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Generalized ρ-Almost Periodic Sequences and Applications

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Abstract: In this paper, we analyze the Bohr ρ-almost periodic type sequences and the generalized ρ-almost periodic type sequences of the form $F : I \times X \to Y$, where $\emptyset \neq I \subseteq \mathbb{Z}^n$, X and Y are complex Banach spaces and ρ is a general binary relation on Y. We provide many structural results, observations and open problems about the introduced classes of ρ-almost periodic sequences. Certain applications of the established theoretical results to the abstract Volterra integro-difference equations are also given.

Keywords: generalized ρ-almost periodic sequence; generalized ρ-almost periodic function; abstract Volterra integro-difference equation; abstract impulsive Volterra integro-differential equation; Banach space

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1. Introduction and Preliminaries

The notion of almost periodicity was introduced by the Danish mathematician H. Bohr around 1924–1926 and later generalized by many others. Let $I = [0, \infty)$ or $I = \mathbb{R}$, let $(X, \|\cdot\|)$ be a complex Banach space and let $f : I \to X$ be a continuous function. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$; the set of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f, \epsilon)$. For further information concerning almost periodic functions and their applications, the interested reader may consult the research monographs [1–9].

An X-valued sequence $(x_k)_{k\in\mathbb{Z}}$ $[(x_k)_{k\in\mathbb{N}}]$ is called (Bohr) almost periodic if and only if, for every $\epsilon > 0$, there exists a natural number $K_0(\epsilon)$ such that among any $K_0(\epsilon)$ consecutive integers in $\mathbb{Z}$ $[\mathbb{N}]$, there exists at least one integer $\tau \in \mathbb{Z}$ $[\tau \in \mathbb{N}]$ satisfying that

$$\left\|x_{k+\tau} - x_k\right\| \leq \epsilon, \quad k \in \mathbb{Z} \ \ [k \in \mathbb{N}];$$

as in the case of functions, this number is said to be an ϵ-period of sequence (x_k). Any almost periodic X-valued sequence is bounded and its range is relatively compact in X. The equivalent concept of Bochner almost periodicity of X-valued sequences can be introduced as well; see, e.g., ([10] Theorem 70, pp. 185–186) and ([10] Theorems 71–73, pp. 186–188). It is well known that a sequence $(x_k)_{k\in\mathbb{Z}}$ in X is almost periodic if and only if there exists an almost periodic function $f : \mathbb{R} \to X$ such that $x_k = f(k)$ for all $k \in \mathbb{Z}$; see, e.g., the proof of ([11] Theorem 2) given in the scalar-valued case. It is not difficult to prove that, for every almost periodic sequence $(x_k)_{k\in\mathbb{N}}$ in X, there exists a unique almost periodic sequence $(\tilde{x_k})_{k\in\mathbb{Z}}$ in X such that $\tilde{x_k} = x_k$ for all $k \in \mathbb{N}$, so that a sequence $(x_k)_{k\in\mathbb{N}}$ in X is almost periodic if and only if there exists an almost periodic function $f : [0,\infty) \to X$ such that $x_k = f(k)$ for all $k \in \mathbb{N}$. The class of almost periodic sequences is essentially important in the analysis of the qualitative properties of solutions for various classes of impulsive Volterra integro-differential equations, Volterra integro-difference equations and ordinary differential equations; cf. the research monographs [10,12,13] and the doctoral dissertation [14] for some results obtained in this direction.

The notion of Stepanov almost periodicity of the sequence $(x_k)_{k\in\mathbb{Z}}$ and its equivalence with the usual almost periodicity of $(x_k)_{k\in\mathbb{Z}}$ have been analyzed for the first time by J. Andres and D. Pennequin in [15]. Further on, the class of equi-Weyl almost periodic sequences $(x_k)_{k\in\mathbb{Z}}$ with values in compact metric spaces has been introduced by A. Iwanik in [16], while the class of Besicovitch almost periodic sequences has been introduced by A. Bellow, V. Losert [17] and further analyzed by V. Bergelson et al. in [18] (cf. also the research article [16] by T. Downarowicz and A. Iwanik, which concerns the notion of quasi-uniform convergence in compact dynamical systems). In our joint study with W.-S. Du and D. Velinov [19], we have recently introduced and analyzed the classes of (equi-)Weyl-p-almost periodic sequences, Doss-p-almost periodic sequences and Besicovitch-p-almost periodic sequences with a general exponent $p \geq 1$, providing also certain applications to the abstract impulsive Volterra integro-differential inclusions.

On the other hand, many structural results about the class of (multi-dimensional) c-almost periodic functions, where $c \in \mathbb{C}$ and $|c| = 1$, have recently been presented in the research article [20] by M. T. Khalladi et al. and the research monograph [6]. The strong motivational factor for the genesis of this paper presents the fact that the class of c-almost periodic sequences has not been explored in the existing literature by now. Furthermore, in a joint research article [21] with M. Fečkan, M. T. Khalladi and A. Rahmani, the first named author has recently introduced and analyzed the class of multi-dimensional ρ-almost periodic-type functions of the form $F : I \times X \to Y$, where $\emptyset \neq I \subseteq \mathbb{R}^n$, X and Y are complex Banach spaces and ρ is a general binary relation on Y. In this paper, we have assumed very mild conditions on the domain $I \times X$; for example, we have not assumed that the interior of I is non-empty or that the set I is unbounded in the direction of some coordinate axes. Here, we specifically analyze the situation in which the following conditions hold true:

$$\emptyset \neq I' \subseteq \mathbb{Z}^n, \ \ \emptyset \neq I \subseteq \mathbb{Z}^n \ \text{ and } \ I + I' \subseteq I. \tag{1}$$

In particular, we introduce and analyze several new classes of Stepanov, Weyl, Besicovitch and Doss ρ-almost periodic-type sequences. Following our research studies carried out in [22–25], we can further analyze many other classes of multi-dimensional ρ-almost periodic-type sequences of the above form.

The organization of paper can be briefly described as follows. Section 1.1 recalls the basic definitions and results about Weyl ρ-almost periodic-type functions, Doss ρ-almost periodic-type functions and Besicovitch almost periodic-type functions in $\mathbb{R}^n$. In Section 2, we remind the readers of the already known notions of (metrical) ρ-almost periodicity for the sequences of the form $F : I \times X \to Y$; the term "sequence" used here is a little bit inappropriate in the case that X is not a trivial space. The first original contribution of ours is Theorem 1, where we analyze the existence of a Bohr I'-almost periodic-type function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$, where $F : I \to Y$ is a given Bohr I'-almost periodic type sequence; cf. also Proposition 1 and Theorem 2. An analogue of Theorem 1 for T-almost periodic sequences, where $T \in L(Y)$ is a linear isomorphism, is clarified in Theorem 3; cf. also Corollary 1 and Problem 2. The main structural results about the introduced classes of generalized ρ-almost periodic sequences are given in Propositions 3 and 4, Theorem 4, Propositions 6 and 7 and Theorem 5; cf. also Corollaries 2 and 3. Concerning the above-mentioned results, we will only note here that it is very difficult to state any satisfactory result concerning the discretization of (equi)-Weyl-p-almost periodic-type functions, Doss-p-almost periodic-type functions and Besicovitch-p-almost periodic-type functions. Several new applications to the abstract Volterra integro-difference equations and the abstract impulsive Volterra integro-differential equations are given in Section 4, which consists of three separate subsections (the theory of difference equations in several variables is still very unexplored (cf. the book chapter [26] by L. Székelyhidi and references cited therein for more details on the subject); this is probably the first research article that investigates the almost periodic solutions of difference equations depending on several variables). In Section 5, we provide several conclusions and final remarks about the introduced classes of (generalized) ρ-almost periodic sequences. In addition to the above, we propose many useful comments, illustrative examples and open problems about the notion under our consideration.

Notation and terminology. Suppose that X, Y, Z and T are given non-empty sets. Let us recall that a binary relation between X into Y is any subset $\rho \subseteq X \times Y$. If $\rho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\rho^{-1} \subseteq Y \times X$ and $\sigma \cdot \rho = \sigma \circ \rho \subseteq X \times T$ by $\rho^{-1} := \{(y,x) \in Y \times X : (x,y) \in \rho\}$ and $\sigma \circ \rho := \{(x,t) \in X \times T : \exists y \in Y \cap Z$ such that $(x,y) \in \rho$ and $(y,t) \in \sigma\}$, respectively. As is well known, the domain and range of ρ are defined by $D(\rho) := \{x \in X : \exists y \in Y$ such that $(x,y) \in X \times Y\}$ and $R(\rho) := \{y \in Y : \exists x \in X$ such that $(x,y) \in X \times Y\}$, respectively; $\rho(x) := \{y \in Y : (x,y) \in \rho\}$ $(x \in X)$, $x\ \rho\ y \Leftrightarrow (x,y) \in \rho$. Set $\rho(X') := \{y : y \in \rho(x)$ for some $x \in X'\}$ $(X' \subseteq X)$ and $\mathbb{N}_n := \{1, \cdot \cdot \cdot, n\}$ $(n \in \mathbb{N})$. An unbounded subset $A \subseteq \mathbb{Z}$ is called syndetic if and only if there exists a strictly increasing sequence (a_n) of integers such that $A = \{a_n : n \in \mathbb{Z}\}$ and $\sup_{n\in\mathbb{Z}}(a_{n+1} - a_n) < +\infty$. Set, for every $\mathbf{t}_0 \in \mathbb{R}^n$ and $l > 0$, $B(\mathbf{t}_0, l) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$, where $|\cdot - \cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$. If $I \subseteq \mathbb{R}^n$ and $M > 0$, we set $I_M := \{\mathbf{t} \in I : |\mathbf{t}| \geq M\}$ and $I'_M := \{\mathbf{t} \in I : |\mathbf{t}| \leq M\}$. If $X_0 \subseteq X$, where $(X, \|\cdot\|)$ is a complex Banach space, then $CH(X_0)$ denotes the convex hull of X_0. In the remainder of the paper, we will always assume that $(Y, \|\cdot\|_Y)$ is likewise a complex Banach space. By I, we denote the identity operator on Y.

1.1. *Weyl ρ-Almost Periodic-Type Functions, Doss ρ-Almost Periodic-Type Functions and Besicovitch Almost Periodic-Type Functions in R^n*

In this subsection, we will always assume that $\rho \subseteq Y \times Y$ is a function. If $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, then $p(\Lambda)$ denotes the collection of all Lebesgue measurable functions from Λ into $[1, \infty]$; for more details about the Lebesgue spaces with variable exponent $L^{p(x)}$, we refer the reader to [6] and the references cited therein.

Let us assume that the following condition holds:

(WM1): Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ and $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Lambda)$, $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0, \infty) \to [0, \infty)$ and $\mathbb{F} : (0, \infty) \times \Lambda \to (0, \infty)$.

We need the following notion ([27]):

Definition 1.

(i) *By* $e - W^{(p(\mathbf{u}),\phi,\mathbb{F},\rho)}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *we denote the set consisting of all functions* $F : \Lambda \times X \to Y$ *such that, for every* $\epsilon > 0$ *and* $B \in \mathcal{B}$*, there exist two finite real numbers* $l > 0$ *and* $L > 0$ *such that for each* $\mathbf{t}_0 \in \Lambda'$ *there exists* $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ *such that, for every* $x \in B$*, the mapping* $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \mathbf{t} + l\Omega$ *is well defined, and*

$$\sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} \mathbb{F}(l, \mathbf{t})\phi\Big(\big\|F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x))\big\|_Y\Big)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \epsilon.$$

(ii) *By* $W^{(p(\mathbf{u}),\phi,\mathbb{F}),\rho}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *we denote the set consisting of all functions* $F : \Lambda \times X \to Y$ *such that, for every* $\epsilon > 0$ *and* $B \in \mathcal{B}$*, there exists a finite real number* $L > 0$ *such that*

for each $\mathbf{t}_0 \in \Lambda'$ *there exists* $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ *such that, for every* $x \in B$, *the mapping* $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \mathbf{t} + l\Omega$ *is well defined, and*

$$\limsup_{l \to +\infty} \sup_{x \in B} \sup_{\mathbf{t} \in \Lambda} \mathbb{F}(l, \mathbf{t}) \phi\Big(\big\| F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x)) \big\|_Y \Big)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \epsilon.$$

Suppose now that Λ is a general non-empty subset of $\mathbb{R}^n$ as well as that $p \in \mathcal{P}(\Lambda)$ and the following condition holds:

$$\phi : [0, \infty) \to [0, \infty) \text{ is measurable, } \mathrm{F} : (0, \infty) \to (0, \infty) \text{ and } p \in \mathcal{P}(\Lambda).$$

Set $\Lambda'' := \{\tau \in \mathbb{R}^n : \tau + \Lambda \subseteq \Lambda\}$ and assume $\emptyset \neq \Lambda' \subseteq \Lambda''$.

We also need the following notion ([27]):

Definition 2. *Suppose that the function* $F : \Lambda \times X \to Y$ *satisfies that* $\phi(\|F(\cdot + \tau; x) - \rho(F(\cdot; x))\|) \in L^{p(\cdot)}(\Lambda'_t)$ *for all* $t > 0$, $x \in X$ *and* $\tau \in \Lambda'$. *Then we say that the function* $F(\cdot; \cdot)$ *is Doss-*$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$*-almost periodic if and only if, for every* $B \in \mathcal{B}$ *and* $\epsilon > 0$, *there exists* $l > 0$ *such that for each* $\mathbf{t}_0 \in \Lambda'$ *there exists a point* $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ *such that, for every* $t > 0$, $x \in B$ *and* $\cdot \in \Lambda_t$, *we have*

$$\limsup_{t \to +\infty} \mathrm{F}(t) \sup_{x \in B} \Big[\phi\big(\|F(\cdot + \tau; x) - \rho(F(\cdot; x))\|_Y \big) \Big]_{L^{p(\cdot)}(\Lambda'_t)} < \epsilon.$$

Suppose, finally, that Λ is a general non-empty subset of $\mathbb{R}^n$ as well as that $p \in \mathcal{P}(\Lambda)$, the function $\phi : [0, \infty) \to [0, \infty)$ is Lebesgue measurable and $\mathrm{F} : (0, \infty) \to (0, \infty)$. Let $\emptyset \neq \Lambda' \subseteq \Lambda''$. Recall, a trigonometric polynomial $P : \Lambda \times X \to Y$ is any linear combination of functions such as $(\mathbf{t}; x) \mapsto e^{i\langle \lambda, \mathbf{t} \rangle} c(x)$, where $c : X \to Y$ is a continuous function.

The following notion has recently been introduced in ([28] Definition 2.1):

Definition 3. *Suppose that* $F : \Lambda \times X \to Y$, $\phi : [0, \infty) \to [0, \infty)$ *and* $\mathrm{F} : (0, \infty) \to (0, \infty)$. *Then we say that the function* $F(\cdot; \cdot)$ *belongs to the class* $e - (\mathcal{B}, \phi, \mathrm{F}) - B^{p(\cdot)}(\Lambda \times X : Y)$ *if and only if for each set* $B \in \mathcal{B}$ *there exists a sequence* $(P_k(\cdot; \cdot))$ *of trigonometric polynomials such that*

$$\lim_{k \to +\infty} \limsup_{t \to +\infty} \mathrm{F}(t) \sup_{x \in B} \Big[\phi\Big(\big\| F(\mathbf{t}; x) - P_k(\mathbf{t}; x) \big\|_Y \Big) \Big]_{L^{p(\mathbf{t})}(\Lambda'_t)} = 0,$$

where we assume that the term in braces belongs to the space $L^{p(\mathbf{t})}(\Lambda'_t)$ *for any compact set K.*

2. Bohr $(\mathcal{B}, I', \rho)$-Almost Periodic Type Sequences

We start our work with the observation that we have recently introduced, in ([21] Definitions 2.1, 2.22 and 2.25), the notions of Bohr $(\mathcal{B}, I', \rho)$-almost periodicity, $(\mathcal{B}, I', \rho)$-uniform recurrence, $\mathbb{D}$-asymptotical Bohr $(\mathcal{B}, I', \rho)$-almost periodicity of type 1 and $\mathbb{D}$-asymptotical $(\mathcal{B}, I', \rho)$-uniform recurrence of type 1 for a function of the form $F : I \times X \to Y$. For the sake of completeness, we will only recall the following notion:

Definition 4. *Suppose that* $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ *is a continuous function,* ρ *is a binary relation on* Y *and* $I + I' \subseteq I$. *Then, we say that:*

(i) $F(\cdot; \cdot)$ *is Bohr* $(\mathcal{B}, I', \rho)$*-almost periodic if and only if for every* $B \in \mathcal{B}$ *and* $\epsilon > 0$ *there exists* $l > 0$ *such that, for each* $\mathbf{t}_0 \in I'$, *there exists* $\tau \in B(\mathbf{t}_0, l) \cap I'$ *such that, for every* $\mathbf{t} \in I$ *and* $x \in B$, *there exists an element* $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ *such that*

$$\big\| F(\mathbf{t} + \tau; x) - y_{\mathbf{t};x} \big\|_Y \leq \epsilon.$$

(ii) *$F(\cdot;\cdot)$ is $(\mathcal{B}, I', \rho)$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ such that*

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \left\| F(\mathbf{t} + \tau_k; x) - y_{\mathbf{t};x} \right\|_Y = 0.$$

If (1) holds, then $F : I \times X \to Y$ is a continuous function if and only if for each $\mathbf{t} \in I$, $x \in B$ and $\epsilon > 0$ there exists $\delta > 0$ such that, for every $y \in X$ with $\|x - y\| < \delta$, we have $\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\|_Y < \epsilon$; in particular, any function $F : I \to Y$ is already continuous. The notion introduced in ([21] Definitions 3.1 and 3.4), with $\omega \in \mathbb{Z}^n \setminus \{0\}$, $\omega_j \in \mathbb{Z} \setminus \{0\}$ for $1 \le j \le n$ and some extra assumptions being satisfied, can serve us to introduce the notion of (ω, ρ)-periodicity and the notion of $(\omega_j, \rho_j)_{j\in\mathbb{N}_n}$-periodicity of a sequence $F : I \to X$. As in all our recent research studies of multi-dimensional almost periodic type functions, we will omit the term "$\mathcal{B}$" from the notation for the sequences of the form $F : I \to Y$, the term "I'" from the notation if $I' = I''$ and the term "ρ" from the notation if $\rho = \mathrm{I}$; for example, a Bohr $\mathcal{B}$-almost periodic sequence is nothing else but a Bohr $(\mathcal{B}, I', \rho)$-almost periodic sequence with $I' = I$ and $\rho = \mathrm{I}$. We also write "c" in place of "$c\mathrm{I}$" if $c \in \mathbb{C}$.

Before proceeding any further, we would like to observe that almost all structural results from the first three sections of [21] hold in the discrete framework. The exceptions are listed below:

(A1) It is clear that the statements of ([21] Corollary 2.4, Theorems 2.14 and 2.16, and Propositions 3.7 and 2.24) cannot be directly formulated in the discrete framework.

(A2) We should further examine the question of whether the statements of ([21] Propositions 2.18 and 2.20) can be formulated with $I = \mathbb{Z}$ or $I = \mathbb{N}_0$ and $I' = \mathbb{N}$.

(A3) We should further examine the question of whether the statements of ([21] Theorem 2.28, and Corollary 2.29) can be formulated with the condition (AP-E) replaced with the condition:

(AP-ED) For every $\mathbf{t}' \in \mathbb{Z}^n$, there exists a finite real number $M > 0$ such that $\mathbf{t}' + I_M \subseteq I$.

Remark 1. *Before considering these questions, let us observe that the notion of strong $\mathcal{B}$-almost periodicity, introduced in ([6] Definition 6.1.24), is meaningful in the discrete setting and that the statement of ([6] Proposition 6.1.25) holds in the discrete framework. Concerning the notion of Bohr $(\mathcal{B}, c)$-almost periodicity and the notion of $(\mathcal{B}, c)$-uniform recurrence introduced in ([6] Definition 7.1.6), we would like to note that the statements of ([6] Proposition 7.1.9, Corollary 7.1.11, Propositions 7.1.13–7.1.16, Theorem 7.1.18) hold in the discrete framework. Keeping this in mind, we can simply prove that the statements of ([20] Propositions 2.2, 2.6–2.9, 2.11 and 2.17; Corollary 2.10; and Theorem 2.13) continue to hold for c-almost periodic sequences (c-uniformly recurrent sequences); in particular, if a sequence $(x_k)_{k\in\mathbb{N}}$ is c-uniformly recurrent for some $c \in \mathbb{C}$, then we must have $|c| = 1$. The statements of ([6] Theorems 6.1.40 and 7.1.25) can be directly formulated in the discrete framework as well.*

Concerning the question (A2), we would like to note that the statements of ([21] Propositions 2.18 and 2.20) continue to hold if $I = \mathbb{Z}$ or $I = \mathbb{N}_0$ and $I' = \mathbb{N}$. This follows from the same argumentation as in the continuous case. Concerning the question (A3), the situation is much more complicated. In connection with this problem, we will first state and prove the following analogue of ([6] Theorem 6.1.37) in the discrete framework:

Theorem 1. *Suppose that $I' \subseteq I \subseteq \mathbb{Z}^n$, $I + I' \subseteq I$, the set I' is unbounded, $S \subseteq \mathbb{Z}^n$ is finite,* (AP-ED) *holds and $\Omega_S := [(I' \cup (-I')) + (I' \cup (-I'))] \cup S$. Then, $F : I \to Y$ is a Bohr I'-almost periodic sequence, resp. an I'-uniformly recurrent sequence if and only if there exists a Bohr I'-almost periodic, resp. an I'-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$. If this is the case, then $\tilde{F}(\cdot)$ is Bohr Ω_S-almost periodic, resp. Ω_S-uniformly recurrent; furthermore, $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$ and the assumption that $F(\cdot)$ is bounded implies that $\tilde{F}(\cdot)$ is uniformly continuous.*

Proof. Suppose first that $F : I \to Y$ is a Bohr I'-almost periodic sequence, resp. an I'-uniformly recurrent sequence. Repeating verbatim the argumentation given in the proof of the above-mentioned result, we find that there exists a Bohr I'-almost periodic, resp. an I'-uniformly recurrent, sequence $\tilde{F}_{\mathbb{Z}} : \mathbb{Z}^n \to Y$ such that $\tilde{F}_{\mathbb{Z}}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$. In order to extend the function $\tilde{F}_{\mathbb{Z}} : \mathbb{Z}^n \to Y$ to a Bohr I'-almost periodic, resp. an I'-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = \tilde{F}_{\mathbb{Z}}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{Z}^n$, we can argue as in the proof of ([11] Theorem 2) with appropriate technical modifications. For the sake of convenience, we will present all relevant details in the case that $n = 2$, extending the proof of ([11] Theorem 2) with $c = 1$ and $\delta = 1/2$ to the two-dimensional setting. If $t = (t_1, t_2) \in \mathbb{R}^2$ is given, then there exist the unique numbers $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$ such that $t_1 \in [k, k+1)$ and $t_2 \in [m, m+1)$. We first define $\tilde{F}(t_1, m) := \tilde{F}_{\mathbb{Z}}(k, m)$ if $t_1 \in [k, k+(1/2))$ and $\tilde{F}(t_1, m) := 2(\tilde{F}_{\mathbb{Z}}(k+1, m) - \tilde{F}_{\mathbb{Z}}(k, m))(t_1 - k - (1/2)) + \tilde{F}_{\mathbb{Z}}(k, m)$ if $t_1 \in [k+(1/2), k+1)$; we similarly define $\tilde{F}(t_1, m+1) := \tilde{F}_{\mathbb{Z}}(k, m+1)$ if $t_1 \in [k, k+(1/2))$ and $\tilde{F}(t_1, m+1) := 2(\tilde{F}_{\mathbb{Z}}(k+1, m+1) - \tilde{F}_{\mathbb{Z}}(k, m+1))(t_1 - k - (1/2)) + \tilde{F}_{\mathbb{Z}}(k, m+1)$ if $t_1 \in [k+(1/2), k+1)$. After that, we define $\tilde{F}(t_1, t_2) := \tilde{F}(t_1, m)$ if $t_2 \in [m, m+(1/2))$ and $\tilde{F}(t_1, t_2) := 2(\tilde{F}(t_1, m+1) - \tilde{F}(t_1, m))(t_2 - m - (1/2)) + \tilde{F}(t_1, m)$ if $t_2 \in [m+(1/2), m+1)$. It can be simply verified that the function $\tilde{F}(\cdot)$ is continuous as well as that $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$ and the function $\tilde{F}(\cdot)$ is uniformly continuous if $F(\cdot)$ is bounded. Further on, let us assume that a point $\mathbf{t}_0 \in I'$ and a number $\epsilon > 0$ are given; then there exist $l > 0$ and $\tau = (\tau_1, \tau_2) \in I' \cap B(\mathbf{t}_0, l)$ such that $\|\tilde{F}_{\mathbb{Z}}(\mathbf{s}+\tau) - \tilde{F}_{\mathbb{Z}}(\mathbf{s})\| < \epsilon/9$, $\mathbf{s} \in \mathbb{Z}^2$. Now, we will prove that $\|\tilde{F}(\mathbf{t}+\tau) - \tilde{F}(\mathbf{t})\| < \epsilon$, $\mathbf{t} \in \mathbb{R}^2$. Suppose that $k \in \mathbb{Z}$, $m \in \mathbb{Z}$, $\mathbf{t} = (t_1, t_2)$, $t_1 \in [k, k+1)$ and $t_2 \in [m, m+1)$. There exist four possibilities:

(i) $t_1 \in [k, k+(1/2))$ and $t_2 \in [m, m+(1/2))$;
(ii) $t_1 \in [k, k+(1/2))$ and $t_2 \in [m+(1/2), m+1)$;
(iii) $t_1 \in [k+(1/2), k+1)$ and $t_2 \in [m, m+(1/2))$;
(iv) $t_1 \in [k+(1/2), k+1)$ and $t_2 \in [m+(1/2), m+1)$.

If (i) holds, then $t_1 + \tau_1 \in [k+\tau_1, k+\tau_1+(1/2))$ and we have

$$\big\|\tilde{F}(\mathbf{t}+\tau) - \tilde{F}(\mathbf{t})\big\| = \big\|\tilde{F}_{\mathbb{Z}}(t_1+\tau_1, m+\tau_2) - \tilde{F}_{\mathbb{Z}}(t_1, m)\big\| \leq \epsilon/3,$$

where the last estimate follows from the estimate $\|\tilde{F}_{\mathbb{Z}}(\mathbf{s}+\tau) - \tilde{F}_{\mathbb{Z}}(\mathbf{s})\| < \epsilon/9$, $\mathbf{s} \in \mathbb{Z}^2$ and the argumentation contained in the proof of ([11] Theorem 2). If (ii) holds, then we have $t_2 + \tau_2 \in [m+\tau_2+(1/2), m+\tau_2+1)$ and therefore

$$\begin{aligned}
&\big\|\tilde{F}(\mathbf{t}+\tau) - \tilde{F}(\mathbf{t})\big\| \\
&= \Big\|2\big[\tilde{F}_{\mathbb{Z}}(t_1+\tau_1, m+1+\tau_2) - \tilde{F}_{\mathbb{Z}}(t_1+\tau_1, m+\tau_2)\big] \cdot (t_2 - m - (1/2)) + \tilde{F}_{\mathbb{Z}}(t_1+\tau_1, m+\tau_2) \\
&- 2\big[\tilde{F}_{\mathbb{Z}}(t_1, m+1) - \tilde{F}_{\mathbb{Z}}(t_1, m)\big] \cdot (t_2 - m - (1/2)) - \tilde{F}_{\mathbb{Z}}(t_1, m)\Big\| \\
&\leq \big\|\tilde{F}_{\mathbb{Z}}(t_1+\tau_1, m+1+\tau_2) - \tilde{F}_{\mathbb{Z}}(t_1, m+1)\big\| + \big\|\tilde{F}_{\mathbb{Z}}(t_1+\tau_1, m+\tau_2) - \tilde{F}_{\mathbb{Z}}(t_1, m)\big\| \\
&+ \big\|\tilde{F}_{\mathbb{Z}}(t_1+\tau_1, m+\tau_2) - \tilde{F}_{\mathbb{Z}}(t_1, m)\big\| \leq 3 \cdot (\epsilon/3) = \epsilon.
\end{aligned}$$

The analysis of cases (iii) and (iv) is similar and therefore $\tilde{F}(\cdot)$ is Bohr I'-almost periodic, resp. I'-uniformly recurrent; as in [6], this simply implies that $\tilde{F}(\cdot)$ is Bohr Ω_S-almost periodic, resp. Ω_S-uniformly recurrent. Finally, it is clear that the existence of a Bohr I'-almost periodic, resp. an I'-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$ implies that $F(\cdot)$ is Bohr I'-almost periodic, resp. I'-uniformly recurrent. □

There exist infinitely many ways to extend the function $\tilde{F}_{\mathbb{Z}}(\cdot)$ to a function $\tilde{F}(\cdot)$ defined on the whole Euclidean plane, obeying all required properties from the formulation of Theorem 1 (we only need to change the values of parameters c and δ from the proof of ([11] Theorem 2)). This readily implies that any non-empty subset I of $\mathbb{Z}^n$ cannot be admissible with respect to the almost periodic extensions (cf. ([6] Definition 6.1.39) for the notion).

Now, we will focus our attention to the case in which $I' = I = \mathbb{Z}^n$. We need the following result of independent interest (cf. also ([1] pp. 54–59) for several related results given in the one-dimensional setting):

Proposition 1. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is a $\mathcal{B}$-almost periodic function, where $\mathcal{B}$ is any collection of compact subsets of X. Then, the function $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \mathbb{Z}^n)$-almost periodic.*

Proof. The statement of proposition is trivial if $Y = \{0\}$; otherwise, there exists an element $y \in Y$ such that $\|y\|_Y = 1$. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be fixed. Then, ([6] Proposition 6.1.22) implies that there exists $\delta \in (0, \epsilon)$ such that the assumption $|\mathbf{t} - \mathbf{t}'| + \|x - x'\| \leq \delta$ for some $\mathbf{t}$, $\mathbf{t}' \in \mathbb{R}^n$ and x, $x' \in X$ imply $\|F(\mathbf{t}; x) - F(\mathbf{t}', x')\|_Y \leq \epsilon$. Furthermore, ([6] Proposition 6.1.19) implies that there exists a relatively dense set of points $\tau = (\tau_1, \ldots, \tau_n)$ in $\mathbb{R}^n$ such that $\|F(\mathbf{t} + \tau; x) - F(\mathbf{t}; x)\|_Y \leq \epsilon$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$, as well as that $\|G_j(\mathbf{t} + \tau; x) - G_j(\mathbf{t}; x)\|_Y \leq \epsilon$ for all $\mathbf{t} \in \mathbb{R}^n$, $j \in \mathbb{N}_n$ and $x \in B$, where the Bohr $\mathcal{B}$-almost periodic function $G_j : \mathbb{R}^n \times X \to Y$ is defined as the usual periodic extension of the function by $G_{j;0}(\mathbf{t}; x) := (1 - |1 - t_j|)y$, $\mathbf{t} = (t_1, \ldots, t_j, \ldots, t_n) \in [0, 2]^n$, $x \in X$ to the space $\mathbb{R}^n \times X$. As in the one-dimensional setting, this simply implies that there exist two vectors $p \in \mathbb{Z}^n$ and $w = (w_1, \ldots, w_n) \in B(0, \delta)$ such that $\tau = 2p + \omega$. Therefore, we have:

$$\begin{aligned}
&\left\|F(\mathbf{t} + 2p; x) - F(\mathbf{t}; x)\right\|_Y \\
&\quad \leq \left\|F(\mathbf{t} + 2p; x) - F(\mathbf{t} + 2p + w; x)\right\|_Y + \left\|F(\mathbf{t} + 2p + w; x) - F(\mathbf{t}; x)\right\|_Y \\
&\quad \leq \epsilon + \delta < 2\epsilon, \quad \mathbf{t} \in \mathbb{R}^n, \ x \in B.
\end{aligned}$$

This simply completes the proof because the set consisting of all points $2p \in \mathbb{Z}^n$ with the above properties is relatively dense in $\mathbb{Z}^n$, which can be trivially shown. □

Keeping in mind Theorem 1 and Proposition 1, we can simply extend the statement of ([11] Theorem 2) to the higher-dimensional setting:

Theorem 2. *Suppose that $F : \mathbb{Z}^n \to Y$. Then, $F(\cdot)$ is a Bohr almost periodic sequence if and only if there exists a Bohr almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $F(\mathbf{t}) = \tilde{F}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{Z}^n$.*

As a simple corollary of Theorem 2, we have that the set of all Bohr almost periodic sequences $F : \mathbb{Z}^n \to Y$ is a linear vector space with the usual operations.

Further on, if $S \subseteq \mathbb{Z}^n$ is finite, $c \in \mathbb{C} \setminus \{1\}$, $|c| = 1$ and $\arg(c)/\pi \in \mathbb{Q}$, then the set of all (Bohr) c-almost periodic sequences $F : \mathbb{Z}^n \to Y$ is not a linear vector space with the usual operations; we define the set Ω_S as it has been done on ([6] p. 467). Arguing as in the proof of Theorem 1, we can similarly deduce the following analogues of ([21] Theorem 2.28, [6] Theorem 7.1.26) in the discrete framework:

Theorem 3. *Suppose that $I' \subseteq I \subseteq \mathbb{Z}^n$, $I + I' \subseteq I$, the set I' is unbounded, $\rho = T \in L(Y)$ is a linear isomorphism, $S \subseteq \mathbb{Z}^n$ is finite and* (AP-ED) *holds. Then, $F : I \to Y$ is a Bohr (I', T)-almost periodic function, resp. an (I', T)-uniformly recurrent function if and only if there exists a Bohr (I', T)-almost periodic, resp. an (I', T)-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$. Furthermore, $R(\tilde{F}(\cdot)) \subseteq CH(T^{-1}\overline{R(F)})$, the boundedness of $F(\cdot)$ implies that $\tilde{F}(\cdot)$ is uniformly continuous and the assumptions* $\arg(c)/\pi \in \mathbb{Q}$ *and $\rho = c\mathrm{I}$ imply that $\tilde{F}(\cdot)$ is Bohr (Ω_S, T)-almost periodic, resp. (Ω_S, T)-uniformly recurrent.*

As an immediate consequence of Theorem 3, we have the following:

Corollary 1. *Suppose that $c \in \mathbb{C}$, $|c| = 1$ and $F : \mathbb{Z}^n \to Y$ is a c-almost periodic sequence. Then, there exists a Bohr $(\mathbb{Z}^n, c)$-almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $F(\mathbf{t}) = \tilde{F}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{Z}^n$.*

Further on, it is logical to ask the following questions with regards to Proposition 1 and Corollary 1:

Proposition 2. *Let $c \in \mathbb{C} \setminus \{1\}$ and $|c| = 1$.*

(Q1) *Suppose that $F : \mathbb{R}^n \times X \to Y$ is a $(\mathcal{B}, c)$-almost periodic function, where $\mathcal{B}$ is any collection of compact subsets of X. Is it true that the function $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \mathbb{Z}^n, c)$-almost periodic?*

(Q2) *Suppose that $F : \mathbb{R}^n \to Y$ is a c-almost periodic function. Is it true that $(F(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^n}$ is a c-almost periodic sequence?*

Suppose, finally, that $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ is a basis of $\mathbb{R}^n$,

$$I = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n : \alpha_i \geq 0 \text{ for all } i \in \mathbb{N}_n\} \cap \mathbb{Z}^n$$

is a convex polyhedral in $\mathbb{R}^n \cap \mathbb{Z}^n$, and $I' \subseteq \mathbb{Z}^n$ is a proper convex subpolyhedral of I. We would like to note that the set Ω_S from the formulation of Theorem 1 is relatively dense in $\mathbb{R}^n$, while the set Ω_S from the formulation of Theorem 1 is relatively dense in $\mathbb{R}^n$ provided that $\arg(c)/\pi \in \mathbb{Q}$. If this is the case, then the mean value $M(F)$, given by the expression (4) below, exists uniformly in $\mathbf{s} \in \mathbb{Z}^n$.

3. Generalized ρ-Almost Periodic Type Sequences

In this section, we analyze various classes of Stepanov, Weyl, Besicovitch and Doss ρ-almost periodic type sequences of the form $F : \Lambda \times X \to Y$, where $\emptyset \neq \Lambda \subseteq \mathbb{Z}^n$. We will always assume here that $\Lambda = \Lambda_1 \times \Lambda_2 \times \ldots \times \Lambda_n$, where for each $j \in \mathbb{N}_n$ there exists an integer $a \in \mathbb{Z}$ such that $\Lambda_j = \mathbb{Z}$, $\Lambda_j = \{\ldots, a-2, a-1, a\}$ or $\Lambda_j = \{a, a+1, a+2, \ldots\}$. Set $\Lambda'' := \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} + \Lambda \subseteq \Lambda\}$. For every integer $l \in \mathbb{N}$, we introduce the set P_l consisting of all closed sub-rectangles of Λ, which contains exactly $(l+1)^n$ points with all integer coordinates. Suppose that a function $\mathbb{F}_l : \{l\} \times P_l \to [0, \infty)$ is given for each integer $l \in \mathbb{N}$.

The following notion generalizes the notion introduced by J. Andres and D. Pennequin [15]:

Definition 5. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $l \in \mathbb{N}$, $1 \leq p < +\infty$, $\Lambda' \subseteq \Lambda''$ and ρ is a binary relation on Y. Then, we say that $F(\cdot;\cdot)$ is Stepanov-$(\mathcal{B}, \Lambda', \mathbb{F}_\cdot, p, \rho, l)$-almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$, which satisfies that, for every $J \in P_l$ and for every $j \in J$, $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that*

$$\sup_{x\in B} \mathbb{F}_l(l, J)\left[\sum_{j\in J}\left\|F(j+\tau;x) - z_{j,x}\right\|^p\right]^{1/p} < \epsilon. \tag{2}$$

In the classical concept, a sequence is almost periodic if and only if it is Stepanov almost periodic (see, e.g., ([15] Consequence 3)). Furthermore, we can simply prove the following result:

Proposition 3.

(i) *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $l \in \mathbb{N}$, $1 \leq p < +\infty$, $\Lambda' \subseteq \Lambda''$ and ρ is a binary relation on Y. If there exists a real number $c_l > 0$ such that $\mathbb{F}_l(l, J) \leq c_l l^{-n/p}$ for all $J \in P_l$ and $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \Lambda', \rho)$-almost periodic, then $F(\cdot;\cdot)$ is Stepanov-$(\mathcal{B}, \Lambda', \mathbb{F}_\cdot, p, \rho, l)$-almost periodic.*

(ii) *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $l \in \mathbb{N}$, $1 \leq p < +\infty$, $\Lambda' \subseteq \Lambda''$ and ρ is a binary relation on Y. If there exists a real number $c_l > 0$ such that $\mathbb{F}_l(l, J) \geq c_l$ for all $J \in P_l$ and $F(\cdot;\cdot)$ is Stepanov-$(\mathcal{B}, \Lambda', \mathbb{F}_\cdot, p, \rho, l)$-almost periodic, then $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \Lambda', \rho)$-almost periodic.*

Keeping in mind the above result, it becomes clear that the concept of Stepanov-$(\mathcal{B}, \Lambda', \mathbb{F}_\cdot, p, \rho, l)$-almost periodicity introduced above is not satisfactory enough. Because of that, in the remainder of paper, we will focus our attention mainly on the Weyl, Besicovitch and Doss classes of generalized ρ-almost periodic sequences.

The following notion generalizes the notion introduced in [16,18,19,29]:

Definition 6. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $1 \leq p < +\infty$, $\Lambda' \subseteq \Lambda''$ and ρ is a binary relation on Y. Then, we say that $F(\cdot;\cdot)$ is:*

(i) *equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist $l \in \mathbb{N}$ and $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$, which satisfies that, for every $J \in P_l$ and for every $j \in J$, $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2) holds;*

(ii) *Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$, which satisfies that there exists an integer $l_\tau \in \mathbb{N}$ such that, for every $l \geq l_\tau$, $J \in P_l$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2) holds.*

It is obvious that any equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic sequence is Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic and any Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic sequence is Doss-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic, where the notion of Doss-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodicity is introduced as follows:

Definition 7. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $1 \leq p < +\infty$, $\Lambda' \subseteq \Lambda''$ and ρ is a binary relation on Y. Then, we say that $F(\cdot;\cdot)$ is Doss-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$, which satisfies that there exists an increasing sequence (l_k) of positive integers such that, for every $k \in \mathbb{N}$, $J \in P_{l_k}$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2) holds with the number l replaced by the number l_k therein.*

As in the recent research studies of multi-dimensional almost periodic type functions, we will omit the term "$\mathcal{B}$" from the notation for the functions of the form $F : \Lambda \to Y$, the term "Λ'" from the notation if $\Lambda' = \Lambda''$ and the term "ρ" from the notation if $\rho = \mathrm{I}$. The situation in which the following condition holds:

(FV) There exists a function $\mathbb{F} : (0, \infty) \to (0, \infty)$ such that $\mathbb{F}(l, J) = \mathbb{F}(l)$ for all $l \in \mathbb{N}$ and $J \in P_l$

will be dominant in our analysis; in this case, an (equi-)Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic (Doss-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic) function is also called (equi-)Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, p, \rho)$-almost periodic (Doss-$(\mathcal{B}, \Lambda', \mathbb{F}, p, \rho)$-almost periodic). The situation in which condition (FV) does not hold is far from being simple for consideration (cf. ([6] Example 6.3.4 and pp. 425–428) for some applications made in the continuous framework).

Remark 2. *We feel it is our duty to emphasize that the notion of a scalar-valued almost periodic sequence in the sense of Weyl, introduced by A. Bellow and V. Losert in [17], is completely misleading; in their approach, an almost periodic sequence $(x_k)_{k \in \mathbb{N}}$ in the sense of Weyl is nothing else but the usual asymptotically almost periodic sequence (by an asymptotically almost periodic sequence we mean a sum of an almost periodic sequence and a sequence vanishing at plus infinity; see ([17] p. 316, Lemma 3.6)). It can be simply proved that any asymptotically almost periodic sequence $(x_k)_{k \in \mathbb{N}}$ is equi-Weyl-$(l^{-1/p}, p)$-almost periodic, i.e., equi-Weyl-p-almost periodic in the usual sense ($p \geq 1$); on the other hand, the sequence $(x_k)_{k \in \mathbb{N}}$ given by $x_k := 1$ if there exists $l \in \mathbb{N}$ such that $k = l^3$, and $x_k := 0$, otherwise, is equi-Weyl-$(l^{-\sigma}, p)$-almost periodic for any $\sigma > 1/2$, but not asymptotically almost periodic.*

We continue by stating the following result:

Proposition 4. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $1 \leq p < +\infty$, $\Lambda' = \Lambda''$ and $\rho : Y \to Y$ is a continuous function. If* (FV) *holds and $F(\cdot;\cdot)$ is equi-Weyl-$(\mathcal{B}, \mathbb{F}, p, \rho)$-almost periodic, then for each bounded set $B \in \mathcal{B}$ we have that the set $\{F(\mathbf{t}; x) : \mathbf{t} \in \Lambda; x \in B\}$ is bounded.*

Proof. Let $B \in \mathcal{B}$ be given and let $\epsilon = 1$. Without loss of generality, we may assume that $\Lambda = \Lambda' = \mathbb{Z}^n$ or $\Lambda = \Lambda' = [0, \infty)^n$. Suppose first that $\Lambda = \Lambda' = \mathbb{Z}^n$. Then, there exist

$l \in \mathbb{N}$ and $L > 0$ such that, for every fixed $\mathbf{t} \in \mathbb{Z}^n$, there exists a point $\tau \in \mathbb{Z}^n \cap B(\mathbf{t}, L)$, which satisfies that, for every $J \in P_l$ and for every $j \in J$ and $x \in B$, (2) holds with $z_{j,x} = \rho(F(j;x))$. Then, $\mathbf{t} - \tau \in B(0, L)$ and, by choosing an appropriate closed rectangle J in $\mathbb{R}^n$ with a vertex $\mathbf{t} - \tau$, we obtain that $\|F(\mathbf{t} - \tau + \tau; x) - \rho(F(\mathbf{t} - \tau; x))\|_Y \leq 1/\mathbb{F}(l)$ for all $x \in B$. This implies $F(\mathbf{t}; x) \in B(\rho(F(\mathbf{t} - \tau; x)), 1/\mathbb{F}(L))$, which gives the required conclusion since B is bounded and $\rho(\cdot)$ is continuous. Suppose now that $\Lambda = \Lambda' = [0, \infty)^n$. If $n = 1$, then the final conclusion follows similarly as in the proof of ([19] Proposition 2) with the corresponding ϵ-period τ belonging to the segment $[t - 2L, t]$ for $t \geq 2L$. In a general case, any of the sequences $t \mapsto F(t, j_2, j_3, \ldots, j_n)$, $t \in \mathbb{N}_0$, $t \mapsto F(j_1, t, j_3, \ldots, j_n)$, $t \in \mathbb{N}_0$, $t \mapsto F(j_1, j_2, t, \ldots, j_n)$, $t \in \mathbb{N}_0$, $\ldots$, $t \mapsto F(j_1, j_2, j_3, \ldots, j_{n-1}, t)$, $t \in \mathbb{N}_0$, is equi-Weyl-$(\mathcal{B}, \mathbb{F}, p, \rho)$-almost periodic (the integers $j_1 \geq 0, \ldots, j_n \geq 0$ are fixed in advance). Taking into account the result established in the one-dimensional setting, it suffices to prove that the set $\{F(\mathbf{t}; x) : t_1 \geq 2L, \ldots, t_n \geq 2L; x \in B\}$ is bounded ($\mathbf{t} = (t_1, t_2, \ldots, t_n)$). This follows as in the case that $\Lambda = \Lambda' = \mathbb{Z}^n$, with the corresponding ϵ-period τ belonging to the cube $[t_1 - 2L, t_1] \times [t_2 - 2L, t_2] \times \ldots \times [t_n - 2L, t_n]$. $\square$

In particular, any equi-Weyl-$(l^{-n/p}, p, \rho)$-almost periodic sequence $F : \mathbb{Z}^n \to Y$, where $\rho : Y \to Y$ is a continuous function and Y is a finite-dimensional space, has a relatively compact range. The interested reader may try to construct an example of an infinite-dimensional Banach space Y and an equi-Weyl-$(l^{-1/p}, p, \mathrm{I})$-almost periodic sequence $F : \mathbb{Z} \to Y$ whose range is not relatively compact in Y.

Example 1.

(i) *Let us observe that there exists a Weyl-$(l^{-1/p}, p, \mathrm{I})$-almost periodic real sequence $(y_k)_{k \in \mathbb{N}}$ (i.e., $(y_k)_{k \in \mathbb{N}}$ is Weyl-p-almost periodic in the usual sense), which is not (Besicovitch-p-)bounded, not equi-Weyl-$(l^{-1/p}, p, \mathrm{I})$-almost periodic and not Besicovitch-p-almost periodic in the sense of ([19] Definition 9); cf. ([19] Example 4(ii)). Concerning the sequence $(y_k)_{k \in \mathbb{N}}$ considered in ([19] Example 4(i)), we would like to note that $(y_k)_{k \in \mathbb{N}}$ is equi-Weyl-$(l^{-\sigma}, p, \mathrm{I})$-almost periodic for any $\sigma > 0$ and $p \geq 1$, as easily approved; let us also recall that for each $p \geq 1$ there exists a Besicovitch-p-almost periodic real sequence $(y_k)_{k \in \mathbb{N}}$ which is not Weyl-p-almost periodic (see ([19] p. 23)).*

(ii) *Let $l \in \mathbb{N}$. Suppose that $(y_k)_{k \in \mathbb{N}}$ is a real sequence defined by $y_k := 0$ for $k = 1, 2, \ldots, l$; $y_{l+2k} := 1$ ($k \in \mathbb{N}_0$) and $y_{l+2k+1} := -1$ ($k \in \mathbb{N}_0$). Then, $(y_k)_{k \in \mathbb{N}}$ is equi-Weyl-$(l^{-\sigma}, p, -\mathrm{I})$-almost periodic for any $\sigma > 0$ and $p \geq 1$, i.e., the sequence $(y_k)_{k \in \mathbb{N}}$ is equi-Weyl-p-almost anti-periodic.*

(iii) *Define the sequence $F : \mathbb{Z}^n \to \mathbb{R}$ by $F(k_1, \ldots, k_n) := 0$ if there exists an index $j \in \mathbb{N}_n$ such that $k_j < 0$ and $F(k_1, \ldots, k_n) := 1$, otherwise. Then, it can be simply proved (cf. ([6] Example 6.3.9) for the continuous version) that $F(\cdot)$ is Weyl-$(l^{-\sigma}, p, \mathrm{I})$-almost periodic for any $p \geq 1$ and $\sigma > (n-1)/p$.*

We continue by raising an issue:

Proposition 5. *In the continuous framework, we know that the space of all complex-valued equi-Weyl-$(l^{-n/p}, p, \mathrm{I})$-almost periodic functions $F : \mathbb{R} \to \mathbb{C}$ is not complete with respect to the Weyl-p-seminorm. If we denote by P the space consisting of all complex-valued equi-Weyl-$(l^{-n/p}, p, \mathrm{I})$-almost periodic sequences $F : \mathbb{Z} \to \mathbb{C}$, then it can be simply proved, as in the continuous framework, that the expression*

$$d(G, H) := \lim_{l \to +\infty} l^{-n/p} \sup_{k \in \mathbb{Z}} \left[\sum_{j=k}^{k+l} \|G(j) - H(j)\|^p \right]^{1/p}, \quad G,\ H \in P$$

defines a pseudometric on P. Is (P, d) complete or not?

Further on, we set $\Lambda'_j := \mathbb{R}$ if $\Lambda_j = \mathbb{Z}$, $\Lambda'_j := (-\infty, a]$ if $\Lambda_j = \{\ldots, a-2, a-1, a\}$ for some $a \in \mathbb{Z}$ and $\Lambda'_j := [a, \infty)$ if $\Lambda_j = \{a, a+1, a+2, \ldots\}$ for some $a \in \mathbb{Z}$ $(1 \leq j \leq n)$. After that, we set $\Lambda_e := \Lambda'_1 \times \Lambda'_2 \times \ldots \times \Lambda'_n$. Now, we are ready to state the following result concerning the extensions of (equi-)Weyl ρ-almost periodic-type sequences and Doss ρ-almost periodic-type sequences:

Theorem 4. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $1 \leq p < +\infty$, $\Lambda' \subseteq \Lambda''$ and $\rho = T \in L(Y)$. If* (FV) *holds and $F(\cdot;\cdot)$ is (equi-)Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, p, \rho)$-almost periodic (Doss-$(\mathcal{B}, \Lambda', \mathbb{F}, p, \rho)$-almost periodic), where for each $j \in \mathbb{N}_n$ we have $\Lambda'_j := [a, \infty)$ for some $a \in \mathbb{Z}$ or $\Lambda'_j = \mathbb{R}$, then there exists a continuous function $\tilde{F} : \Lambda_e \times X \to Y$ such that $\tilde{F} \in (e-)W^{p,x,\mathbb{F}}_{[0,1]^n,\Lambda',\mathcal{B}}(\Lambda_e \times X : Y)$ ($\tilde{F}(\cdot;\cdot)$ is Doss-$(p, x, \mathbb{F}, \mathcal{B}, \Lambda', T)$-almost periodic) and $\tilde{F}(\mathbf{t}; x) = F(\mathbf{t}; x)$ for all $\mathbf{t} \in \Lambda$ and $x \in X$.*

Proof. We will present the proof only in the one-dimensional setting, for the class of equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, p, \rho)$-almost periodic sequences with $X = \{0\}$; the general result can be deduced similarly, following the argumentation contained in the proof of Theorem 1. Suppose first that $\Lambda_e = [a, \infty)$ for some $a \in \mathbb{Z}$. If $t \in [b, b+1)$ for some $b \in \mathbb{Z}$ with $b \geq a$, then we set $\tilde{F}(t) := F(b)$ for $t \in [b, b+(1/2))$ and $\tilde{F}(t) := 2(F(b+1) - F(b))(t - b - (1/2)) + F(b)$ for $t \in [b+(1/2), b+1)$. Let $\epsilon > 0$ be given. By our assumption, we can find an integer $l \in \mathbb{N}$ and a real number $L > 0$ such that, for every $t_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(t_0, L)$, which satisfies that, for every $j \in \mathbb{N} \cap [a, \infty)$, we have $\sum_{k=j}^{j+l} \|F(k+\tau) - TF(k)\|^p \leq \epsilon^p [\mathbb{F}(l)]^{-1}$. We need to prove that, for every fixed real number $x \geq a$, we have:

$$\int_x^{x+l} \|\tilde{F}(s+\tau) - T\tilde{F}(s)\|^p \, ds \leq \text{Const.} \cdot \epsilon^p [\mathbb{F}(l)]^{-1}. \tag{3}$$

In order to see this, observe first that for each $t \in \mathbb{R}$ we have $\int_{t+(1/2)}^{t+1} (s - t - (1/2)) \, ds = 1/8$; keeping this and the definition of $\tilde{F}(\cdot)$ in mind, we can compute as follows:

$$\begin{aligned}
&\int_x^{x+l} \|\tilde{F}(s+\tau) - T\tilde{F}(s)\|^p \, ds \\
&\leq \int_{\lfloor x \rfloor}^{\lfloor x \rfloor + 1} \|\tilde{F}(s+\tau) - T\tilde{F}(s)\|^p \, ds + \ldots + \int_{\lfloor x \rfloor + l}^{\lfloor x \rfloor + l + 1} \|\tilde{F}(s+\tau) - T\tilde{F}(s)\|^p \, ds \\
&\leq \left[\int_{\lfloor x \rfloor}^{\lfloor x \rfloor + (1/2)} \|\tilde{F}(s+\tau) - T\tilde{F}(s)\|^p \, ds + \ldots + \int_{\lfloor x \rfloor + l}^{\lfloor x \rfloor + l + (1/2)} \|\tilde{F}(s+\tau) - T\tilde{F}(s)\|^p \, ds \right] \\
&+ \left[\int_{\lfloor x \rfloor + (1/2)}^{\lfloor x \rfloor + 1} \|\tilde{F}(s+\tau) - T\tilde{F}(s)\|^p \, ds + \ldots + \int_{\lfloor x \rfloor + l + (1/2)}^{\lfloor x \rfloor + l + 1} \|\tilde{F}(s+\tau) - T\tilde{F}(s)\|^p \, ds \right] \\
&\leq c_p \left[\|F(\lfloor x \rfloor + \tau) - TF(\lfloor x \rfloor)\|_Y^p + \ldots + \|F(\lfloor x \rfloor + l + \tau) - TF(\lfloor x \rfloor + l)\|_Y^p \right] \\
&+ c_p \cdot \left[\|F(\lfloor x \rfloor + \tau) - TF(\lfloor x \rfloor)\|_Y^p + \ldots + \|F(\lfloor x \rfloor + \tau + l + 1) - TF(\lfloor x \rfloor + l + 1)\|_Y^p \right] \\
&\leq 3c_p \sup_{j \geq a} \sum_{k=j}^{j+l} \|F(k+\tau) - TF(k)\|^p \leq 3c_p \epsilon^p [\mathbb{F}(l)]^{-1},
\end{aligned}$$

where $c_p > 0$ is a finite real constant. This proves (3) and completes the proof in this case. The consideration is quite similar in the case that $\Lambda_e = \mathbb{R}$. □

Remark 3. *It is also possible to assume that $\Lambda'_{j_0} := (-\infty, a]$ for some $a \in \mathbb{Z}$ and $j_0 \in \mathbb{N}$, but then we must replace the set $\Omega = [0,1]^n$ with the direct product of the sets $\Omega_j = [0,1]$ or $\Omega_j = [-1,0]$ for $1 \leq j \leq n$, with the obvious choice $\Omega_{j_0} = [-1,0]$.*

Keeping in mind Proposition 4, Theorem 4, Remark 3 and the construction given in the proof of Theorem 1, we can formulate the following:

Corollary 2. *Suppose that $F : \Lambda \to Y$ is a given sequence, $1 \leq p < +\infty$, $\Lambda' = \Lambda''$ and $\rho = \mathrm{I}$. Suppose, further, that for each $j \in \mathbb{N}_n$ we have $\Lambda'_j := [a, \infty)$ ($\Lambda'_j := (-\infty, a]$) for some $a \in \mathbb{Z}$ or $\Lambda'_j = \mathbb{R}$, and $F(l, J) \equiv l^{-n/p}$ for all $l \in \mathbb{N}$ and $J \in P_l$. Define $\Omega := \Omega_1 \times \ldots \times \Omega_n$, where $\Omega_j = [0, 1]$ if $\Lambda'_j = [a, \infty)$ and $\Omega_j = [-1, 0]$ if $\Lambda'_j = (-\infty, a]$ for some $a \in \mathbb{Z}$ ($1 \leq j \leq n$). If $F(\cdot)$ is equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, p, \rho)$-almost periodic, then the mean value*

$$M(F) := \lim_{T \to +\infty} \frac{1}{T^n} \sum_{\mathbf{t} \in (\mathbf{s} + T\Omega) \cap \mathbb{Z}^n} F(\mathbf{t}) \tag{4}$$

exists uniformly on $\mathbf{s} \in \Lambda$.

Proof. Without loss of generality, we may assume that $\Lambda_e = \mathbb{R}^n$. Let the function $\tilde{F}(\cdot; \cdot)$ be given by Theorem 4; then we know that the mean value

$$M(\tilde{F}) := \lim_{T \to +\infty} \frac{1}{T^n} \int_{\mathbf{s} + T\Omega} \tilde{F}(\mathbf{t}) \, d\mathbf{t},$$

exists uniformly on $\mathbf{s} \in [0, \infty)^n$; cf. the proof of ([6] Theorem 6.3.32) and ([6] Remark 6.3.33). Keeping in mind the way of construction of $\tilde{F}(\cdot)$, this implies the required conclusion after a simple computation involving the boundedness of sequence $F(\cdot)$. $\square$

Remark 4. *In contrast with the statements of Theorems 1 and 3, it is very difficult to state a satisfactory converse in Theorem 4 for the corresponding Weyl (Doss) class. For example, due to the conclusions established in ([15] Example 4), we know that there exists an infinitely differentiable Stepanov-1-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ such that the sequence $(f(k))_{k \in \mathbb{Z}}$ is unbounded and the sequence $(f(k + (1/2)))_{k \in \mathbb{Z}}$ is almost periodic. Due to Proposition 4, it follows that the sequence $(f(k))_{k \in \mathbb{Z}}$ cannot be equi-Weyl-almost periodic, i.e., equi-Weyl-$(\mathbb{Z}, l^{-1}, 1, \mathrm{I})$-almost periodic.*

For the sequel, let us recall that A. Iwanik has investigated the equi-Weyl-1-almost periodic sequences with values in compact metric spaces ([29]). We would like to point out that the statement of ([29] Lemma 1) holds for an arbitrary equi-Weyl-1-almost periodic sequence $g : \mathbb{Z} \to X$ such that $R(g)$ is contained in a compact convex subset of X as well as that the assumption that $R(g)$ is a relatively compact subset of X is slightly redundant in our framework. In the present situation, the best we can do is to state and prove the following extension of ([29] Lemma 1):

Proposition 6. *Suppose that $F : \Lambda \to Y$ is a given sequence such that $R(F) \subseteq K$ for some compact convex subset K of Y, $1 \leq p < +\infty$, $\Lambda' = \Lambda''$ and $\rho = \mathrm{I}$. Suppose, further, that $F(l, J) \equiv l^{-n/p}$ for all $l \in \mathbb{N}$ and $J \in P_l$. If $F(\cdot)$ is equi-Weyl-$(\Lambda', \mathbb{F}., p, \rho)$-almost periodic, then for each $\epsilon > 0$ there exist a Bohr almost periodic function $H : \Lambda \to Y$ with values in K and an integer $l \in \mathbb{N}$ such that, for every $J \in P_l$, we have*

$$l^{-n/p} \left[\sum_{j \in J} \| F(j; x) - H(j; x) \|^p \right]^{1/p} \leq \epsilon. \tag{5}$$

Proof. We will outline the main details of the proof only. If $\Lambda'_{j_0} := (-\infty, a]$ ($\Lambda'_{j_0} = [a, +\infty)$) for some $a \in \mathbb{Z}$, then we set $\Omega_j = [-1, 0]$ ($\Omega_j = [0, 1]$); if $\Lambda'_{j_0} := \mathbb{R}$, then we set $\Omega_j = [-1, 1]$; cf. also Remark 3. Set $\Omega := \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$. Let $\epsilon > 0$ be given. Then, we know that there exist $l \in \mathbb{N}$ and $L \in \mathbb{N}$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$, which satisfies that, for every $J \in P_l$ and for every $j \in J$, (2) holds with $z_{j,x} = F(j; x)$. We write the region Λ as a countable union of the closed rectangles $(\Lambda_j)_{j \in \mathbb{N}}$, which are translations of the cube $L\Omega$ in $\mathbb{R}^n$. Then, for each $j \in \mathbb{N}$ there exists a point $\tau_j \in \Lambda_j$ such that, for every $J \in P_l$ and for every $j \in J$, (2) holds with $z_{j,x} = F(j;x)$ and $\tau = \tau_j$. It is clear that the set $J = \{\tau_j : j \in \mathbb{N}\}$ is syndetic in Λ, with the meaning clear. Define $J_k := J \cap k\Omega$ for all $k \in \mathbb{N}$. Let a point $j \in \Lambda$ be fixed. Then, any member of the sequence $(|J_{kl}|^{-1} \sum_{\mathbf{t} \in J_{kl}} F(j+\mathbf{t}))_{k \in \mathbb{N}}$ belongs to K since $R(F) \subseteq K$ and K is convex. Since K is a compact subset of X, we obtain the existence of a strictly increasing sequence (k_m) of positive integers such that

$$\lim_{m \to +\infty} \frac{1}{|J_{k_m l}|} \sum_{\mathbf{t} \in J_{k_m l}} F(j+\mathbf{t}) =: H(j)$$

exists in K. Keeping in mind that

$$\liminf_{m \to +\infty}(a_m + b_m) \geq \liminf_{m \to +\infty} a_m + \liminf_{m \to +\infty} b_m$$

for any two sequences (a_m) and (b_m) of positive real numbers and the well-known inequality between the means

$$\left(\frac{a_1 + \ldots + a_m}{m}\right)^p \leq \frac{a_1^p + \ldots + a_m^p}{m}, \quad m \in \mathbb{N};\ a_j \geq 0,\ 1 \leq j \leq m,$$

we can argue in the same way as in [29] to conclude that the function $H : \Lambda \to Y$ is Bohr almost periodic and satisfies the required properties. $\square$

Remark 5. *The foregoing argumentation shows that, for every equi-Weyl-p-almost periodic sequence $g : \mathbb{Z} \to X$, there exists a uniformly continuous equi-Weyl-p-almost periodic function $\tilde{g} : \mathbb{R} \to X$ such that $\tilde{g}(t) = g(t)$ for all $t \in \mathbb{Z}$ as well as that $\tilde{g}(t) \in CH(R(g))$, $t \in \mathbb{R}$ $(1 \leq p < +\infty)$. Then, we can argue as in the proof of ([6] Theorem 6.3.23) in order to see that for each $\epsilon > 0$ there exists an almost periodic function $h : \mathbb{R} \to X$ such that $R(h) \subseteq \overline{CH(R(g))}$ and $D_W(\tilde{g}, h) < \epsilon$, where $D_W(\cdot;\cdot)$ denotes the Weyl distance of functions. However, it is not clear how to prove that the last estimate implies that for each $\epsilon > 0$ there exists $l > 0$ such that*

$$D_{S_l}^p\big((\tilde{g}(k))_{k \in \mathbb{Z}}, (h(k))_{k \in \mathbb{Z}}\big) := \sup_{k \in \mathbb{Z}} \frac{1}{l} \sum_{j=k}^{k+l-1} \|\tilde{g}(j) - h(j)\|^p < \epsilon.$$

Of course, if $g : \mathbb{Z} \to X$ is an almost periodic sequence, then we have $\|\tilde{g}(t) - h(t)\| < \epsilon$ for all $t \in \mathbb{R}$; the same result can be clarified for the almost periodic sequences $g : \mathbb{Z}^n \to X$, thus providing an extension of ([17] Fundamental Theorem II, p. 319) to the higher-dimensional setting.

The converse statement in Proposition 6 can be proved using a simple argumentation and the decomposition

$$\begin{aligned}&\|F(\mathbf{t}+\tau) - F(\mathbf{t})\|_Y \\ &\leq \|F(\mathbf{t}+\tau) - H(\mathbf{t}+\tau)\|_Y + \|H(\mathbf{t}+\tau) - H(\mathbf{t})\|_Y + \|H(\mathbf{t}) - F(\mathbf{t})\|_Y,\ \mathbf{t} \in \Lambda,\ \tau \in \Lambda' :\end{aligned}$$

Proposition 7. *Suppose that $F : \Lambda \to Y$ is a given sequence, $1 \leq p < +\infty$, $\Lambda' = \Lambda''$ and $\rho = \mathrm{I}$. Suppose, further, that $F(l, J) \equiv l^{-n/p}$ for all $l \in \mathbb{N}$ and $J \in P_l$. If for each $\epsilon > 0$ there exist a Bohr almost periodic function $H : \Lambda \to Y$ and an integer $l \in \mathbb{N}$ such that, for every $J \in P_l$, we have (5), then $F(\cdot)$ is equi-Weyl-$(\Lambda', \mathbb{F}., p, \rho)$-almost periodic.*

Since the sum of two compact (convex) subsets of Y is likewise a compact (convex) subset of Y, combining Propositions 6 and 7, we obtain:

Proposition 8. *Denote by $e - W^{p,\Lambda'}_{ap;cc}(\Lambda : Y)$ the collection of all equi-Weyl-$(\Lambda', \mathbb{F}., p, \rho)$-almost periodic sequences such that $F(l, J) \equiv l^{-n/p}$ for all $l \in \mathbb{N}$ and $J \in P_l$, $1 \leq p < +\infty$, $\Lambda' = \Lambda''$, $\rho = \mathrm{I}$ and $R(F)$ is contained in a compact convex subset of Y. Then, $e - W^{p,\Lambda'}_{ap;cc}(\Lambda : Y)$ is a vector space with the usual operations.*

Remark 6. *Suppose that Y is a finite-dimensional space and the assumptions of Proposition 7 hold. Since the convex hull of a compact subset K of Y is compact, Proposition 4 implies that $F(\cdot)$ is equi-Weyl-$(\Lambda', \mathbb{F}., p, \rho)$-almost periodic if and only if for each $\epsilon > 0$ there exist a Bohr almost periodic function $H : \Lambda \to Y$ and an integer $l \in \mathbb{N}$ such that, for every $J \in P_l$, we have (5). If this is the case, then $F(\cdot)$ is Besicovitch-$(l^{-n/p}, p)$-almost periodic in the sense of Definition 8 below.*

In connection with Proposition 8 and Remark 6, we would like to ask the following question (the interested reader may also try to formulate an analogue of ([29] Lemma 3) in our framework):

Proposition 9. *Denote by $e - W^{p,\Lambda'}_{ap}(\Lambda : Y)$ the collection of all equi-Weyl-$(\Lambda', \mathbb{F}., p, \rho)$-almost periodic sequences such that $F(l, J) \equiv l^{-n/p}$ for all $l \in \mathbb{N}$ and $J \in P_l$, $1 \leq p < +\infty$, $\Lambda' = \Lambda''$ and $\rho = \mathrm{I}$. Is it true that $e - W^{p,\Lambda'}_{ap}(\Lambda : Y)$ is a vector space with the usual operations? Furthermore, is it true that the equivalence relation clarified in Remark 6 holds if the space Y is infinite-dimensional?*

Now, we will introduce the class of Besicovitch-$(\mathcal{B}, \mathbb{F}, p)$-almost periodic sequences:

Definition 8. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $\mathbb{F} : (0, \infty) \to [0, \infty)$ and $1 \leq p < +\infty$. Then, we say that $F(\cdot;\cdot)$ is Besicovitch-$(\mathcal{B}, \mathbb{F}, p)$-almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a trigonometric polynomial $P(\cdot;\cdot)$ such that*

$$\limsup_{l \to +\infty} \mathbb{F}(l) \sup_{x \in B} \left[\sum_{j \in [-l,l]^n \cap \Lambda} \left\| F(j;x) - P(j;x) \right\|^p \right]^{1/p} < \epsilon.$$

If $\mathbb{F}(l) \equiv l^{-n/p}$, then we omit the term "$\mathbb{F}$" from the notation.

Since $\limsup_{l \to +\infty} \cdot$ is sub-additive, it follows that the set of all Besicovitch-$(\mathcal{B}, \mathbb{F}, p)$-almost periodic sequences is a vector space with the usual operations. The usual example of a Besicovitch-p-almost periodic sequence ($\Lambda = \mathbb{Z}^n$, $X = \{0\}$, $\mathbb{F}(l) \equiv l^{-n/p}$) is obtained in the one-dimensional framework by taking the Fourier coefficients of a complex Borel measure on the unit circle (see ([17] p. 315)).

The following results can be established for the Besicovitch class (Corollary 3(ii) can be deduced using the argumentation contained in the proof of ([17] Lemma 3.4(1))):

Theorem 5. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $\mathbb{F} : (0, \infty) \to [0, \infty)$, $1 \leq p < +\infty$ and $\Lambda' \subseteq \Lambda''$. If $F(\cdot;\cdot)$ is Besicovitch-$(\mathcal{B}, \mathbb{F}, p)$-almost periodic, where for each $j \in \mathbb{N}_n$ we have $\Lambda'_j := [a, \infty)$ for some $a \in \mathbb{Z}$ or $\Lambda'_j = \mathbb{R}$, then there exists a continuous function $\tilde{F} : \Lambda_e \times X \to Y$ such that $\tilde{F} \in e - (\mathcal{B}, x, \mathbb{F}) - B^p(\Lambda_e \times X : Y)$ and $\tilde{F}(\mathbf{t}; x) = F(\mathbf{t}; x)$ for all $\mathbf{t} \in \Lambda$ and $x \in X$.*

Corollary 3. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $1 \leq p < +\infty$ and $\Lambda' = \Lambda''$. Suppose, further, that for each $j \in \mathbb{N}_n$ we have $\Lambda'_j := [a, \infty)$ ($\Lambda'_j := (-\infty, a]$) for some $a \in \mathbb{Z}$ or $\Lambda'_j = \mathbb{R}$, and $\mathbb{F}(l) \equiv l^{-n/p}$ for all $l > 0$. Define $\Omega := \Omega_1 \times \ldots \times \Omega_n$, where $\Omega_j = [0, 1]$ if $\Lambda'_j = [a, \infty)$ and $\Omega_j = [-1, 0]$ if $\Lambda'_j = (-\infty, a]$ for some $a \in \mathbb{Z}$ ($1 \leq j \leq n$). If $F(\cdot)$ is Besicovitch-$(\mathcal{B}, \mathbb{F}, p)$-almost periodic, then the following holds:*

(i) *The set $\{F(\mathbf{t};x) : \mathbf{t} \in \Lambda,\ x \in B\}$ is Besicovitch-p-bounded for each bounded subset B of the collection $\mathcal{B}$, i.e.,*

$$\limsup_{l \to +\infty} \frac{1}{l^n} \sup_{x \in B} \sum_{\mathbf{t} \in [-l,l]^n \cap \Lambda} \|F(\mathbf{t};x)\|^p < +\infty.$$

(ii) *If $X = \{0\}$, then the mean value $M(F)$, given by (4), exists uniformly on $\mathbf{s} \in \Lambda$.*

It is worth noting that, besides the mean value $M(F)$, we can also define the Bohr–Fourier coefficients of $F(\cdot)$ following our approach; cf. ([17] Lemma 3.4(1)). We ought to observe that the proofs of ([17] Lemma 3.11: (1)(b); (2)) are not correct: speaking-matter-of-factly, the argumentation given in the cited monograph ([1] pp. 107–109) of A. S. Besicovitch only shows that, for a given Besicovitch-p-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ and a given number $\epsilon > 0$, we have the existence of a sufficiently large positive real number $t_0(\epsilon) > 0$ and a corresponding Bochner–Fejér trigonometric polynomial $\sigma_B^f(\cdot)$ such that

$$\int_{-t}^{t} \left\|f(s) - \sigma_B^f(s)\right\|^p ds \leq 2\epsilon^p t, \quad t \geq t_0(\epsilon).$$

However, it is not clear why the last inequality would imply the existence of an integer $k_0(\epsilon) \in \mathbb{N}$ such that

$$\sum_{j=-k}^{k} \left\|f(j) - \sigma_B^f(j)\right\|^p \leq 2\epsilon^p k, \quad k \geq k_0(\epsilon),$$

even if the all above terms are well-defined and $f(\cdot)$ is continuous. Therefore, it is clear that we must follow another approach in the discrete setting.

Remark 7. *In the question ([19] (Q4)), we have asked the following:*

Is it true that the sequence $(y_k)_{k \in \mathbb{Z}}$ [$(y_k)_{k \in \mathbb{N}}$] is (equi-)Weyl-p-almost periodic (Doss-p-almost periodic/Besicovitch-p-almost periodic) ($1 \leq p < \infty$) if and only if there exists a continuous (equi-)Weyl-p-almost periodic (Doss-p-almost periodic/Besicovitch-p-almost periodic) function $f : \mathbb{R} \to X$ ($f : [0,\infty) \to X$) such that $y_k = f(k)$ for all $k \in \mathbb{Z}$ [$k \in \mathbb{N}$] (cf. the notion introduced above with $n = 1$, $\mathbb{F}.(l) \equiv l^{-1/p}$ and $\rho = \mathrm{I}$)?

In Theorems 4 and 5, we have proved the existence of a continuous (equi-)Weyl-p-almost periodic (Doss-p-almost periodic/Besicovitch-p-almost periodic) function $f(\cdot)$ obeying the required properties. On the other hand, in Remark 4, we have shown that the converse statement is not true for the class of equi-Weyl-p-almost periodic sequences; it seems very plausible that the same statement is not true for the classes of Doss-p-almost periodic sequences and Besicovitch-p-almost periodic sequences.

Concerning the completeness of the space of Besicovitch-$(\mathcal{B}, p)$-almost periodic sequences, denoted here simply by P, we will only state the following direct consequence of ([28] Theorem 2.3), which provides a discrete analogue of the famous result established by J. Marcinkiewicz in [30]:

Theorem 6. *Suppose that $1 \leq p < +\infty$, $\Lambda' = \Lambda''$ and for each $j \in \mathbb{N}_n$ we have $\Lambda'_j := [a,\infty)$ ($\Lambda'_j := (-\infty,a]$) for some $a \in \mathbb{Z}$ or $\Lambda'_j = \mathbb{R}$, and $\mathbb{F}(l) \equiv l^{-n/p}$ for all $l > 0$. Define $\Omega := \Omega_1 \times \ldots \times \Omega_n$, where $\Omega_j = [0,1]$ if $\Lambda'_j = [a,\infty)$ and $\Omega_j = [-1,0]$ if $\Lambda'_j = (-\infty,a]$ for some $a \in \mathbb{Z}$ ($1 \leq j \leq n$). Then, for every bounded set B of the collection $\mathcal{B}$, we have that (P, d_B) is a complete pseudometric space, where*

$$d_B(F,G) := \limsup_{l \to +\infty} l^{-n/p} \sup_{x \in B} \left[\sum_{j \in [-l,l]^n \cap \Lambda} \left\|F(j) - G(j)\right\|^p\right]^{1/p}, \quad F,\ G \in P.$$

Before proceeding to the next section, we will only note that the statements of ([28] Propositions 1 and 2) can be formulated in the discrete setting; cf. also ([17] Lemma 3.1), which can be formulated if one of the corresponding sequences $\mathbf{b}(\cdot)$ or $\mathbf{c}(\cdot)$ is vector-valued. Details can be left to the interested readers.

4. Applications to the Abstract Volterra Integro-Difference Equations

In this section, we will consider certain applications of the introduced notion to the abstract Volterra integro-difference equations. Before doing this, we would like to notice that we have recently analyzed the existence and uniqueness of Weyl, Besicovitch and Doss almost periodic-type solutions to the abstract impulsive Volterra integro-differential equations in [19]. Concerning the statement of ([19] Theorem 8), we would like to make the following comment: Let us replace the condition (ew-M1), resp., (w-M1), in the formulation of this result with the following condition:

(ew-M1-T) For every $\epsilon > 0$, there exist $s \in \mathbb{N}$ and $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I'$, which satisfies that there exists an integer $q_\tau \in \mathbb{N}$ such that $|t_{i+q_\tau} - t_i - \tau| < \epsilon$ for all $i \in \mathbb{N}$ and

$$\sup_{|J|=s} \left[\frac{1}{s} \sum_{j \in J} \left\| y_{j+q_\tau} - T y_j \right\|^p \right]^{1/p} < \epsilon, \tag{6}$$

where the supremum is taken over all segments $J \subseteq \mathbb{N}$ of length s and $\rho = T \in L(X)$.

(w-M1-T) For every $\epsilon > 0$, there exists $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I'$, which satisfies that there exist an integer $q_\tau \in \mathbb{N}$ and an integer $s_\tau \in \mathbb{N}$ such that $|t_{i+q_\tau} - t_i - \tau| < \epsilon$ for all integers $i \in \mathbb{N}$ and (6) holds for all integers $s \geq s_\tau$, with $\rho = T \in L(X)$.

Then, the function $G_2 : [0, \infty) \to X$ from the formulation of the above-mentioned result will be (equi-)Weyl-(p, T)-almost periodic (see [6] for the notion); a similar comment can be made in the case of consideration of ([19] Theorem 9). Observe, finally, that it would be very difficult to say anything relevant if the term $1/s$ in (6) is replaced with the term $1/s^\sigma$, where $\sigma \in (0, 1)$.

In connection with our results established in [19], we will also provide the following illustrative example:

Example 2. *Suppose that the family of sequences $(t_k^j)_{k \in \mathbb{Z}}$, $j \in \mathbb{Z}$ is equipotentially almost periodic. Then, we know that there exist $\zeta \in \mathbb{R} \setminus \{0\}$ and an almost periodic function $a : \mathbb{R} \to \mathbb{R}$ such that $t_k = \zeta k + a(k)$ for all $k \in \mathbb{Z}$. The function $g(x) := \zeta x + a(x)$, $x \in \mathbb{R}$ has the property that the function $x \mapsto f(x) = g(x) - \zeta x$, $x \in \mathbb{R}$ is almost periodic; then, we can apply the theorem of H. Bohr concerning the argument of a complex-valued almost periodic function in order to see that the function $x \mapsto \exp(ig(x))$, $x \in \mathbb{R}$ is almost periodic. This, in particular, implies that $(e^{it_k})_{k \in \mathbb{Z}}$ is an almost periodic sequence. Of course, the converse statement is not true since $(e^{i[2\pi k^2]})_{k \in \mathbb{Z}}$ is an almost periodic sequence, but the sequence $(t_k \equiv 2\pi k^2)_{k \in \mathbb{Z}}$ does not satisfy that the family of sequences $(t_k^j)_{k \in \mathbb{Z}}$, $j \in \mathbb{Z}$ is equipotentially almost periodic.*

It would be very difficult to summarize all relevant results concerning the almost periodic type solutions to the abstract Volterra integro-difference equations. For more details on the subject, we refer the reader to the research monographs [31] by R. P. Agarwal, [32] by R. P. Agarwal, C. Cuevas, C. Lizama, [33] by S. Elaydi as well as to the research articles [15,34–45]; cf. also the references quoted in ([5] Section 3.11) and ([6] p. 303).

We will divide the remainder of this section into three individual subsections.

4.1. On the Abstract Difference Equation $u(k+1) = Au(k) + f(k)$

In ([36] Section 3), D. Araya, R. Castro and C. Lizama have considered the almost automorphic solutions of the first-order linear difference equation

$$u(k+1) = Au(k) + f(k), \quad k \in \mathbb{Z}, \tag{7}$$

where $A \in L(X)$ and $(f_k \equiv f(k))_{k\in\mathbb{Z}}$ is an almost automorphic sequence. In this subsection, we will expand the above-mentioned research study by assuming that $(f_k)_{k\in\mathbb{Z}}$ is a generalized almost periodic sequence.

We will first assume that $A = \lambda \mathrm{I}$, where $\lambda \in \mathbb{C}$ and $|\lambda| \neq 1$. Due to ([36] Theorem 3.1), we know that the almost automorphy of sequence $(f_k)_{k\in\mathbb{Z}}$ implies the existence of an almost automorphic solution $u(\cdot)$ of (7), which is given by

$$u(k) = \sum_{m=-\infty}^{k} \lambda^{k-m} f(k-1), \quad k \in \mathbb{Z}, \tag{8}$$

if $|\lambda| < 1$, and

$$u(k) = -\sum_{m=k}^{\infty} \lambda^{k-m-1} f(k), \quad k \in \mathbb{Z}, \tag{9}$$

if $|\lambda| > 1$. Before proceeding any further, we would like to note that this is a unique almost automorphic solution of (7); in actual fact, any almost automorphic sequence is bounded and we only need to show the uniqueness of bounded solutions of (7). However, this can be proved in an almost trivial way; furthermore, there exists a unique polynomially bounded solution of (8).

We will first prove the following theorem:

Theorem 7. *Suppose that $\mathbb{F} : (0, \infty) \to (0, \infty)$, $1 \leq p < +\infty$, $\rho = T \in L(X)$, (FV) holds and $f(\cdot)$ is equi-Weyl-$(\mathbb{F}, p, T)$-almost periodic (polynomially bounded Weyl-$(\mathbb{F}, p, T)$-almost periodic; polynomially bounded Doss-$(\mathbb{F}, p, T)$-almost periodic). Then, a unique (equi-)Weyl-$(\mathbb{F}, p, T)$-almost periodic (polynomially bounded Weyl-$(\mathbb{F}, p, T)$-almost periodic; polynomially bounded Doss-$(\mathbb{F}, p, T)$-almost periodic) solution of (7) is given by (8) if $|\lambda| < 1$, and (9) if $|\lambda| > 1$.*

Proof. We will consider the class of equi-Weyl-$(\mathbb{F}, p, T)$-almost periodic sequences, only. Due to Proposition 4, we know that $(f_k)_{k\in\mathbb{Z}}$ is a bounded sequence and the above argument shows that we only need to prove that the function $u(\cdot)$, given by (8) if $|\lambda| < 1$, and (9) if $|\lambda| > 1$, is an (equi-)Weyl-$(\mathbb{F}., p, T)$-almost periodic solution of (7). Clearly, the function $u(\cdot)$ is well-defined and bounded; for simplicity, we will consider henceforth the case $|\lambda| < 1$, only. Let $\epsilon > 0$ be given. Then, we know that there exist $l \in \mathbb{N}$ and $L > 0$ such that, for every $t_0 \in \mathbb{Z}$, there exists $\tau \in B(t_0, l) \cap \mathbb{Z}$ such that

$$\sup_{k\in\mathbb{Z}} \mathbb{F}(l) \left[\sum_{j=k}^{k+l} \left\| f(j+\tau) - Tf(j) \right\|^p \right]^{1/p} \leq \epsilon. \tag{10}$$

Let $k \in \mathbb{Z}$ be fixed. Then, we have:

$$\begin{aligned}
&\mathbb{F}(l)\left[\sum_{j=k}^{k+l}\|u(j+\tau)-Tu(j)\|^{p}\right]^{1/p} \\
&= \mathbb{F}(l)\left[\sum_{j=k}^{k+l}\left\|\sum_{v=0}^{\infty}\lambda^{v}\left[f(j+\tau-v-1)-Tf(j-v-1)\right]\right\|^{p}\right]^{1/p} \\
&\leq \mathbb{F}(l)\left[\sum_{j=k}^{k+l}\left|\sum_{v=0}^{\infty}|\lambda|^{v}\|f(j+\tau-v-1)-Tf(j-v-1)\|\right|^{p}\right]^{1/p}.
\end{aligned}$$

Suppose now that $\zeta > 1/p$. Using the Hölder inequality, we obtain that there exists a finite real constant $c_{\lambda} > 0$ such that

$$\left|\sum_{v=0}^{\infty}|\lambda|^{v}\|f(j+\tau-v-1)-Tf(j-v-1)\|\right|^{p} \leq c_{\lambda}^{p}\sum_{v=0}^{\infty}\frac{1}{(1+v^{\zeta})^{p}}\|f(j+\tau-v-1)-Tf(j-v-1)\|^{p},$$

so that the last estimate in the above calculation and (10) together imply that

$$\begin{aligned}
&\mathbb{F}(l)\left[\sum_{j=k}^{k+l}\|u(j+\tau)-Tu(j)\|^{p}\right]^{1/p} \\
&\leq c_{\lambda}\mathbb{F}(l)\left[\sum_{j=k}^{k+l}\sum_{v=0}^{\infty}\frac{1}{(1+v^{\zeta})^{p}}\|f(j+\tau-v-1)-Tf(j-v-1)\|^{p}\right]^{1/p} \\
&= c_{\lambda}\mathbb{F}(l)\left[\sum_{v=0}^{\infty}\frac{1}{(1+v^{\zeta})^{p}}\sum_{j=k}^{k+l}\|f(j+\tau-v-1)-Tf(j-v-1)\|^{p}\right]^{1/p} \\
&\leq c_{\lambda}\mathbb{F}(l)\left[\sum_{v=0}^{\infty}\frac{1}{(1+v^{\zeta})^{p}}\left(\epsilon/\mathbb{F}(l)\right)^{p}\right]^{1/p} = \epsilon c_{\lambda}\left[\sum_{v=0}^{\infty}\frac{1}{(1+v^{\zeta})^{p}}\right]^{1/p},
\end{aligned}$$

finishing the proof. □

In general case, we have the following result:

Theorem 8. *Suppose that $\mathbb{F}:(0,\infty)\rightarrow(0,\infty)$, $1 \leq p < +\infty$, $\rho = T \in L(X)$, (FV) holds and $f(\cdot)$ is equi-Weyl-$(\mathbb{F},p,T)$-almost periodic (polynomially bounded Weyl-$(\mathbb{F},p,T)$-almost periodic; polynomially bounded Doss-$(\mathbb{F},p,T)$-almost periodic). Then, there exists an (equi-)Weyl-$(\mathbb{F},p,T)$-almost periodic (polynomially bounded Weyl-$(\mathbb{F},p,T)$-almost periodic; polynomially bounded Doss-$(\mathbb{F},p,T)$-almost periodic) solution of (7), provided that $A \in L(X)$ and $\|A\| < 1$.*

Keeping in mind Theorem 7, the statement of ([36] Theorem 3.2) can be simply reformulated for the generalized almost periodic solutions considered in this paper. For simplicity, we will only state the following result:

Theorem 9. *Suppose that A is a complex matrix, which satisfies that its point spectrum is disjoint from the unit sphere S_1. Suppose, further, that $\mathbb{F}:(0,\infty)\rightarrow(0,\infty)$, $1 \leq p < +\infty$, (FV) holds and $f(\cdot)$ is equi-Weyl-$(\mathbb{F},p,\mathrm{I})$-almost periodic. Then, there exists a unique equi-Weyl-$(\mathbb{F},p,\mathrm{I})$-almost periodic solution of (7).*

We can similarly prove the analogues of Theorems 7–9 for the generalized Besicovitch almost periodic solutions of (8). For the sake of brevity, we will only state the following analogue of Theorem 7 here:

Theorem 10. *Suppose that* $\mathbb{F} : (0, \infty) \to (0, \infty)$, $1 \leq p < +\infty$, $\mathbb{F}_1 : (0, \infty) \to (0, \infty)$ *and* $f(\cdot)$ *is polynomially bounded Besicovitch-*$(\mathbb{F}, p)$*-almost periodic. Suppose, further, that for each* $\epsilon > 0$ *there exist* $l_0 > 0$, $c > 0$ *and* $k > 0$ *such that, for every* $l \geq l_0$*, we have*

$$\frac{\mathbb{F}_1(l)}{\mathbb{F}(l+v)} \leq c(1+v)^k.$$

Then, a unique polynomially bounded Besicovitch-$(\mathbb{F}_1, p)$*-almost periodic solution of* (7) *is given by* (8) *if* $|\lambda| < 1$*, and* (9) *if* $|\lambda| > 1$.

In this paper, we will not reconsider the statements of ([36] Theorems 3.6–3.9). Concerning the generalized almost periodic type solutions of the following semilinear analogue of (7):

$$u(k+1) = Au(k) + f(k; u(k)), \quad k \in \mathbb{Z},$$

where $A \in L(X)$, we will only note that our already established results can be used to deduce the variants of ([36] Theorems 4.1–4.3) for a class of bounded c-uniformly recurrent sequences (cf. ([20] Theorems 2.28–3.1)) and a class of bounded slowly oscillating sequences (cf. ([23] Theorem 5) and the paragraph following it). We will consider the composition principles for the generalized Weyl, Besicovitch and Doss almost periodic sequences somewhere else (cf. [5–27] for continuous versions).

Before proceeding to the next subsection, we would like to recommend for the readers the doctoral dissertation of M. Veselý [14], where we have located a great number of results about the existence and uniqueness of almost periodic solutions of the abstract difference equation $u(k+1) = A_k u(k) + f(k)$, $k \in \mathbb{Z}$, where $(A_k)_{k\in\mathbb{Z}}$ is a sequence of closed linear operators satisfying certain properties.

4.2. On the Abstract Fractional Difference Equation $\Delta^\alpha u(k) = Au(k+1) + f(k)$

Discrete fractional calculus is a very attractive field of applied mathematics and computation, which is incredibly important in the modeling of various phenomena concerning interval-valued systems, chaotic systems with short memory and image encryption and discrete-time recurrent neural networks. In the recent research article [46], E. Alvarez, S. Díaz and C. Lizama have analyzed the existence and uniqueness of (N, λ)-periodic solutions for the abstract fractional difference equation

$$\Delta^\alpha u(k) = Au(k+1) + f(k), \quad k \in \mathbb{Z}, \tag{11}$$

where A is a closed linear operator on X, $0 < \alpha < 1$ and $\Delta^\alpha u(k)$ denotes the Caputo fractional difference operator of order α; see ([46] Definition 2.3). In this subsection, we will use the same notion and notation as in the above-mentioned paper, providing a few important observations about ([46] Theorem 4.2) only.

Suppose that A is a closed linear operator on X such that $1 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A, and $\|(\mathrm{I} - A)^{-1}\| < 1$. Then, ([46] Theorem 3.4) implies that A generates a discrete (α, α)-resolvent sequence $\{S_{\alpha,\alpha}(v)\}_{v\in\mathbb{N}_0}$ such that $\sum_{v=0}^{+\infty} \|S_{\alpha,\alpha}(v)\| < +\infty$. If $(f_k)_{k\in\mathbb{Z}}$ is a bounded sequence, then we know that the function

$$u(k) = \sum_{l=-\infty}^{k-1} S_{\alpha,\alpha}(k-1-l)f(l), \quad k \in \mathbb{Z} \tag{12}$$

is a mild solution of (11). Since $\sum_{v=0}^{+\infty} \|S_{\alpha,\alpha}(v)\| < +\infty$, we can almost directly conclude that $u(t)$ will be T-almost periodic (T-uniformly recurrent), where $\rho = T \in L(X)$, provided that $(f_k)_{k\in\mathbb{Z}}$ is T-almost periodic (T-uniformly recurrent). Concerning the generalized almost periodic type solutions of (11), we are in a position to directly clarify certain results in the case that (FV) holds and the forcing term $f(\cdot)$ is equi-Weyl-$(\mathbb{F}, 1, T)$-almost periodic (bounded Weyl-$(\mathbb{F}, 1, T)$-almost periodic; bounded Doss-$(\mathbb{F}, 1, T)$-almost periodic; bounded Besicovitch-$(\mathbb{F}, 1)$-almost periodic). Speaking-matter-of-factly, if $f(\cdot)$ enjoys this feature, then a mild solution $u(\cdot)$ of (11), given by (12), enjoys the same feature as well. In order to see this, let us assume that $f(\cdot)$ is equi-Weyl-$(\mathbb{F}, 1, T)$-almost periodic, for example. Then, $f(\cdot)$ is bounded and we have

$$\begin{aligned}
&\mathbb{F}(l) \sum_{j=k}^{k+l} \|u(j+\tau) - Tu(j)\| \\
&\quad = \mathbb{F}(l) \sum_{j=k}^{k+l} \left\| \sum_{v=0} S_{\alpha,\alpha}(v) \left[f(j+\tau-1-v) - Tf(j-1-v) \right] \right\| \\
&\quad \leq \mathbb{F}(l) \sum_{j=k}^{k+l} \sum_{v=0} \|S_{\alpha,\alpha}(v)\| \cdot \|f(j+\tau-1-v) - Tf(j-1-v)\| \\
&\quad = \mathbb{F}(l) \sum_{v=0} \sum_{j=k}^{k+l} \|S_{\alpha,\alpha}(v)\| \cdot \|f(j+\tau-1-v) - Tf(j-1-v)\|,
\end{aligned}$$

which simply implies the required ($k,\ \tau \in \mathbb{Z}; l \in \mathbb{N}$). Unfortunately, the existence of an equi-Weyl-$(\mathbb{F}, p, T)$-almost periodic solution of (11), where $p > 1$, requires further investigations of the solution family $\{S_{\alpha,\alpha}(v)\}_{v \in \mathbb{N}_0}$; basically, the mild solution $u(\cdot)$ of (11), given by (12), enjoys the same feature as the forcing term $f(\cdot)$, provided that $\sum_{v=0}^{\infty} [\|S_{\alpha,\alpha}(v)\|^q \cdot (1 + v^{\zeta})^q] < +\infty$ for some $\zeta > 1/p$, where $1/p + 1/q = 1$ (this follows from our previous considerations from Section 4.1).

Concerning the generalized almost periodic type solutions of the following semilinear analogue of (11):

$$\Delta^{\alpha} u(k) = Au(k+1) + f(k; u(k)), \quad k \in \mathbb{Z},$$

we will only note that an analogue of ([46] Theorem 4.5) can be formulated, e.g., for bounded c-uniformly recurrent sequences and bounded slowly oscillating sequences.

Without going into further details, we will only note that the similar conclusions can be given in the case of consideration of the following class of Volterra difference equations with infinite delay:

$$u(k+1) = \alpha \sum_{l=-\infty}^{k} a(k-l)u(l) + f(k), \quad k \in \mathbb{Z},\ \alpha \in \mathbb{C};$$

cf. ([47] Theorems 3.1 and 3.3).

4.3. Two Multi-Dimensional Analogues of the Abstract Difference Equation $u(k+1) = Au(k) + f(k)$

It is worth noticing that the statements of ([36] Theorems 3.1, 3.2 and 3.5) can be formulated in the multi-dimensional setting. Without going into full details, we will only explain here how one can formulate some multi-dimensional extensions of ([36] Theorem 3.1(i)).

Suppose that $f : \mathbb{Z}^n \to X$, $\lambda_1,\ \lambda_2, \ldots,\ \lambda_n$ are given complex numbers and

$$\max(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|) < 1.$$

Consider the function

$$\begin{aligned}
u(k_1, k_2, \ldots, k_n) &:= \sum_{l_1 \leq k_1, l_2 \leq k_2, \ldots, l_n \leq k_n} \lambda_1^{k_1-l_1} \lambda_2^{k_2-l_2} \cdot \ldots \cdot \lambda_n^{k_n-l_n} f(l_1-1, l_2-1, \ldots, l_n-1) \\
&= \sum_{v_1 \geq 0, v_2 \geq 0, \ldots, v_n \geq 0} \lambda_1^{v_1} \lambda_2^{v_2} \cdot \ldots \cdot \lambda_n^{v_n} f(k_1-v_1-1, k_2-v_2-1, \ldots, k_n-v_n-1)
\end{aligned}$$

defined for any $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Using the same argumentation as in Section 4.1, we can simply prove the following: If (FV) holds and the sequence $f(\cdot)$ is equi-Weyl-$(\Lambda', \mathbb{F}, p, T)$-almost periodic (polynomially bounded Weyl-$(\Lambda', \mathbb{F}, p, T)$-almost periodic; polynomially bounded Doss-$(\Lambda', \mathbb{F}, p, T)$-almost periodic), then the sequence $u(\cdot)$ enjoys the same property as $f(\cdot)$; a similar statement can be deduced for the class of generalized Besicovitch-p-almost periodic sequences ($\rho = T \in L(X)$).

On the other hand, it is very simple to find the form of function $F : \mathbb{Z}^n \to X$ such that

$$u\big(k_1+1, k_2+1, \ldots, k_n+1\big) = \lambda_1\lambda_2\ldots\lambda_n \cdot u\big(k_1, k_2, \ldots, k_n\big) + F\big(k_1, k_2, \ldots, k_n\big),$$

for all $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. More precisely, we have:

$$\begin{aligned}
&u\big(k_1+1, k_2+1, \ldots, k_n+1\big) \\
&= \sum_{l_1 \le k_1+1, l_2 \le k_2+1, \ldots, l_n \le k_n+1} \lambda_1^{k_1+1-l_1}\lambda_2^{k_2+1-l_2} \cdot \ldots \cdot \lambda_n^{k_n+1-l_n} f\big(l_1-1, l_2-1, \ldots, l_n-1\big) \\
&= \sum_{l_2 \le k_2+1, \ldots, l_n \le k_n+1} \lambda_2^{k_2+1-l_2} \cdot \ldots \cdot \lambda_n^{k_n+1-l_n} f\big(k_1, l_2-1, \ldots, l_n-1\big) \\
&+ \lambda_1 \sum_{l_1 \le k_1, l_2 \le k_2+1, \ldots, l_n \le k_n+1} \lambda_1^{k_1+1-l_1}\lambda_2^{k_2+1-l_2} \cdot \ldots \cdot \lambda_n^{k_n+1-l_n} f\big(l_1-1, l_2-1, \ldots, l_n-1\big) \\
&= \sum_{l_2 \le k_2+1, \ldots, l_n \le k_n+1} \lambda_2^{k_2+1-l_2} \cdot \ldots \cdot \lambda_n^{k_n+1-l_n} f\big(k_1, l_2-1, \ldots, l_n-1\big) \\
&+ \lambda_1 \sum_{l_1 \le k_1, l_3 \le k_3+1, \ldots, l_n \le k_n+1} \lambda_1^{k_1-l_1}\lambda_3^{k_3+1-l_3} \cdot \ldots \cdot \lambda_n^{k_n+1-l_n} f\big(l_1-1, k_2, \ldots, l_n-1\big) \\
&+ \lambda_1\lambda_2 \sum_{l_1 \le k_1, l_2 \le k_2, l_3 \le k_3+1, \ldots, l_n \le k_n+1} \lambda_1^{k_1-l_1}\lambda_2^{k_2-l_2}\lambda_3^{k_3+1-l_3} \cdot \ldots \cdot \lambda_n^{k_n+1-l_n} \\
&\times f\big(l_1-1, l_2-1, \ldots, l_n-1\big) \\
&= \ldots .
\end{aligned}$$

In the second approach, we consider the solution $u_j : \mathbb{Z} \to X$ of the equation $u_j(k+1) = \lambda u_j(k) + f_j(k)$, $k \in \mathbb{Z}$, where $f_j(\cdot)$ is a generalized almost periodic sequence ($1 \le j \le n$) and $\lambda \in \mathbb{C}$ satisfies $|\lambda| < 1$. Define $u(k_1, \ldots, k_n) = u_1(k_1) + u_2(k_2) + \ldots + u_n(k_n)$ and $f(k_1, \ldots, k_n) = f_1(k_1) + f_2(k_2) + \ldots + f_n(k_n)$ for all $k_j \in \mathbb{Z}$ ($1 \le j \le n$). Then, we have

$$u\big(k_1+1, \ldots, k_n+1\big) = \lambda u\big(k_1, \ldots, k_n\big) + f\big(k_1, \ldots, k_n\big), \quad \big(k_1, \ldots, k_n\big) \in \mathbb{Z}^n;$$

moreover, the sequence $u(\cdot)$ has a similar almost periodic behavior as the forcing terms $f_j(\cdot)$. For example, if all sequences $f_j(\cdot)$ are equi-Weyl-p-almost periodic in the usual sense, then the sequence $u(\cdot)$ is likewise equi-Weyl-p-almost periodic in the usual sense ($1 \le p < \infty$).

In [41], C. Lizama and L. Roncal have investigated the almost periodicity for semidiscrete equations with the (fractional) Laplacian. It is worth noting that the argument contained in the proof of Theorem 7 can serve one to formulate an analogue of ([41] Theorem 1.5(1)) for the forcing terms $g(t, \cdot)$, which are generalized Weyl almost periodic for each fixed number $t \ge 0$; an extension to the higher dimensions can be also formulated following the consideration given in ([41], Remark 14). We close this section with the observation that we can also analyze the pointwise products of generalized almost periodic sequences and the invariance of generalized almost periodicity under the actions of the infinite convolution products (see, e.g., ([36] Theorem 2.13)).

5. Conclusions and Final Remarks

In this research article, we have examined the class of Bohr ρ-almost periodic-type sequences and several classes of generalized ρ-almost periodic-type sequences of the form $F : I \times X \to Y$, where $\emptyset \ne I \subseteq \mathbb{Z}^n$, X and Y are complex Banach spaces and ρ is a general binary relation on Y. We have provided a great number of structural results, illustrative examples and open problems about the introduced classes of ρ-almost periodic sequences.

Some applications of the obtained results are given to the abstract Volterra integro-difference equations and the abstract impulsive Volterra integro-differential equations.

Let us finally mention some conclusions and final remarks about the introduced notion, some topics not considered here and some perspectives for further expansion of the theory. Before proceeding any further, we would like to emphasize that many other classes of generalized ρ-almost periodic type sequences are introduced and analyzed in our recent research studies [22–25,48,49]. For example, suppose that $F : I \times X \to Y$ and (1) holds. Then, the notion introduced in ([23] Definitions 1–6) can be used to provide the definitions of:

(1) $(S, \mathbb{D}, \mathcal{B})$-asymptotical (ω, ρ)-periodicity of $F(\cdot;\cdot)$;
(2) $(S, \mathcal{B})$-asymptotical $(\omega_j, \rho_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodicity of $F(\cdot;\cdot)$;
(3) $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', \rho)$-almost periodicity of $F(\cdot;\cdot)$;
(4) $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', \rho)$-uniform recurrence of $F(\cdot;\cdot)$;
(5) $\mathbb{D}$-remotely $(\mathcal{B}, I', \rho)$-almost periodicity of $F(\cdot;\cdot)$;
(6) $\mathbb{D}$-remotely $(\mathcal{B}, I', \rho)$-uniform recurrence of $F(\cdot;\cdot)$;
(7) $(\mathbb{D}, \mathcal{B}, \rho)$-slowly oscillating property of $F(\cdot;\cdot)$, and
(8) $(\mathcal{B}, (\mathbb{D}_j, \rho_j)_{j\in\mathbb{N}_n})$-slowly oscillating property of $F(\cdot;\cdot)$.

Furthermore, the notion introduced in ([22] Definition 2.1) can be used to provide the definition of a Levitan almost periodic sequence $F : I \times X \to Y$. The applications of Levitan almost periodic sequences (and the sequences introduced in ([23] Definitions 1–6) mentioned above) to the abstract Volterra integro-difference equations have not been considered in the existing literature by now; we will examine this problematic somewhere else.

Concerning some other topics, we would like to emphasize that we have not considered here the generalized ρ-uniformly recurrent type sequences. More precisely, we can introduce the following notion (compare to Definitions 6 and 7):

Definition 9. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $1 \le p < +\infty$, ρ is a binary relation on Y and $(\tau_k)_{k\in\mathbb{N}}$ is a sequence in Λ'' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$. Then, we say that $F(\cdot;\cdot)$ is:*

(i) equi-Weyl-$(\mathcal{B}, (\tau_k), \mathbb{F}., p, \rho)$-uniformly recurrent if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist $l \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ such that, for every $k \ge k_0$, $J \in P_l$ and for every $j \in J$, $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2) holds with the point τ replaced by the point τ_k therein;

(ii) Weyl-$(\mathcal{B}, (\tau_k), \mathbb{F}., p, \rho)$-uniformly recurrent if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $k_0 \in \mathbb{N}$ such that, for every $k \ge k_0$, there exists an integer $l_k \in \mathbb{N}$ such that, for every $l \ge l_k$, $J \in P_l$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2) holds with the point τ replaced by the point τ_k therein.

Definition 10. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $1 \le p < +\infty$, ρ is a binary relation on Y and $(\tau_k)_{k\in\mathbb{N}}$ is a sequence in Λ'' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$. Then, we say that $F(\cdot;\cdot)$ is Doss-$(\mathcal{B}, (\tau_k), \mathbb{F}., p, \rho)$-uniformly recurrent if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $k_0 \in \mathbb{N}$ such that, for every $k \ge k_0$, there exists an increasing sequence (l_m^k) of positive integers such that, for every $m \in \mathbb{N}$, $J \in P_{l_m^k}$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2) holds with the point τ replaced by the point τ_k and the number l replaced by the number l_m^k therein.*

It is very simple to reformulate Theorem 4 and the conclusion from Remark 3 to the Weyl and Doss generalized classes of ρ-uniformly recurrent-type sequences; for the sake of simplicity, we will skip all details concerning this question here because we have not recalled, in Section 1.1, the definitions of Weyl and Doss classes of ρ-uniformly recurrent-type functions.

Concerning some possibilities for further expansion of the theory, we would like to note that almost automorphic-type sequences are also incredibly important. We will explore multi-dimensional almost automorphic-type sequences and their applications somewhere else. It is our strong belief that the almost periodic-type solutions and the almost automorphic-type solutions of difference equations in several variables will receive considerable attention from authors in the near future.

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