brent for her help in obtaining some of the numerical results, testing the algorithms, plotting graphs, reading proofs, and in many other ways.

Finally I wish to thank the Commonwealth Scientific and Industrial Research Organization, Australia, for its generous support during my stay at Stanford.

This work is dedicated to Oscar and Nancy Brent, who laid the foundations; and to George Forsythe, who guided the construction.

R. BRENT

and as N is the least positive n such that $x_n \geq b$, this gives

$$N = \left\lceil \frac{\sqrt{\frac{1}{2}M}}{k}\left(\sqrt{k(b-a)+t} - \sqrt{t}\right) \right\rceil. \tag{4.12}$$

(4.12) shows that N is essentially proportional to $\sqrt{M}$.

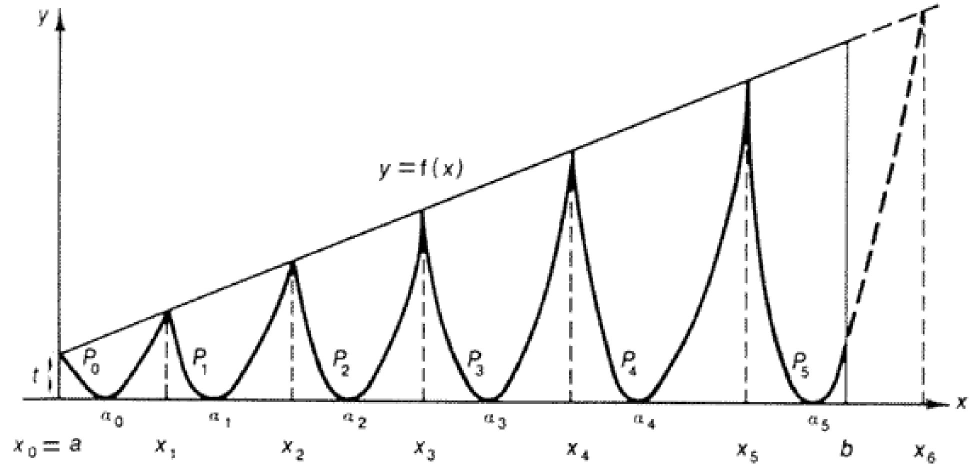

DIAGRAM 4.1 **A straight line**

Two limiting cases of (4.12) are interesting. If t is small and k not too small, so that $k(b-a) \gg t$, then

$$N \simeq \sqrt{\frac{M(b-a)}{2k}}, \tag{4.13}$$

which is independent of t. (In this section we are neglecting the effect of rounding errors, but these should not be important if t satisfies the weak condition (3.68).)

If k is very small, so that $k(b-a) \ll t$, then (4.12) gives

$$N \simeq \frac{b-a}{2\delta}, \tag{4.14}$$

and the algorithm proceeds in steps of size about 2δ, where δ is given by (4.1).

A parabola

If the global minimum of f occurs at an interior point μ, then $f'(\mu) = 0$. If $f''(\mu) \neq 0$ we may analyze the behavior of the algorithm near μ by considering the parabolic approximation $f(\mu) + \frac{1}{2}f''(\mu)(x-\mu)^2$ to $f(x)$. Thus, suppose that

$$M > m > 0 \tag{4.15}$$

and

$$f(x) = \tfrac{1}{2}m(x-\mu)^2 + t, \tag{4.16}$$

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