Non determinism through type isomophism

Alejandro Díaz-Caro
LIPN, Université Paris 13, Sorbonne Paris Cité

Gilles Dowek
INRIA – Paris–Rocquencourt

7th LSFA
Rio de Janeiro, September 29-30, 2012

Motivation: Di Cosmo's isomorphisms [Di Cosmo'95]

- $A \wedge B \equiv B \wedge A$
- $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$
- $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$
- $(A \wedge B) \Rightarrow C \equiv A \Rightarrow (B \Rightarrow C)$
- $A \Rightarrow (B \Rightarrow C) \equiv B \Rightarrow (A \Rightarrow C)$
- $A \wedge \mathbf{T} \equiv A$
- $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$
- $\mathbf{T} \Rightarrow A \equiv A$
- $\forall X.\forall Y.A \equiv \forall Y.\forall X.A$
- $\forall X.A \equiv \forall Y.A[Y = X]$
- $\forall X.(A \Rightarrow B) \equiv A \Rightarrow \forall X.B$ if $X \notin FV(A)$
- $\forall X.(A \wedge B) \equiv \forall X.A \wedge \forall X.B$
- $\forall X.\mathbf{T} \equiv \mathbf{T}$
- $\forall X.(A \wedge B) \equiv \forall X.\forall Y.(A \wedge (B[Y = X]))$

Motivation: Di Cosmo's isomorphisms [Di Cosmo'95]

- $A \wedge B \equiv B \wedge A$
- $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$
- $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$
- $(A \wedge B) \Rightarrow C \equiv A \Rightarrow (B \Rightarrow C)$
- $A \Rightarrow (B \Rightarrow C) \equiv B \Rightarrow (A \Rightarrow C)$
- $A \wedge \mathbf{T} \equiv A$
- $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$
- $\mathbf{T} \Rightarrow A \equiv A$
- $\forall X.\forall Y.A \equiv \forall Y.\forall X.A$
- $\forall X.A \equiv \forall Y.A[Y = X]$
- $\forall X.(A \Rightarrow B) \equiv A \Rightarrow \forall X.B$ if $X \notin FV(A)$
- $\forall X.(A \wedge B) \equiv \forall X.A \wedge \forall X.B$
- $\forall X.\mathbf{T} \equiv \mathbf{T}$
- $\forall X.(A \wedge B) \equiv \forall X.\forall Y.(A \wedge (B[Y = X]))$

We want a proof-system where isomorphic propositions have the same proofs

Minimal second order propositional logic

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash \mathbf{t} : B}{\Gamma \vdash \lambda x.\mathbf{t} : A \Rightarrow B} \qquad \frac{\Gamma \vdash \mathbf{t} : A \Rightarrow B \qquad \Gamma \vdash \mathbf{s} : A}{\Gamma \vdash \mathbf{ts} : B}$$

$$\frac{\Gamma \vdash \mathbf{t} : A}{\Gamma \vdash \mathbf{t} : \forall X.A}\ {\scriptstyle X \notin FV(\Gamma)} \qquad \frac{\Gamma \vdash \mathbf{t} : \forall X.A}{\Gamma \vdash \mathbf{t} : A[B/X]}$$

Minimal second order propositional logic

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash \mathbf{t} : B}{\Gamma \vdash \lambda x.\mathbf{t} : A \Rightarrow B} \qquad \frac{\Gamma \vdash \mathbf{t} : A \Rightarrow B \qquad \Gamma \vdash \mathbf{s} : A}{\Gamma \vdash \mathbf{ts} : B}$$

$$\frac{\Gamma \vdash \mathbf{t} : A}{\Gamma \vdash \mathbf{t} : \forall X.A}\ {\scriptstyle X \notin FV(\Gamma)} \qquad \frac{\Gamma \vdash \mathbf{t} : \forall X.A}{\Gamma \vdash \mathbf{t} : A[B/X]}$$

Adding conjunction

$$\frac{\Gamma \vdash \mathbf{t} : A \qquad \Gamma \vdash \mathbf{r} : B}{\Gamma \vdash \langle \mathbf{t}, \mathbf{r} \rangle : A \wedge B}$$

Minimal second order propositional logic

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash \mathbf{t} : B}{\Gamma \vdash \lambda x.\mathbf{t} : A \Rightarrow B} \qquad \frac{\Gamma \vdash \mathbf{t} : A \Rightarrow B \qquad \Gamma \vdash \mathbf{s} : A}{\Gamma \vdash \mathbf{ts} : B}$$

$$\frac{\Gamma \vdash \mathbf{t} : A}{\Gamma \vdash \mathbf{t} : \forall X.A}\ {\scriptstyle X \notin FV(\Gamma)} \qquad \frac{\Gamma \vdash \mathbf{t} : \forall X.A}{\Gamma \vdash \mathbf{t} : A[B/X]}$$

Adding conjunction

$$\frac{\Gamma \vdash \mathbf{t} : A \qquad \Gamma \vdash \mathbf{r} : B}{\Gamma \vdash \langle \mathbf{t}, \mathbf{r} \rangle : A \wedge B}$$

We want $A \wedge B = B \wedge A$

$A \wedge (B \wedge C) = (A \wedge B) \wedge C$

so $\langle \mathbf{t}, \mathbf{r} \rangle = \langle \mathbf{r}, \mathbf{t} \rangle$

$\langle \mathbf{t}, \langle \mathbf{r}, \mathbf{s} \rangle \rangle = \langle \langle \mathbf{t}, \mathbf{r} \rangle, \mathbf{s} \rangle$

Minimal second order propositional logic

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash \mathbf{t} : B}{\Gamma \vdash \lambda x.\mathbf{t} : A \Rightarrow B} \qquad \frac{\Gamma \vdash \mathbf{t} : A \Rightarrow B \qquad \Gamma \vdash \mathbf{s} : A}{\Gamma \vdash \mathbf{ts} : B}$$

$$\frac{\Gamma \vdash \mathbf{t} : A}{\Gamma \vdash \mathbf{t} : \forall X.A}\ {\scriptstyle X \notin FV(\Gamma)} \qquad \frac{\Gamma \vdash \mathbf{t} : \forall X.A}{\Gamma \vdash \mathbf{t} : A[B/X]}$$

Adding conjunction

$$\frac{\Gamma \vdash \mathbf{t} : A \qquad \Gamma \vdash \mathbf{r} : B}{\Gamma \vdash \ \mathbf{t} + \mathbf{r}\ : A \wedge B}$$

We want $A \wedge B = B \wedge A$

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$

so $\langle \mathbf{t}, \mathbf{r} \rangle = \langle \mathbf{r}, \mathbf{t} \rangle$

$$\langle \mathbf{t}, \langle \mathbf{r}, \mathbf{s} \rangle \rangle = \langle \langle \mathbf{t}, \mathbf{r} \rangle, \mathbf{s} \rangle$$

We write

$$\mathbf{t} + \mathbf{r} = \mathbf{r} + \mathbf{t}$$

$$\mathbf{t} + (\mathbf{r} + \mathbf{s}) = (\mathbf{t} + \mathbf{r}) + \mathbf{s}$$

Minimal second order propositional logic

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash \mathbf{t} : B}{\Gamma \vdash \lambda x.\mathbf{t} : A \Rightarrow B} \qquad \frac{\Gamma \vdash \mathbf{t} : A \Rightarrow B \qquad \Gamma \vdash \mathbf{s} : A}{\Gamma \vdash \mathbf{ts} : B}$$

$$\frac{\Gamma \vdash \mathbf{t} : A}{\Gamma \vdash \mathbf{t} : \forall X.A}\ {\scriptstyle X \notin FV(\Gamma)} \qquad \frac{\Gamma \vdash \mathbf{t} : \forall X.A}{\Gamma \vdash \mathbf{t} : A[B/X]}$$

Adding conjunction

$$\frac{\Gamma \vdash \mathbf{t} : A \qquad \Gamma \vdash \mathbf{r} : B}{\Gamma \vdash \mathbf{t} + \mathbf{r} : A \wedge B}$$

We want $A \wedge B = B \wedge A$

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$

so $\langle \mathbf{t}, \mathbf{r} \rangle = \langle \mathbf{r}, \mathbf{t} \rangle$

$$\langle \mathbf{t}, \langle \mathbf{r}, \mathbf{s} \rangle \rangle = \langle \langle \mathbf{t}, \mathbf{r} \rangle, \mathbf{s} \rangle$$

We write

$$\mathbf{t} + \mathbf{r} = \mathbf{r} + \mathbf{t}$$

$$\mathbf{t} + (\mathbf{r} + \mathbf{s}) = (\mathbf{t} + \mathbf{r}) + \mathbf{s}$$

Also $A \Rightarrow (B \wedge C) = (A \Rightarrow B) \wedge (A \Rightarrow C)$ induces

$$\lambda x.(\mathbf{t} + \mathbf{r}) = \lambda x.\mathbf{t} + \lambda x.\mathbf{r}$$

$$(\mathbf{t} + \mathbf{r})\mathbf{s} = \mathbf{ts} + \mathbf{rs}$$

What about $\wedge$-elimination?

$$\frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : A \wedge B}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : A}$$

What about $\wedge$-elimination?

$$\frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : A \wedge B}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : A} \qquad \text{But } A \wedge B = B \wedge A \text{ !!} \qquad \frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : B \wedge A}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : B}$$

Moreover

$\mathbf{t} + \mathbf{r} = \mathbf{r} + \mathbf{t}$ so $\pi_1(\mathbf{t} + \mathbf{r}) = \pi_1(\mathbf{r} + \mathbf{t})$!!

What about $\wedge$-elimination?

$$\frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : A \wedge B}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : A} \qquad \text{But } A \wedge B = B \wedge A \text{ !!} \qquad \frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : B \wedge A}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : B}$$

Moreover
$\mathbf{t} + \mathbf{r} = \mathbf{r} + \mathbf{t}$ so $\pi_1(\mathbf{t} + \mathbf{r}) = \pi_1(\mathbf{r} + \mathbf{t})$!!

Workaround: Church-style. Project w.r.t. a type

$$\text{If} \quad \Gamma \vdash \mathbf{t} : A \quad \text{then} \quad \pi_A(\mathbf{t} + \mathbf{r}) \to \mathbf{t}$$

What about $\wedge$-elimination?

$$\frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : A \wedge B}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : A} \qquad \text{But } A \wedge B = B \wedge A \text{ !!} \qquad \frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : B \wedge A}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : B}$$

Moreover

$\mathbf{t} + \mathbf{r} = \mathbf{r} + \mathbf{t}$ so $\pi_1(\mathbf{t} + \mathbf{r}) = \pi_1(\mathbf{r} + \mathbf{t})$!!

Workaround: Church-style. Project w.r.t. a type

$$\text{If} \quad \Gamma \vdash \mathbf{t} : A \quad \text{then} \quad \pi_A(\mathbf{t} + \mathbf{r}) \to \mathbf{t}$$

This induces non-determinism

$$\text{If} \quad \begin{array}{l} \Gamma \vdash \mathbf{t} : A \\ \Gamma \vdash \mathbf{r} : A \end{array} \quad \text{then} \quad \begin{array}{l} \pi_A(\mathbf{t} + \mathbf{r}) \to \mathbf{t} \\ \pi_A(\mathbf{t} + \mathbf{r}) \to \mathbf{r} \end{array}$$

What about $\wedge$-elimination?

$$\frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : A \wedge B}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : A} \qquad \text{But } A \wedge B = B \wedge A \text{ !!} \qquad \frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : B \wedge A}{\Gamma \vdash \pi_1(\mathbf{t} + \mathbf{r}) : B}$$

Moreover

$\mathbf{t} + \mathbf{r} = \mathbf{r} + \mathbf{t}$ so $\pi_1(\mathbf{t} + \mathbf{r}) = \pi_1(\mathbf{r} + \mathbf{t})$!!

Workaround: Church-style. Project w.r.t. a type

$$\text{If} \quad \Gamma \vdash \mathbf{t} : A \quad \text{then} \quad \pi_A(\mathbf{t} + \mathbf{r}) \to \mathbf{t}$$

This induces non-determinism

$$\text{If} \quad \begin{array}{l} \Gamma \vdash \mathbf{t} : A \\ \Gamma \vdash \mathbf{r} : A \end{array} \quad \text{then} \quad \begin{array}{l} \pi_A(\mathbf{t} + \mathbf{r}) \to \mathbf{t} \\ \pi_A(\mathbf{t} + \mathbf{r}) \to \mathbf{r} \end{array}$$

*We are interested in the proof theory and both **t** and **r** are valid proofs of A*

The calculus

Types

$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B \mid \forall X.A$

Equivalences

$$
\begin{aligned}
A \wedge B &\equiv B \wedge A \\
(A \wedge B) \wedge C &\equiv A \wedge (B \wedge C) \\
A \Rightarrow (B \wedge C) &\equiv (A \Rightarrow B) \wedge (A \Rightarrow C)
\end{aligned}
$$

The calculus

Types

$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B \mid \forall X.A$

Equivalences

$$
\begin{array}{rcl}
A \wedge B & \equiv & B \wedge A \\
(A \wedge B) \wedge C & \equiv & A \wedge (B \wedge C) \\
A \Rightarrow (B \wedge C) & \equiv & (A \Rightarrow B) \wedge (A \Rightarrow C)
\end{array}
$$

Terms

$$
\begin{array}{rl}
\mathbf{t}, \mathbf{r}, \mathbf{s} ::= & x^A \mid \lambda x^A.\mathbf{t} \mid \mathbf{tr} \mid \Lambda X.\mathbf{t} \mid \mathbf{t}\{A\} \\
& \mid \mathbf{t} + \mathbf{r} \mid \pi_A(\mathbf{t})
\end{array}
$$

The calculus

Types

$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B \mid \forall X.A$

Equivalences

$$\begin{array}{rcl} A \wedge B & \equiv & B \wedge A \\ (A \wedge B) \wedge C & \equiv & A \wedge (B \wedge C) \\ A \Rightarrow (B \wedge C) & \equiv & (A \Rightarrow B) \wedge (A \Rightarrow C) \end{array}$$

Terms

$$\begin{array}{rl} \mathbf{t}, \mathbf{r}, \mathbf{s} ::= & x^A \mid \lambda x^A.\mathbf{t} \mid \mathbf{tr} \mid \Lambda X.\mathbf{t} \mid \mathbf{t}\{A\} \\ & \mid \mathbf{t} + \mathbf{r} \mid \pi_A(\mathbf{t}) \end{array}$$

Reduction rules

$$\begin{array}{rl} (\lambda x^A.\mathbf{t})\mathbf{r} \hookrightarrow & \mathbf{t}[\mathbf{r}/x] \\ (\Lambda X.\mathbf{t})\{A\} \hookrightarrow & \mathbf{t}[A/X] \\ \pi_A(\mathbf{t} + \mathbf{r}) \hookrightarrow & \mathbf{t} \quad (\text{if } \Gamma \vdash \mathbf{t} : A) \end{array}$$

The calculus

Types

$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B \mid \forall X.A$

Equivalences

$$\begin{aligned} A \wedge B &\equiv B \wedge A \\ (A \wedge B) \wedge C &\equiv A \wedge (B \wedge C) \\ A \Rightarrow (B \wedge C) &\equiv (A \Rightarrow B) \wedge (A \Rightarrow C) \end{aligned}$$

Terms

$$\begin{aligned} \mathbf{t}, \mathbf{r}, \mathbf{s} ::= {} & x^A \mid \lambda x^A.\mathbf{t} \mid \mathbf{tr} \mid \Lambda X.\mathbf{t} \mid \mathbf{t}\{A\} \\ & \mid \mathbf{t} + \mathbf{r} \mid \pi_A(\mathbf{t}) \end{aligned}$$

Reduction rules

$$\begin{aligned} (\lambda x^A.\mathbf{t})\mathbf{r} &\hookrightarrow \mathbf{t}[\mathbf{r}/x] \\ (\Lambda X.\mathbf{t})\{A\} &\hookrightarrow \mathbf{t}[A/X] \\ \pi_A(\mathbf{t} + \mathbf{r}) &\hookrightarrow \mathbf{t} \quad (\text{if } \Gamma \vdash \mathbf{t} : A) \end{aligned}$$

$$\begin{aligned} \mathbf{t} + \mathbf{r} &\leftrightarrows \mathbf{r} + \mathbf{t} \\ (\mathbf{t} + \mathbf{r}) + \mathbf{s} &\leftrightarrows \mathbf{t} + (\mathbf{r} + \mathbf{s}) \\ (\mathbf{t} + \mathbf{r})\mathbf{s} &\leftrightarrows \mathbf{ts} + \mathbf{rs} \\ \lambda x^A.(\mathbf{t} + \mathbf{r}) &\leftrightarrows \lambda x^A.\mathbf{t} + \lambda x^A.\mathbf{r} \\ \pi_{A \Rightarrow B}(\mathbf{t})\mathbf{r} &\leftrightarrows \pi_B(\mathbf{tr}) \\ & \quad (\text{if } \Gamma \vdash \mathbf{t} : A \Rightarrow (B \wedge C)) \end{aligned}$$

The calculus

Types

$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B \mid \forall X.A$

Equivalences

$$\begin{aligned} A \wedge B &\equiv B \wedge A \\ (A \wedge B) \wedge C &\equiv A \wedge (B \wedge C) \\ A \Rightarrow (B \wedge C) &\equiv (A \Rightarrow B) \wedge (A \Rightarrow C) \end{aligned}$$

Terms

$\mathbf{t}, \mathbf{r}, \mathbf{s} ::= x^A \mid \lambda x^A.\mathbf{t} \mid \mathbf{tr} \mid \Lambda X.\mathbf{t} \mid \mathbf{t}\{A\} \mid \mathbf{t} + \mathbf{r} \mid \pi_A(\mathbf{t})$

Reduction rules

$$\begin{aligned} (\lambda x^A.\mathbf{t})\mathbf{r} &\hookrightarrow \mathbf{t}[\mathbf{r}/x] \\ (\Lambda X.\mathbf{t})\{A\} &\hookrightarrow \mathbf{t}[A/X] \\ \pi_A(\mathbf{t} + \mathbf{r}) &\hookrightarrow \mathbf{t} \quad (\text{if } \Gamma \vdash \mathbf{t} : A) \end{aligned}$$

$$\begin{aligned} \mathbf{t} + \mathbf{r} &\leftrightarrows \mathbf{r} + \mathbf{t} \\ (\mathbf{t} + \mathbf{r}) + \mathbf{s} &\leftrightarrows \mathbf{t} + (\mathbf{r} + \mathbf{s}) \\ (\mathbf{t} + \mathbf{r})\mathbf{s} &\leftrightarrows \mathbf{ts} + \mathbf{rs} \\ \lambda x^A.(\mathbf{t} + \mathbf{r}) &\leftrightarrows \lambda x^A.\mathbf{t} + \lambda x^A.\mathbf{r} \\ \pi_{A \Rightarrow B}(\mathbf{t})\mathbf{r} &\leftrightarrows \pi_B(\mathbf{tr}) \\ & \quad (\text{if } \Gamma \vdash \mathbf{t} : A \Rightarrow (B \wedge C)) \end{aligned}$$

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax \qquad \frac{\Gamma, x : A \vdash \mathbf{t} : B}{\Gamma \vdash \lambda x^A.\mathbf{t} : A \Rightarrow B}\ \Rightarrow_I$$

$$\frac{\Gamma \vdash \mathbf{t} : A \Rightarrow B \qquad \Gamma \vdash \mathbf{s} : A}{\Gamma \vdash \mathbf{ts} : B}\ \Rightarrow_E$$

$$\frac{\Gamma \vdash \mathbf{t} : A \quad X \notin FV(\Gamma)}{\Gamma \vdash \Lambda X.\mathbf{t} : \forall X.A}\ \forall_I \qquad \frac{\Gamma \vdash \mathbf{t} : \forall X.A}{\Gamma \vdash \mathbf{t}\{B\} : A[B/X]}\ \forall_E$$

$$\frac{\Gamma \vdash \mathbf{t} : A \quad \Gamma \vdash \mathbf{r} : B}{\Gamma \vdash \mathbf{t} + \mathbf{r} : A \wedge B}\ \wedge_I \qquad \frac{\Gamma \vdash \mathbf{t} : A \wedge B}{\Gamma \vdash \pi_A(\mathbf{t}) : A}\ \wedge_E$$

$$\frac{\Gamma \vdash \mathbf{t} : A \quad A \equiv B}{\Gamma \vdash \mathbf{t} : B}\ \equiv$$

Theorem (Subject reduction)

If $\Gamma \vdash \boldsymbol{t} : A$ *and* $\boldsymbol{t} \to \boldsymbol{r}$ *then* $\Gamma \vdash \boldsymbol{r} : A$

with $\to := \hookrightarrow$ or $\leftrightarrows$

Example (I)

$$\vdash \lambda x^{A \wedge B}.x : (A \wedge B) \Rightarrow (A \wedge B)$$

Example (I)

$$\vdash \lambda x^{A \wedge B}.x : (A \wedge B) \Rightarrow (A \wedge B)$$

$$(A \wedge B) \Rightarrow (A \wedge B) \qquad \equiv \qquad ((A \wedge B) \Rightarrow A) \wedge ((A \wedge B) \Rightarrow B)$$

Example (I)

$$\vdash \lambda x^{A \wedge B}.x : (A \wedge B) \Rightarrow (A \wedge B)$$

$$(A \wedge B) \Rightarrow (A \wedge B) \qquad \equiv \qquad ((A \wedge B) \Rightarrow A) \wedge ((A \wedge B) \Rightarrow B)$$

Hence

$$\vdash \pi_{(A \wedge B) \Rightarrow A}(\lambda x^{A \wedge B}.x) : (A \wedge B) \Rightarrow A$$

Example (I)

$$\vdash \lambda x^{A \wedge B}.x : (A \wedge B) \Rightarrow (A \wedge B)$$

$$(A \wedge B) \Rightarrow (A \wedge B) \qquad \equiv \qquad ((A \wedge B) \Rightarrow A) \wedge ((A \wedge B) \Rightarrow B)$$

Hence

$$\vdash \pi_{(A \wedge B) \Rightarrow A}(\lambda x^{A \wedge B}.x) : (A \wedge B) \Rightarrow A$$

Let $\vdash \mathbf{r} : A \wedge B$

Example (I)

$$\vdash \lambda x^{A \wedge B}.x : (A \wedge B) \Rightarrow (A \wedge B)$$

$$(A \wedge B) \Rightarrow (A \wedge B) \qquad \equiv \qquad ((A \wedge B) \Rightarrow A) \wedge ((A \wedge B) \Rightarrow B)$$

Hence

$$\vdash \pi_{(A \wedge B) \Rightarrow A}(\lambda x^{A \wedge B}.x) : (A \wedge B) \Rightarrow A$$

Let $\vdash \mathbf{r} : A \wedge B$

$$\vdash \pi_{(A \wedge B) \Rightarrow A}(\lambda x^{A \wedge B}.x) \;\; \mathbf{r} : A$$

Example (I)

$$\vdash \lambda x^{A\wedge B}.x : (A\wedge B) \Rightarrow (A\wedge B)$$

$$(A\wedge B) \Rightarrow (A\wedge B) \qquad \equiv \qquad ((A\wedge B) \Rightarrow A) \wedge ((A\wedge B) \Rightarrow B)$$

Hence

$$\vdash \pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x) : (A\wedge B) \Rightarrow A$$

Let $\quad \vdash \mathbf{r} : A \wedge B$

$$\vdash \pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x)\ \ \mathbf{r} : A$$

$$\pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x)\mathbf{r} \qquad \leftrightarrows \qquad \pi_A((\lambda x^{A\wedge B}.x)\mathbf{r}) \qquad \hookrightarrow \qquad \pi_A(\mathbf{r})$$

Example (II)

$$\mathbf{TF} = \lambda x^{\mathbb{B}}.\lambda y^{\mathbb{B}}.(x + y)$$

$$\vdash \mathbf{TF} \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B})$$

Example (II)

$$\mathbf{TF} = \lambda x^{\mathbb{B}}.\lambda y^{\mathbb{B}}.(x + y)$$

$$\vdash \mathbf{TF} \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B})$$

$$\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B}) \qquad \equiv \qquad (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}) \wedge (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B})$$

Example (II)

$$\mathbf{TF} = \lambda x^{\mathbb{B}}.\lambda y^{\mathbb{B}}.(x + y)$$

$$\vdash \mathbf{TF} \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B})$$

$$\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B}) \qquad \equiv \qquad (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}) \wedge (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B})$$

$$\vdash \pi_{\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}}(\mathbf{TF}) \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}$$

Example (II)

$$\textbf{TF} = \lambda x^{\mathbb{B}}.\lambda y^{\mathbb{B}}.(x + y)$$

$$\vdash \textbf{TF} \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B})$$

$$\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B}) \qquad \equiv \qquad (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}) \wedge (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B})$$

$$\vdash \pi_{\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}}(\textbf{TF}) \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}$$

Let $\quad \vdash \mathbf{t} : \mathbb{B} \quad$ and $\quad \vdash \mathbf{f} : \mathbb{B}$

Example (II)

$$\mathbf{TF} = \lambda x^{\mathbb{B}}.\lambda y^{\mathbb{B}}.(x + y)$$

$$\vdash \mathbf{TF} \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B})$$

$$\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B}) \qquad \equiv \qquad (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}) \wedge (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B})$$

$$\vdash \pi_{\mathbb{B}\Rightarrow\mathbb{B}\Rightarrow\mathbb{B}}(\mathbf{TF}) \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}$$

Let $\quad \vdash \mathbf{t} : \mathbb{B} \quad$ and $\quad \vdash \mathbf{f} : \mathbb{B}$

$$\pi_{\mathbb{B}\Rightarrow\mathbb{B}\Rightarrow\mathbb{B}}(\mathbf{TF})\ \mathbf{t}\ \mathbf{f}$$

Example (II)

$$\mathbf{TF} = \lambda x^{\mathbb{B}}.\lambda y^{\mathbb{B}}.(x + y)$$

$$\vdash \mathbf{TF} \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B})$$

$$\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B}) \qquad \equiv \qquad (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}) \wedge (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B})$$

$$\vdash \pi_{\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}}(\mathbf{TF}) \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}$$

Let $\quad \vdash \mathbf{t} : \mathbb{B} \quad$ and $\quad \vdash \mathbf{f} : \mathbb{B}$

$$\pi_{\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}}(\mathbf{TF})\ \mathbf{t}\ \mathbf{f} \quad \leftrightarrows \quad \pi_{\mathbb{B} \Rightarrow \mathbb{B}}((\mathbf{TF})\mathbf{t})\ \mathbf{f}$$

Example (II)

$$\mathbf{TF} = \lambda x^{\mathbb{B}}.\lambda y^{\mathbb{B}}.(x + y)$$

$$\vdash \mathbf{TF} \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B})$$

$$\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B}) \qquad \equiv \qquad (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}) \wedge (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B})$$

$$\vdash \pi_{\mathbb{B}\Rightarrow\mathbb{B}\Rightarrow\mathbb{B}}(\mathbf{TF}) \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}$$

Let $\quad \vdash \mathbf{t} : \mathbb{B} \quad$ and $\quad \vdash \mathbf{f} : \mathbb{B}$

$$\pi_{\mathbb{B}\Rightarrow\mathbb{B}\Rightarrow\mathbb{B}}(\mathbf{TF})\ \mathbf{t}\ \mathbf{f} \quad \leftrightarrows \quad \pi_{\mathbb{B}\Rightarrow\mathbb{B}}((\mathbf{TF})\mathbf{t})\ \mathbf{f} \quad \leftrightarrows \quad \pi_{\mathbb{B}}((\mathbf{TF})\mathbf{tf})$$

Example (II)

$$\mathbf{TF} = \lambda x^{\mathbb{B}}.\lambda y^{\mathbb{B}}.(x + y)$$

$$\vdash \mathbf{TF} \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B})$$

$$\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow (\mathbb{B} \wedge \mathbb{B}) \qquad \equiv \qquad (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}) \wedge (\mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B})$$

$$\vdash \pi_{\mathbb{B}\Rightarrow\mathbb{B}\Rightarrow\mathbb{B}}(\mathbf{TF}) \ : \ \mathbb{B} \Rightarrow \mathbb{B} \Rightarrow \mathbb{B}$$

Let $\quad \vdash \mathbf{t} : \mathbb{B} \quad$ and $\quad \vdash \mathbf{f} : \mathbb{B}$

$$\pi_{\mathbb{B}\Rightarrow\mathbb{B}\Rightarrow\mathbb{B}}(\mathbf{TF})\ \mathbf{t}\ \mathbf{f} \quad \leftrightarrows \quad \pi_{\mathbb{B}\Rightarrow\mathbb{B}}((\mathbf{TF})\mathbf{t})\ \mathbf{f} \quad \leftrightarrows \quad \pi_{\mathbb{B}}((\mathbf{TF})\mathbf{tf})$$

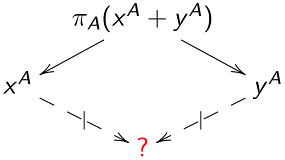
$$\hookrightarrow \ \pi_{\mathbb{B}}(\mathbf{t} + \mathbf{f}) \ \hookrightarrow \ \mathbf{t}$$

$$\hookrightarrow \ \pi_{\mathbb{B}}(\mathbf{t} + \mathbf{f}) \ \hookrightarrow \ \mathbf{f}$$

Confluence (some ideas)

Of course, a **non-deterministic** calculus is **not confluent**!

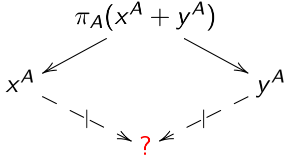
Counterexample



Confluence (some ideas)

Of course, a **non-deterministic** calculus is **not confluent**!

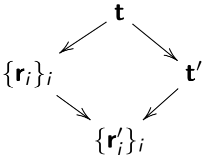
Counterexample



However, we can prove it keeps some coherence

- Confluence of the deterministic fragment
- Confluence of the "term ensembles"

e.g.



[Arrighi,Díaz-Caro,Gadella,Grattage'08]

Conclusions (with some examples)

Proof system

Let $\mathbf{t}$ be a proof of A
and $\mathbf{r}$ be a proof of B
so $\mathbf{t} + \mathbf{r}$ is a proof of both $A \wedge B$ and $B \wedge A$

Non deterministic calculus

$$\mathit{tt} = \Lambda X.\lambda x^X.\lambda y^X.x \qquad \mathit{ff} = \Lambda X.\lambda x^X.\lambda y^X.y$$

$$\mathbb{B} = \forall X.X \Rightarrow X \Rightarrow X$$

$$\frac{\vdash \mathit{tt} + \mathit{ff} : \mathbb{B} \wedge \mathbb{B}}{\vdash \pi_{\mathbb{B}}(\mathit{tt} + \mathit{ff}) : \mathbb{B}}$$

$$\pi_{\mathbb{B}}(\mathit{tt} + \mathit{ff}) \to \mathit{tt} \qquad \pi_{\mathbb{B}}(\mathit{tt} + \mathit{ff}) \to \mathit{ff}$$

So far:

- Proof system where (three) isomorphic types get the same proofs
- Non-deterministic calculus

Future directions (open problems)

Can we continue adding Di Cosmo's isomorphisms?

e.g. $A \wedge \mathbf{T} \equiv A$ induces $\mathbf{t} + \mathbf{0} \leftrightarrows \mathbf{t}$ and $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$ induces $\lambda x.\mathbf{0} \leftrightarrows \mathbf{0}$

Future directions (open problems)

Can we continue adding Di Cosmo's isomorphisms?

e.g. $A \wedge \mathbf{T} \equiv A$ induces $\mathbf{t} + \mathbf{0} \leftrightarrows \mathbf{t}$ and $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$ induces $\lambda x.\mathbf{0} \leftrightarrows \mathbf{0}$
But if $\mathbf{T} \Rightarrow \mathbf{T} \equiv \mathbf{T}$ then $\vdash (\lambda x^{\mathbf{T}}.xx)(\lambda x^{\mathbf{T}}.xx) : \mathbf{T}$ (wrong)

Future directions (open problems)

Can we continue adding Di Cosmo's isomorphisms?

e.g. $A \wedge \mathbf{T} \equiv A$ induces $\mathbf{t} + \mathbf{0} \leftrightarrows \mathbf{t}$ and $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$ induces $\lambda x.\mathbf{0} \leftrightarrows \mathbf{0}$
But if $\mathbf{T} \Rightarrow \mathbf{T} \equiv \mathbf{T}$ then $\vdash (\lambda x^{\mathbf{T}}.xx)(\lambda x^{\mathbf{T}}.xx) : \mathbf{T}$ (wrong)

A more interesting open question:

Can we use this no determinism to define a probabilistic/quantum language?

Future directions (open problems)

Can we continue adding Di Cosmo's isomorphisms?

e.g. $A \wedge \mathbf{T} \equiv A$ induces $\mathbf{t} + \mathbf{0} \leftrightarrows \mathbf{t}$ and $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$ induces $\lambda x.\mathbf{0} \leftrightarrows \mathbf{0}$
But if $\mathbf{T} \Rightarrow \mathbf{T} \equiv \mathbf{T}$ then $\vdash (\lambda x^{\mathbf{T}}.xx)(\lambda x^{\mathbf{T}}.xx) : \mathbf{T}$ (wrong)

A more interesting open question:

Can we use this no determinism to define a probabilistic/quantum language?

Some clues:

- Similar to the linear-algebraic lambda-calculus [Arrighi,Dowek]

Future directions (open problems)

Can we continue adding Di Cosmo's isomorphisms?

e.g. $A \wedge \mathbf{T} \equiv A$ induces $\mathbf{t} + \mathbf{0} \leftrightarrows \mathbf{t}$ and $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$ induces $\lambda x.\mathbf{0} \leftrightarrows \mathbf{0}$
But if $\mathbf{T} \Rightarrow \mathbf{T} \equiv \mathbf{T}$ then $\vdash (\lambda x^{\mathbf{T}}.xx)(\lambda x^{\mathbf{T}}.xx) : \mathbf{T}$ (wrong)

A more interesting open question:

Can we use this no determinism to define a probabilistic/quantum language?

Some clues:

- Similar to the linear-algebraic lambda-calculus [Arrighi,Dowek]
- We need call-by-value (no-cloning)

Future directions (open problems)

Can we continue adding Di Cosmo's isomorphisms?

e.g. $A \wedge \mathbf{T} \equiv A$ induces $\mathbf{t} + \mathbf{0} \leftrightarrows \mathbf{t}$ and $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$ induces $\lambda x.\mathbf{0} \leftrightarrows \mathbf{0}$
But if $\mathbf{T} \Rightarrow \mathbf{T} \equiv \mathbf{T}$ then $\vdash (\lambda x^{\mathbf{T}}.xx)(\lambda x^{\mathbf{T}}.xx) : \mathbf{T}$ (wrong)

A more interesting open question:

Can we use this no determinism to define a probabilistic/quantum language?

Some clues:

- Similar to the linear-algebraic lambda-calculus [Arrighi,Dowek]
- We need call-by-value (no-cloning)
  - In call-by-value,

$$\mathbf{t}(\mathbf{r} + \mathbf{s}) \leftrightarrows \mathbf{tr} + \mathbf{ts}$$

Future directions (open problems)

Can we continue adding Di Cosmo's isomorphisms?

e.g. $A \wedge \mathbf{T} \equiv A$ induces $\mathbf{t} + \mathbf{0} \leftrightarrows \mathbf{t}$ and $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$ induces $\lambda x.\mathbf{0} \leftrightarrows \mathbf{0}$
But if $\mathbf{T} \Rightarrow \mathbf{T} \equiv \mathbf{T}$ then $\vdash (\lambda x^{\mathbf{T}}.xx)(\lambda x^{\mathbf{T}}.xx) : \mathbf{T}$ (wrong)

A more interesting open question:

Can we use this no determinism to define a probabilistic/quantum language?

Some clues:

- Similar to the linear-algebraic lambda-calculus [Arrighi,Dowek]
- We need call-by-value (no-cloning)
  - In call-by-value,

$$\mathbf{t}(\mathbf{r} + \mathbf{s}) \leftrightarrows \mathbf{tr} + \mathbf{ts}$$

But $(A \wedge B) \Rightarrow C \quad \not\equiv \quad (A \Rightarrow C) \wedge (B \Rightarrow C)$

Future directions (open problems)

Can we continue adding Di Cosmo's isomorphisms?

e.g. $A \wedge \mathbf{T} \equiv A$ induces $\mathbf{t} + \mathbf{0} \leftrightarrows \mathbf{t}$ and $A \Rightarrow \mathbf{T} \equiv \mathbf{T}$ induces $\lambda x.\mathbf{0} \leftrightarrows \mathbf{0}$
But if $\mathbf{T} \Rightarrow \mathbf{T} \equiv \mathbf{T}$ then $\vdash (\lambda x^{\mathbf{T}}.xx)(\lambda x^{\mathbf{T}}.xx) : \mathbf{T}$ (wrong)

A more interesting open question:

Can we use this no determinism to define a probabilistic/quantum language?

Some clues:

- Similar to the linear-algebraic lambda-calculus [Arrighi,Dowek]
- We need call-by-value (no-cloning)
  - In call-by-value,

$$\mathbf{t}(\mathbf{r} + \mathbf{s}) \leftrightarrows \mathbf{tr} + \mathbf{ts}$$

But $\quad (A \wedge B) \Rightarrow C \quad \not\equiv \quad (A \Rightarrow C) \wedge (B \Rightarrow C)$

Workaround: Use polymorphism: $\forall X.X \Rightarrow C_X$ [Arrighi,Díaz-Caro]