High Accuracy Geometric Hermite Interpolation

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Computer Sciences Technical Report #692
May 1987

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ABSTRACT

We describe a parametric cubic spline interpolation scheme for planar curves which is based on an idea of Sabin for the construction of C^1 bicubic parametric spline surfaces. The method is a natural generalization of [standard] Hermite interpolation. In addition to position and tangent, the curvature is prescribed at each knot. This ensures that the resulting interpolating piecewise cubic curve is twice continuously differentiable and can be constructed locally. Moreover, under appropriate assumptions, the interpolant preserves convexity and is 6-th order accurate.

AMS (MOS) Subject Classifications: 41A15, 41A25, 53A04

Key Words: splines, curves, interpolation, geometric smoothness, accuracy

[1] supported by the United States Army under Contract No. DAAG29-80-C-0041

[2] sponsored by the National Science Foundation under Grant No. DMS-8351187

1. Description of the method

The [standard] cubic Hermite interpolant of a [planar] curve

$$t \mapsto f(t) : \mathbb{R} \mapsto \mathbb{R}^2$$

matches $f(t_i)$ and $f'(t_i)$ at a given [increasing] sequence of knots t_i. While very simple to compute, this interpolant depends on the parametrization of f and is, in general, merely C^1. On the other hand, the construction of twice continuously differentiable cubic spline interpolants usually involves the solution of a global system of equations (cf. e.g. [Bo78]).

We show in this paper that curvature continuity and high accuracy can be achieved by a simple algorithm which is based on the geometric characterization of C^2-continuity.

$\mathbf{C^2}$-condition. Denote by $a \times b := a_1 b_2 - a_2 b_1$ the cross product of two vectors in $\mathbb{R}^2$. A planar curve f is twice continuously differentiable if the unit tangent vector $f^* := f'/|f'|$ and the signed curvature $f^{**} := f' \times f''/|f'|^3$ are continuous.

The form of the continuity conditions suggests the definition of a geometric Hermite interpolant $p := p_f$ to a planar curve f via the conditions

$$p(i) = f_i, \quad p^*(i) = d_i, \quad p^{**}(i) = \kappa_i$$

where $f_i := f(t_i)$, $d_i := f^*(t_i)$ and $\kappa_i := f^{**}(t_i)$ and the components of p are cubic polynomials on each of the parameter intervals $[i, i+1]$. The idea for this method is due to Sabin [S68] who used a similar construction for C^1 interpolation of surfaces. Later Manning [M73] and Bär [B74] described interpolation methods using curvature data, but their schemes involve the solution of global nonlinear systems.

From the definition of the method it is clear that the piecewise polynomial curve p is twice continuously differentiable. Moreover, the polynomial segments of p can be computed individually from the corresponding data at two consecutive knots. Considering for example $p_{|[0,1]}$, this is best described using the Bézier form (cf. Figure 1) [Bö84],

$$t \mapsto p(t) =: \sum_{\nu=0}^{3} b_\nu B_\nu(t), \quad 0 \le t \le 1,$$

with $B_\nu(t) := \binom{3}{\nu} t^\nu (1-t)^{3-\nu}$.

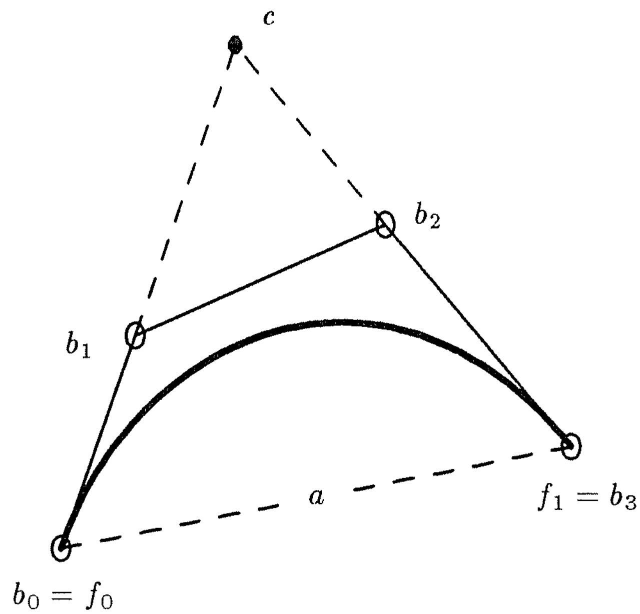

Figure 1. Bézier polygon of a cubic curve segment

By the interpolation conditions,

$$
\begin{aligned}
b_0 = f_0, \quad & b_1 = b_0 + \delta_0 d_0 \\
b_3 = f_1, \quad & b_2 = b_3 - \delta_1 d_1
\end{aligned}
$$

and

$$
\kappa_0 = 2d_0 \times (b_2 - b_1)/(3\delta_0^2), \quad \kappa_1 = 2d_1 \times (b_1 - b_2)/(3\delta_1^2)
$$

as one verifies from the definitions. Replacing $b_2 - b_1$ by $(f_1 - f_0) - \delta_0 d_0 - \delta_1 d_1$ in the last equations and setting $a := f_1 - f_0$ yields a system of quadratic equations for δ,

$$
\begin{aligned}
(d_0 \times d_1)\delta_0 &= (a \times d_1) - (3/2)\kappa_1 \delta_1^2 \\
(d_0 \times d_1)\delta_1 &= (d_0 \times a) - (3/2)\kappa_0 \delta_0^2
\end{aligned}
\qquad (Q)
$$

Unfortunately, this system does not always have solutions with $\delta_\nu > 0$ and, as is illustrated in Figure 2, in general there is no unique positive solution. However, if the data are "consistent", positive solutions exist and are easily computed numerically. The restrictions on the data will be made precise in the next section. Moreover, we will show that, asymptotically, the scheme performs exceptionally well.

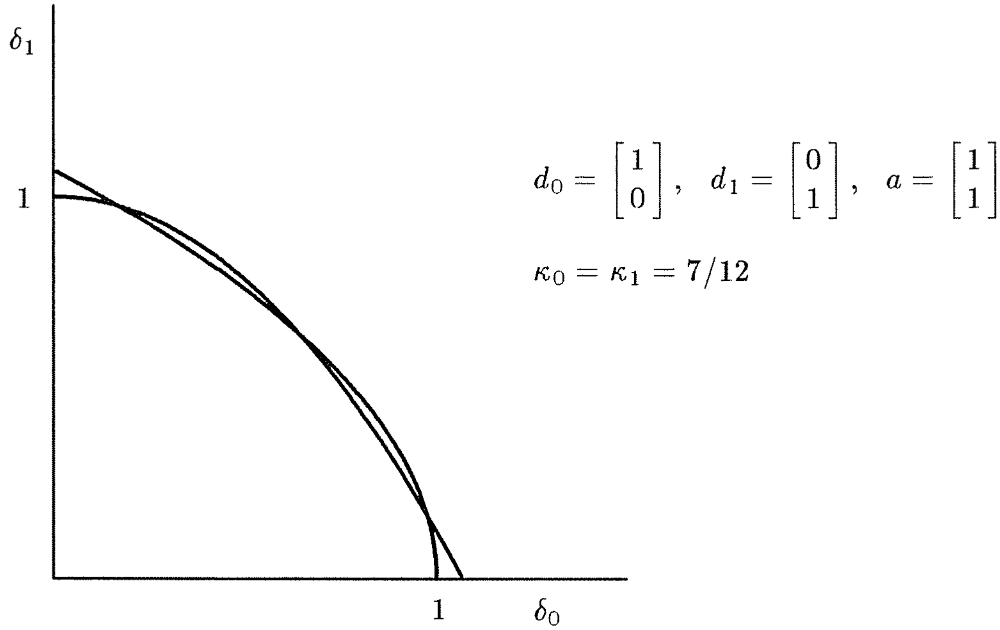

Figure 2. Nonuniqueness of positive solutions for (Q)

Theorem. If f is a smooth curve with nonvanishing curvature and

$$h := \sup_i |f_{i+1} - f_i|$$

is sufficiently small, then positive solutions of the system (Q) exist and the corresponding interpolant[s] satisfy

$$\text{dist}(f, p_f) = O(h^6).$$

This is an interesting improvement over linear interpolation techniques. But, two drawbacks of the method should be noted. Even for h small, the interpolant needs not be unique. As stated above, in this case the assertion of the Theorem is valid for any interpolant corresponding to a solution of (Q). Moreover, the positivity of the curvature is essential. If $f^{**}(t) = 0$, for some t, then, as the example in section 3 shows, the approximation order may drop to the standard rate $O(h^4)$ in a neighborhood of this point.

2. Solvability of the quadratic system

In this section we discuss the restrictions on the data f_i, d_i and κ_i which guarantee solvability of (Q). First we note that in the "exceptional" case when $d_0 \times d_1 = 0$ the restriction on the curvatures is

$$\operatorname{sign}(\kappa_\nu) = (-)^\nu \operatorname{sign}(d_\nu \times a).$$

Now, throughout the following, assume that $d_0 \times d_1 \neq 0$. In this case, the system (Q) can be simplified by introducing the new parameters ϱ defined by

$$\delta_0 =: \varrho_0 \frac{a \times d_1}{d_0 \times d_1}, \quad \delta_1 =: \varrho_1 \frac{d_0 \times a}{d_0 \times d_1}. \tag{1}$$

With

$$c := f_\nu + \frac{a \times d_{1-\nu}}{d_0 \times d_1} d_\nu$$

denoting the intersection of the tangents through f_0 and f_1 (cf. Figure 1), the parameters ϱ are the weights in writing b_1, b_2 as convex combinations of f_0, f_1 and c,

$$b_1 = f_0 + \varrho_0(c - f_0), \quad b_2 = f_1 + \varrho_1(c - f_1). \tag{2}$$

Defining

$$R_0 := \frac{3}{2} \frac{\kappa_0 \, (a \times d_1)^2}{(d_0 \times a)(d_0 \times d_1)^2}, \quad R_1 := \frac{3}{2} \frac{\kappa_1 \, (d_0 \times a)^2}{(a \times d_1)(d_0 \times d_1)^2} \tag{3}$$

and substituting (1) into (Q) we obtain the equivalent system

$$\begin{aligned} \varrho_0 &= 1 - R_1 \varrho_1^2 \\ \varrho_1 &= 1 - R_0 \varrho_0^2. \end{aligned} \tag{Q$'$}$$

As is illustrated in Figure 3, depending on R, this system can have 0, 1, 2 or 3 positive solutions ϱ which correspond to positive δ if

$$d_0 \times d_1, \quad a \times d_1, \quad d_0 \times a$$

are of the same sign. For data corresponding to a curve with nonvanishing curvature, the coefficients R_i are positive. Hence, in this case, a simple condition for solvability of (Q$'$) is that

$$(1 - R_0) \cdot (1 - R_1) \geq 0. \tag{4}$$

If the data f_i and d_i are prescribed, this condition can always be satisfied by selecting appropriate values for κ_i which can be viewed as shape parameters. The condition also guarantees that the interpolant preserves convexity. For, if $R_i \geq 0$, then there exists a solution of (Q$'$) with $0 \leq \varrho_i \leq 1$. By (2) and the definition of c this implies convexity or concavity of the Bézier polygon.

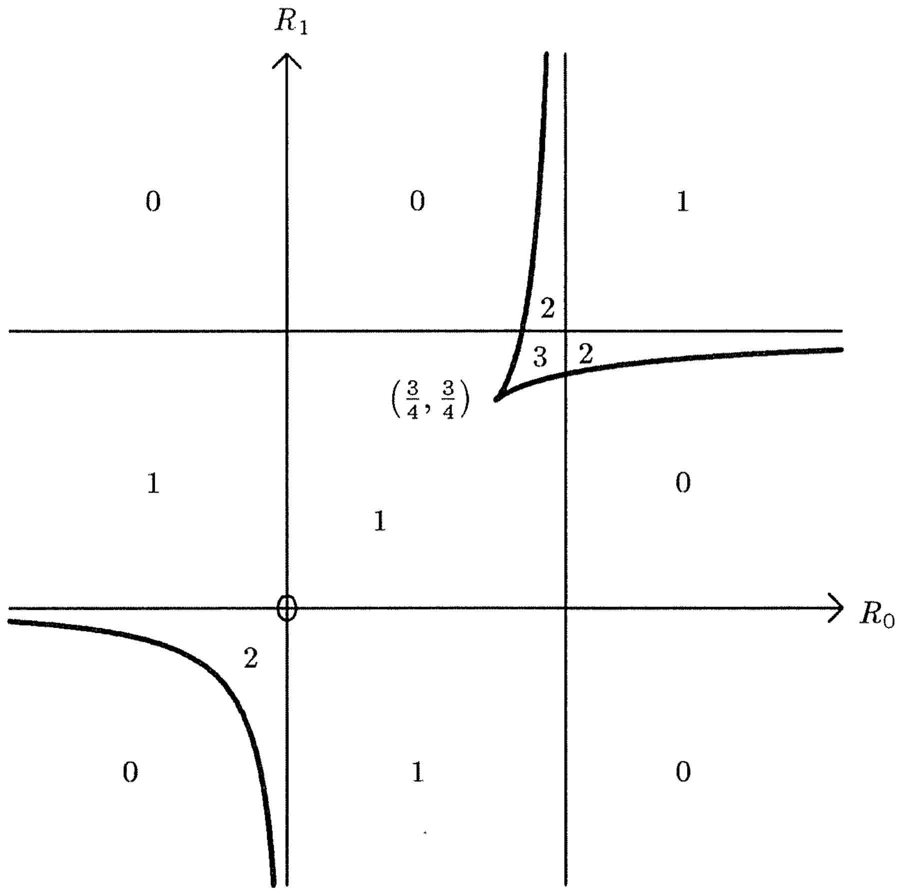

Figure 3. Number of positive solutions for (Q′)

3. Asymptotic behavior

In this section we analyze the system (Q′) as the distance of the interpolation points f_i tends to zero and prove the Theorem. We assume throughout that the curve f is smooth with $|\kappa| \geq c > 0$. Parametrizing f by arclength s,

$$f^*(s) = f'(s) =: \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix}$$

with θ the indefinite integral of the curvature,

$$\theta'(s) = \kappa(s).$$

We consider the system (Q′) for the data corresponding to the points $f_0 = f(-h)$ and $f_1 = f(h)$. Assuming without loss that $\theta(0) = 0$ and expanding θ to second order at $s = 0$,

$$\theta(s) = \theta_1 s + \theta_2 s^2 + O(s^3),$$

we determine the principal part of the system (Q′) as $h \mapsto 0$. From the definitions we see that

$$a = \int_{-h}^{h} \begin{bmatrix} \cos\theta(\sigma) \\ \sin\theta(\sigma) \end{bmatrix} d\sigma = \begin{bmatrix} 2h \\ 0 \end{bmatrix} + O(h^3)$$

$$d_0 = \begin{bmatrix} \cos\theta(-h) \\ \sin\theta(-h) \end{bmatrix} = \begin{bmatrix} 1 \\ -\theta_1 h \end{bmatrix} + O(h^2) \tag{5}$$

$$d_1 = \begin{bmatrix} 1 \\ \theta_1 h \end{bmatrix} + O(h^2)$$

and obtain the expansions

$$d_0 \times d_1 = \sin[\theta(h) - \theta(-h)] = 2\theta_1 h + O(h^3)$$

$$d_0 \times a = \int_{-h}^{h} \sin[\theta(\sigma) - \theta(-h)] d\sigma = 2\theta_1 h^2 - (4/3)\theta_2 h^3 + O(h^4) \tag{6}$$

$$a \times d_1 = \int_{-h}^{h} \sin[\theta(h) - \theta(\sigma)] d\sigma = 2\theta_1 h^2 + (4/3)\theta_2 h^3 + O(h^4).$$

For the coefficients R in the system (Q′) this gives

$$R_0(h) = R_1(-h) = (3/4) + |\theta_1|^{-1} O(h^2). \tag{7}$$

While this shows that (Q′) is solvable for small h, the system becomes singular in the limit, i.e. for $R = (3/4, 3/4)$ the system has a triple root

$$\varrho = (2/3,\ 2/3)$$

(cf. Figure 3). This somewhat complicates the proof of the Theorem which is based on an asymptotic expansion of the solutions ϱ_i.

Lemma. For any solution ϱ of (Q′) an expansion of the form

$$\varrho_\nu = (2/3) + (-)^\nu \epsilon(h) + O(h^2) \tag{8}$$

is valid with $\epsilon(h) = O(h)$.

Proof. We substitute $1 - R_1\varrho_1$ for ϱ_0 in the second equation of (Q′) and replace ϱ_1 by $(2/3) - \epsilon$. This yields

$$\Omega(\epsilon, h) := 1 - R_0(h)(1 - R_1(h)((2/3) - \epsilon)^2)^2 - ((2/3) - \epsilon) = 0.$$

Since $\Omega(0,0) = 0$ we conclude [H77] that any solution branch has an expansion of the form

$$\epsilon(h) = \epsilon_0 h^\alpha (1 + o(1))$$

with some rational exponent α determined from the Newton polygon of Ω. For the proof of the Lemma it is sufficient to show that $\alpha \geq 1$ since then (8) holds for $i = 1$ and substituting this expansion into the first equation of (Q′) gives

$$\varrho_0 = 1 - ((3/4) + O(h^2))((2/3) - \epsilon(h) + O(h^2))^2$$

which establishes (8) also in case $i = 0$. We now turn to the computation of α, i.e. a discussion of the Newton diagram for Ω. To this end, Figure 4 displays the indices of the Taylor coefficients

$$\Omega_{i,j} := \partial_\epsilon^i \partial_h^j \Omega\,(0,0)$$

which are zero (circle), nonzero (asterisk) and which depend on the Taylor coefficients θ of κ (dot). It is easily verified from (7) that

$$\begin{aligned} \Omega_{3,0} &= 27/4 \\ \Omega_{i,j} &= 0, \quad i + j \leq 2. \end{aligned}$$

However, the explicit form of the higher order coefficients depends in a complicated manner on the coefficients θ_i. For example, with the aid of MACSYMA,

$$\Omega_{0,3} = \frac{864\theta_1^2\theta_4 - 2160\theta_1\theta_2\theta_3 + 1280\theta_2^3 + 288\theta_1^4\theta_2}{135\theta_1^3}$$

and, while nonzero in general, this coefficient will vanish for certain combinations of the θ_i. Fortunately, these details are irrelevant. Newton's polygon bounds the convex hull of the indices (i,j) for which $\Omega_{i,j} \neq 0$ and, for each branch, α equals the absolute value of one of the slopes of the polygon segments which connect the smallest nonzero indices on the axis. The polygon corresponding to a maximal number of nonzero $\Omega_{i,j}$ is shown in Figure 4. In any case, since $\Omega_{3,0} \neq 0$, it is clear from the Figure that α must be at least one.

♣

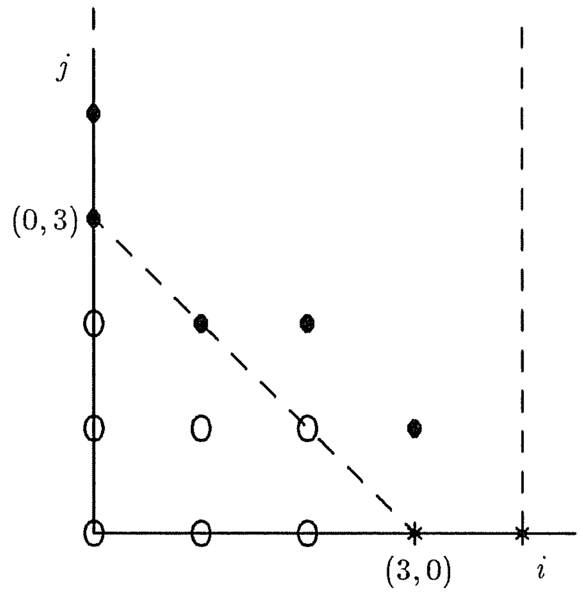

Figure 4. Newton's polygon for Ω

With the aid of the Lemma we can now give the

Proof of the Theorem. We first obtain bounds on the derivatives of the interpolant p. We claim that

$$p'(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + O(h) \tag{9}$$

and that

$$|p^{(i)}(t)| = O(h^i) , \quad i = 1,2,3. \tag{10}$$

It is sufficient to prove the corresponding assertions for the Bézier coefficients of the derivatives of p which are given by

$$\begin{aligned} p'/3 &: \delta_0 d_0, \quad a - \delta_0 d_0 - \delta_1, \quad \delta_1 d_1 \\ p^{(2)}/6 &: a - 2\delta_0 d_0 - \delta_1 d_1, \quad 2\delta_1 d_1 + \delta_0 d_0 - a \\ p^{(3)}/6 &: 3\delta_0 d_0 + 3\delta_1 d_1 - 2a. \end{aligned}$$

From (1), (6) and (8) we see that

$$\begin{aligned} \delta_0 &= (2/3)h + \epsilon h - (4/9)(\theta_1/\theta_2)h^2 + O(h^3) \\ \delta_1 &= (2/3)h - \epsilon h + (4/9)(\theta_1/\theta_2)h^2 + O(h^3). \end{aligned}$$

Using these expansions, the assertions (9) and (10) are easily verified. Setting

$$\tilde{\epsilon} := \epsilon h - (4/9)(\theta_1/\theta_2)h^2$$

it follows from (5) that

$$\delta_0 d_0 = \begin{bmatrix} (2/3)h + \tilde{\epsilon} \\ -(2/3)\theta_1 h^2 \end{bmatrix} + O(h^3), \quad \delta_1 d_1 = \begin{bmatrix} (2/3)h - \tilde{\epsilon} \\ (2/3)\theta_1 h^2 \end{bmatrix} + O(h^3). \tag{11}$$

Therefore all Bézier coefficients of p' are of the form

$$\begin{bmatrix} (2/3)h \\ 0 \end{bmatrix} + O(h^2)$$

which proves (9) and (10) for $i = 1$. Comparing (11) with the expansion for a in (5) yields the remaining cases in (10).

We now estimate the error by choosing an appropriate parametrization. To this end we write

$$p(t) =: \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad f(t) =: \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}.$$

By (9) and since

$$f'(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + O(t)$$

the functions x and X are invertible on $[x(0), x(1)] = [X(-h), X(h)]$, i.e. there exist functions x^{-1} and X^{-1} with

$$x(x^{-1}(\xi)) = \xi, \quad X(X^{-1}(\xi)) = \xi.$$

This provides the parametrizations

$$p : \xi \mapsto \begin{bmatrix} \xi \\ z(\xi) \end{bmatrix}, \quad f : \xi \mapsto \begin{bmatrix} \xi \\ Z(\xi) \end{bmatrix}$$

with $z := y\, x^{-1}$, $Z := Y\, X^{-1}$ which we use to compare the two curves. By the chain rule,

$$\begin{aligned} z' &= \frac{y'}{x'} = \cot\theta \\ z'' &= \frac{x'y'' - y'x''}{(x')^3} = \kappa\left(1 + \left(\frac{y'}{x'}\right)^2\right)^{3/2}, \end{aligned}$$

and the corresponding formulas hold for the derivatives of Z. Therefore, the functions z and Z, together with their first and second derivatives, match at $\xi_0 := x(0) = X(-h)$ and $\xi_1 := x(1) = X(h)$. Since $\xi_1 - \xi_0 = a = 2h + O(h^3)$, it follows that

$$\max_{\xi_0 \le \xi \le \xi_1} |z(\xi) - Z(\xi)| = O(h^6)$$

if the 6-th derivative of z is bounded, independently of h. But, since y and x are cubic polynomials, $z^{(6)}$ is a sum of terms

$$y^{(i)} \prod_{\nu} x^{(j_\nu)} \; / \; (x')^{i+\sum j_\nu}$$

where $i, j_\nu \leq 3$. The boundedness of these expressions follows from (9) and (10).

♣

The Theorem does not extend to curves with singular points as is shown by the following

Example. For the curve f defined by

$$\theta(t) = t^3 \; , \quad f(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

denote, as before, by p the interpolant corresponding to the data at $f(-h)$ and $f(h)$. This curve is strictly convex, but the curvature $\kappa(t) = 3t^2$ vanishes at $t = 0$. We claim that

$$p(1/2) = \begin{bmatrix} 0 \\ \gamma h^4 \end{bmatrix} \; + \; O(h^5) \tag{12}$$

with $\gamma \neq 0$. This proves that the approximation can be at most fourth order accurate since both s and p are symmetric with respect to the line $\{(x, y) : x = 0\}$.

To verify (12), we calculate the Bézier coefficients of p from (Q′) using (3), (5) and (6). Using the abbreviation $q \doteq r$ for $q = (1 + O(h^6))r$ we obtain

$$\begin{aligned} d_0 \times d_1 \; &\doteq \; 2h^3, \\ d_0 \times z \; = \; z \times d_1 \; &\doteq \; 2h^4, \\ R_0 \; = \; R_1 \; &\doteq \; 9/4, \\ \varrho_0 \; = \; \varrho_1 \; &\doteq \; \frac{2}{9}(\sqrt{10} - 1). \end{aligned}$$

This implies that the Bézier coefficients for p satisfy

$$\begin{aligned} b_0 \; &= \; p(-h) \; = \; -\int_{-h}^{0} \begin{bmatrix} \cos\theta(\sigma) \\ \sin\theta(\sigma) \end{bmatrix} d\sigma \; \doteq \; \begin{bmatrix} -h \\ h^4/4 \end{bmatrix} \\ b_1 \; &= \; b_0 + \varrho_0 \frac{z \times d_1}{d_0 \times d_1} d_0 \; \doteq \; \begin{bmatrix} -h \\ h^4/4 \end{bmatrix} + \frac{2}{9}(\sqrt{10} - 1)\frac{2h^4}{2h^3} \begin{bmatrix} 1 \\ -h^3 \end{bmatrix} \; =: \; \begin{bmatrix} -\gamma_1 h \\ -\gamma_2 h^4 \end{bmatrix}, \end{aligned}$$

and, by symmetry,

$$b_2 \; \doteq \; \begin{bmatrix} \gamma_1 h \\ -\gamma_2 h^4 \end{bmatrix} \; , \quad b_3 \; \doteq \; \begin{bmatrix} h \\ h^4/4 \end{bmatrix} .$$

Since

$$p(1/2) \;=\; \frac{1}{8}(b_0 + b_3) + \frac{3}{8}(b_1 + b_2),$$

(12) follows with

$$\gamma := \frac{1}{8}\;(2\;\frac{1}{4}) - \frac{3}{8}\;(2\;\gamma_2),$$

and γ is nonzero since γ_2 is irrational.

♣

4. Examples

In this section we discuss a few examples which illustrate the performance of the method. Figure 5 shows the interpolant for a circle with radius 1. Already for four knots, the deviation in curvature is less than one percent. Moreover, as the number n of interpolation points is increased, the error decays at the predicted rate $O(n^{-6})$ as is shown in the table below.

number of points	error	rate
4	.14E-2	
8	.21E-4	−6.07
16	.32E-6	−6.02
32	.49E-8	−6.01

In the second example the data are

$$\begin{aligned} f_i &= (\cos(4i\pi/5),\ \sin(4i\pi/5)) \\ d_i &= (-\sin(4i\pi/5),\ \cos(4i\pi/5)) \\ \kappa_i &= 2 \end{aligned}$$

and this yields the periodic curve in Figure 6.

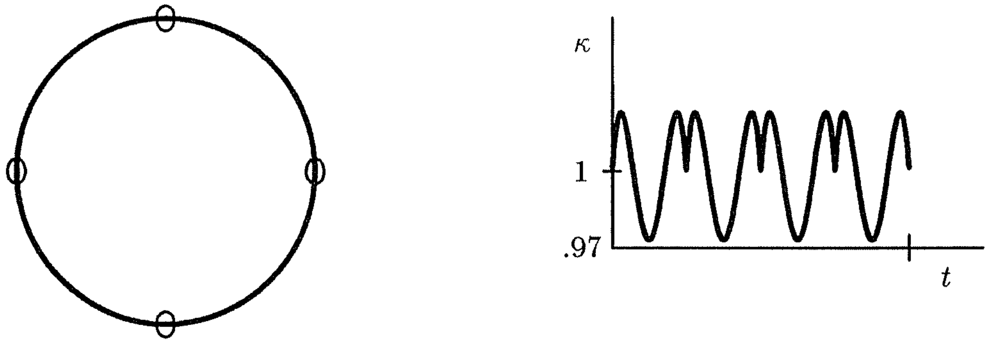

Figure 5. Interpolation of a circle

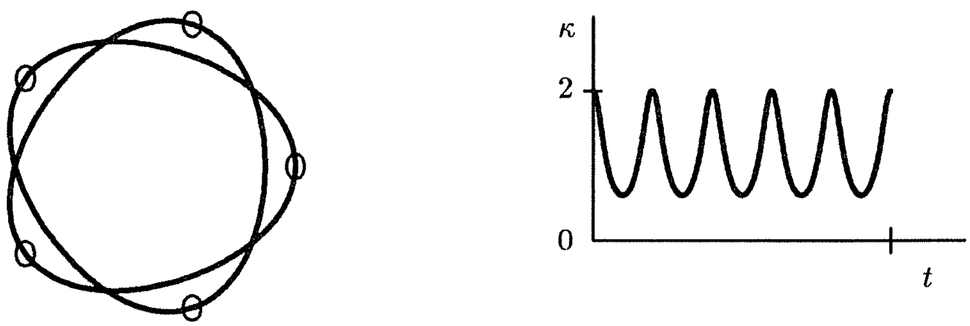

Figure 6. Interpolation of a pentagon

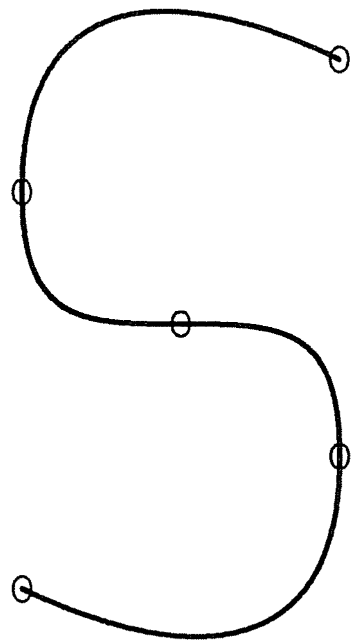

Figure 7.

Our final example (Figure 7) illustrates the interpolation of inflection points showing that the shape of the data is preserved.

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