Internship Proposal in Computational Geometry
Bowyer-Watson Algorithm for Delaunay Triangulation on Hyperbolic Surfaces

Advisors	Vincent Despré	Marc Pouget
Emails	vincent.despre@inria.fr	marc.pouget@inria.fr
Location	INRIA LORIA, Nancy, France. Equipe Gamble	

1 Context

Delaunay triangulations in the Euclidean space $\mathbb{R}^d$ appeared to be an extremely strong tool in many areas of computer science and, thus, they have been extensively studied [1]. Their mathematical properties are well understood, many algorithms to construct them have been proposed and analyzed in various contexts. The following simple incremental Bowyer-Watson algorithm [2, 3] is considered the best algorithm for practical implementations. For each new point p:

- Find the simplices in conflict with p, i.e. the simplices whose circumscribing ball contains p
- Remove these simplices from the triangulation. This creates a hole.
- Fill the conflict hole by “starring” it from p.

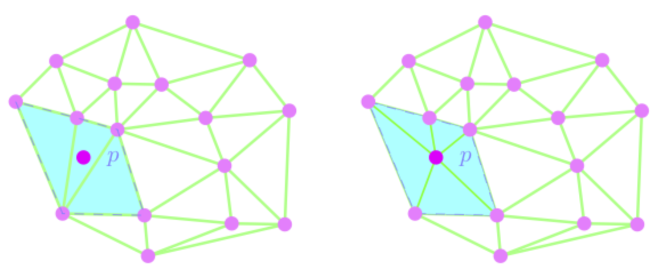

Figure 1: Incremental Bowyer-Watson Algorithm in the plane.

A hyperbolic surface is a closed and orientable topological surface equipped with some hyperbolic metric of constant curvature -1. Recently, motivated in part by applications in other sciences and its ubiquity, there has been an increased effort to understand the hyperbolic geometry of surfaces from a computational geometry point of view.

An algorithm was proposed for a class of symmetric hyperbolic surfaces that are the quotient of the hyperbolic plane by a regular 4g-gon [4]. This algorithm is based on the abovementioned incremental algorithm, which subsumes that the conflict hole is a topological disk for any new point p. This is a priori not always the case on a surface, see Fig. 2.

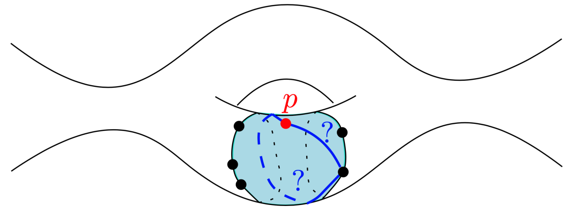

Figure 2: Bowyer's insertion is not well defined when the conflict region is not a topological disk.

This has led to the necessity of ensuring that the Delaunay triangulation is maintained as a simplicial complex throughout the incremental construction, i.e., its edges do not form cycles of length 1 (loops, i.e., edges whose two vertices are equal) or 2 (two edges sharing the same two vertices) [4].

A package of CGAL, the Computational Geometry Algorithms Library, implements this incremental algorithm in the case of the Bolza surface which is a symmetric genus 2 hyperbolic surface.

2 Objectives

A general hyperbolic surface may be given by a polygon in the hyperbolic plane with edges identifications. The objective is to generalized the result of [4] to any hyperbolic surfaces. This generalization has been made possible by a recent result [5] that allows the input polygon to be a Dirichlet fundamental domain of the surface. To make the generalization works and give a precise complexity, there are subtle difficulties to overcome:

- The Dirichlet domain ensures [6] that the output Delaunay triangulation can be represented with finitely many isometric copies of the polygon in the hyperbolic plane. However, a brutal use of this result would give a disproportionate number of domains of the order of g^g, where g is the genus of the underlying surface.
- We need to respect the simplicial property required by the algorithm. Thus, it is necessary to find a small set of initialisation points that ensures this property.

The candidate should have interest in low-dimensional topology and geometry together with algorithmics. Since the topic is between mathematics and computer science, the candidate should have a master in one of the two topics and a taste for the other aspect.

The work may extend to a PhD on a related topic thanks to an ANR grant.

References

[1] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars. *Computational Geometry: Algorithms and Applications*. Springer Berlin Heidelberg, 2008.

[2] A. Bowyer. Computing Dirichlet tessellations. *The Computer Journal*, 24(2):162–166, 01 1981.

[3] D. F. Watson. Computing the n-dimensional Delaunay tessellation with application to Voronoi polytopes. *The Computer Journal*, 24(2):167–172, 01 1981.

[4] Matthijs Ebbens, Iordan Iordanov, Monique Teillaud, and Gert Vegter. Delaunay triangulations of generalized Bolza surfaces. *Journal of Computational Geometry*, 13(1):125–177, April 2022.

[5] Vincent Despré, Benedikt Kolbe, Hugo Parlier, and Monique Teillaud. Computing a Dirichlet Domain for a Hyperbolic Surface. In *39th International Symposium on Computational Geometry (SoCG 2023)*, pages 27:1–27:15, Dallas, United States, June 2023.

[6] Vincent Despré, Benedikt Kolbe, and Monique Teillaud. Representing infinite hyperbolic periodic delaunay triangulations using finitely many Dirichlet domains. *Preprint*, 2021.

[7] Matthijs Ebbens, Hugo Parlier, and Gert Vegter. Minimal delaunay triangulations of hyperbolic surfaces, 2020.