Multi-criteria trapezoidal valued intuitionistic fuzzy decision making with Choquet integral based TOPSIS

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Abstract A generalized trapezoidal-valued intuitionistic fuzzy geometric aggregation operator is proposed which is then used to aggregate decision makers' opinions in group decision making process. An extension of TOPSIS, a multi-criteria trapezoidal-valued intuitionistic fuzzy decision making technique, to a group decision environment is also proposed, where inter-dependent or interactive characteristics among criteria and preference of decision makers are under consideration. Furthermore, Choquet integral-based distance between trapezoidal-valued intuitionistic fuzzy values is defined. Combining the trapezoidal-valued intuitionistic fuzzy geometric aggregation operator with Choquet integral-based distance, an extension of TOPSIS method is developed to deal with a multi-criteria trapezoidal-valued intuitionistic fuzzy group decision making problems. Finally, an illustrative example is provided to understand the proposed method.

Keywords: Multi-criteria group decision making; Interval-valued intuitionistic fuzzy sets; Fuzzy measures; GIIFGA operator; TOPSIS.

1 Introduction

Technique for order preference by similarity to ideal solution (TOPSIS) is a useful and practical technique for selection and ranking of alternatives through distance measures. The basic principle is that the chosen alternative should have the shortest distance from the positive-ideal solution and the farthest distance from the negative-ideal solution. In the TOPSIS theory, crisp values used for weights and performance ratings of the criterias. Hwang and Youn [25] developed a classical approach to multi-attribute/multi-criteria decision making (MADM/MCDM) problems by using TOPSIS. Human judgment and preference are often ambiguous and cannot be estimated with exact numeric value, so the crisp values are not suitable to model real-world situations. Fuzzy set theory [67] has been successfully used to handle imprecision (or uncertainty) in decision making problems, to solve the ambiguity in information from human judgement and preference. Since fuzzy numbers applied and used to establish a prototype fuzzy TOPSIS ([11], [38]), Recently a lot of work on fuzzy TOPSIS has been developed by several authors ([10], [14], [27], [31], [36], [54], [53], [55], [65], [66]). Atanassov gave the notion of intuitionistic fuzzy sets (IFS) which is an extension of Zadeh's [67] fuzzy set. IFS has proved to be a very suitable tool to describe the uncertain or imprecise decision information. Recently, IFS has received more attention and has been applied in the field of decision making and fuzzy TOPSIS has been extended to IFS TOPSIS ([3], [8], [12], [35]). The concept of interval-valued intuitionistic fuzzy sets (IVIFS) was introduced by Atanassov in [7], as a generalization of IFS. The basic characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers. Some operational laws of the IVIFS are defined in [5]. In [46] and [47] a novel method for multiple attribute decision making based on IVIFS and TOPSIS method in uncertain environments is presented. In [58] some geometric aggregation operators, such as the interval-valued intuitionistic fuzzy weighted geometric averaging (IIFWGA) operator and the interval-valued intuitionistic fuzzy ordered weighted geometric averaging (IIFOWGA) operator are defined and applications of the IIFWGA and

IIFOWGA operators to multiple attribute group decision making with interval-valued intuitionistic fuzzy information are given. Wei [57] applied IIFWGA aggregation functions to dealing with dynamic multiple attribute decision making where all the attribute values are expressed in intuitionistic fuzzy numbers or interval-valued intuitionistic fuzzy numbers.

These aggregation process are based on the assumption that the criteria (attribute) or preferences of decision makers are independent, and the aggregation operators are linear operators based on additive measures, which is characterized by an independence axiom ([28], [52]). For real decision making problems, there is a phenomenon that there exists some degree of inter-dependent or interactive characteristics between criteria ([17], [18], [22]). Decision makers invited usually come from same or similar fields for a decision problem. They have similar knowledge, preference and social status. Their subjective preference always shows non-linearity. Independence phenomena among these criteria and mutual preferential independence of decision makers are violated. Sugeno [45] introduced the concept of non-additive fuzzy measure, which only make a monotonicity instead of additivity property. It is most effective tool to modeling interaction phenomena ([19], [20], [26], [30], [40]) and deal with decision making problems ([17], [18], [21], [22], [39]). Liginlal and Ow [34] is an excellent review on analyzing decision maker behavior using fuzzy measure. In the real decision making problems, the attributes of the problem are often correlated or inter-dependent. Choquet integral [13] is a useful tool to model the correlation or inter-dependence. It has been studied and applied in the decision making methods ([1], [2], [9], [15], [16], [24], [32], [33], [37], [41], [43], [48], [49], [50], [51], [59], [60], [61], [62], [63]). Aggregation of decision makers' opinions is very important in group decision making problems to perform evaluation process. Group decision making involves weighted aggregation of all individual decisions to obtain a single collective decision. In [44], aggregation operator of intuitionistic fuzzy group decision making is proposed with the weights of decision makers. The weights of decision makers plays an important role in the process of aggregation. In [46], [47] and [64], aggregation of the interval-valued intuitionistic fuzzy group decision making environment with the Choquet integral is studied. Until now, we do not have any aggregation of the trapezoidal-valued intuitionistic fuzzy group decision making environment with Choquet integral. In this paper, we first develop a generalized trapezoidal-valued intuitionistic fuzzy geometric aggregation operator for aggregating all individual decision makers' opinions under trapezoidal-valued intuitionistic fuzzy group decision making environment. Combining this operator with TOPSIS on Choquet integral-based distance, a multi-criteria trapezoidal-valued intuitionistic fuzzy group decision making is proposed, where interaction phenomena among the decision making problem and weights of decision makers are taken into account.

Rest of the paper is organized as follows: In Section 2, we review ρ-fuzzy measure, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and trapezoidal fuzzy numbers. In Section 3, we introduce trapezoidal-valued intuitionistic fuzzy set and some operational laws on trapezoidal-valued intuitionistic fuzzy values. Order relation and some of its properties are also studied in this section. In Section 4, based on these operational laws, a generalized trapezoidal-valued intuitionistic fuzzy geometric aggregation operator is proposed, and some of its properties are examined. In Section 5, according to definition of Choquet integral, we define the

Choquet integral-based distance between any two trapezoidal-valued intuitionistic fuzzy sets. Combining the generalized trapezoidal-valued intuitionistic fuzzy geometric aggregation operator with Choquet integral-based distance, an extension of TOPSIS is developed to deal with a multi-criteria trapezoidal-valued intuitionistic fuzzy group decision making problems where inter-dependent or interactive characteristics among criteria and preference of decision makers are considered. In Section 6, an illustrative example is constructed to understand the application of the method and to demonstrate its practicality and feasibility.

2 Preliminaries

As preparation for introducing our new aggregation operators, some preliminary concepts are given in this section.

Let X be a crisp universe of generic elements, a *fuzzy set* A in the universe X is a mapping from X to $[0, 1]$. For any $x \in X$, the value $A(x)$ is called the *degree of membership of* x in A.

Let $X = \{x_1, x_2, ..., x_n\}$ be the set of the attributes, $P(X)$ be the power set of X.

Definition 2.1 [56] A ρ-fuzzy measure μ on the set X is a function $\mu : P(X) \to [0, 1]$ satisfying the following axioms:

1. $\mu(\phi) = 0$, $\mu(X) = 1$;
2. $B \subseteq C$ implies $\mu(B) \le \mu(C)$, for all $B, C \subseteq X$;
3. $\mu(B \cup C) = \mu(B) + \mu(C) + \rho\mu(B)\mu(C)$ for all $B, C \subseteq X$ and $B \cap C = \phi$, where $\rho \in (-1, +\infty)$.

In the above definition, if $\rho = 0$, then the third condition reduces to the axiom of the additive measure:

$\mu(B \cup C) = \mu(B) + \mu(C)$ for all $B, C \subseteq X$ and $B \cap C = \phi$. Also $\rho \neq 0$ indicates that the ρ-fuzzy measure μ is non-additive and there is interaction between B and C.

If $\rho > 0$, then $\mu(B \cup C) > \mu(B) + \mu(C)$, which implies that μ is a super-additive measure. If $\rho < 0$, then $\mu(B \cup C) < \mu(B) + \mu(C)$, which implies that μ is a sub-additive measure.

If X is a finite set, then $\bigcup\limits_{i=1}^{n} x_i = X$. To determine ρ-fuzzy measure μ on X avoiding the computational complexity, Sugeno [45] gave the following Eq. (2.1)

$$\mu(X) = \mu\left(\bigcup_{i=1}^{n} x_i\right) = \begin{cases} \frac{1}{\rho}\left\{\prod\limits_{i=1}^{n}[1 + \rho\mu(x_i)] - 1\right\} & \text{if } \rho \neq 0, \\ \sum\limits_{i=1}^{n} \mu(x_i) & \text{if } \rho = 0. \end{cases} \tag{2.1}$$

It can be noted that $\mu(x_i)$ for a subset with a single element x_i is called a fuzzy density.

Especially for every subset $A \subset X$, we have

$$\mu(A) = \begin{cases} \frac{1}{\rho}\left\{\prod\limits_{x_i \in A}[1 + \rho\mu(x_i)] - 1\right\} & \text{if } \rho \neq 0, \\ \sum\limits_{x_i \in A} \mu(x_i) & \text{if } \rho = 0. \end{cases} \tag{2.2}$$

Based on Eq. (2.1), the value of ρ can be uniquely determined from $\mu(X) = 1$, which is equivalent to solving

$$1 = \frac{1}{\rho}\left\{\prod_{i=1}^{n}[1 + \rho\mu(x_i)] - 1\right\}. \tag{2.3}$$

If the elements of B in X are independent, we have

$$\mu(B) = \sum_{x_i \in B} \mu(x_i), \text{ for all } B \subseteq X. \tag{2.4}$$

Definition 2.2 [29] A binary operation $T : [0,1]^2 \rightarrow [0,1]$ is a triangular norm (t-norm) if it satisfies the following:

1. $T(1, x) = x$ for all $x \in X$. (Boundary condition)
2. $T(x, y) = T(y, x)$ for all $x, y \in X$. (Commutativity)
3. $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in X$. (Associativity)
4. if $w \leq x$ and $y \leq z$ then $T(w, y) \leq T(x, z)$ for all $w, x, y, z \in X$. (Monotonicity)

Definition 2.3 [29] A binary operation $S : [0,1]^2 \rightarrow [0,1]$ is a triangular conorm (t-conorm) if it satisfies the following:

1. $S(0, x) = x$ for all $x \in X$. (Boundary condition)
2. $S(x, y) = S(y, x)$ for all $x, y \in X$. (Commutativity)
3. $S(x, S(y, z)) = S(S(x, y), z)$ for all $x, y, z \in X$. (Associativity)
4. if $w \leq x$ and $y \leq z$ then $S(w, y) \leq S(x, z)$ for all $w, x, y, z \in X$. (Monotonicity)

Let X be a universe of discourse, a fuzzy set in X is an expression A given by $A = \{\langle x, t_A(x)\rangle \,|x \in X\}$, where $t_A : X \rightarrow [0,1]$ is a membership function which characterizes the degree of membership of the element x to the set A. The main characteristic of fuzzy sets is that: the membership function assigns to each element x in a universe of discourse X a membership degree in interval $[0, 1]$ and the non-membership degree equals one minus the membership degree, i.e., this single membership degree combines the evidence for x and the evidence against x, without indicating how much there is of each. The single membership value tells us nothing about the lack of knowledge. In real applications, however, the information of an object corresponding to a fuzzy concept may be incomplete, i.e., the sum of the membership degree and the non-membership degree of an element in a universe corresponding to a fuzzy concept may be less than one. In fuzzy set theory, there is no means to incorporate the lack of knowledge with the membership degrees. In 1986, Atanassov [4] generalized the concept of fuzzy set, and defined the concept of intuitionistic fuzzy set as follows.

Definition 2.4 [4] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, an intuitionistic fuzzy set in X is an expression A given by $A = \{(x_i, t_A(x_i), f_A(x_i))|x_i \in X\}$, where $t_A : X \to [0, 1]$, $f_A : X \to [0, 1]$ with the condition: $0 \leq t_A(x_i)+f_A(x_i) \leq 1$, for all x_i in X. The numbers $t_A(x_i)$ and $f_A(x_i)$ represent the degree of membership and the degree of non-membership of the element x_i in the set A, respectively.

For each intuitionistic fuzzy set A in X, if $\pi_A(x) = 1-t_A(x)-f_A(x)$, for all $x \in X$. Then $\pi_A(x)$ is called the degree of indeterminacy of x to A.
Specially, if $\pi_A(x) = 1 - t_A(x) - f_A(x) = 0$, for all $x \in X$. Then the intuitionistic fuzzy set A is reduced to a fuzzy set.

Atanassov and Gargov [7] subsequently introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers.

Definition 2.5 [7] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. An interval-valued intuitionistic fuzzy set A in X is an expression given by $A = \{(x_i, t_A(x_i), f_A(x_i))|x_i \in X\}$, where $t_A : X \to D[0,1]$, $f_A : X \to D[0,1]$ with the condition: $0 \leq \sup t_A(x_i) + \sup f_A(x_i) \leq 1$, for all x_i in X. The intervals $t_A(x_i)$ and $f_A(x_i)$ denote, respectively, the degree of belongingness and the degree of non-belongingness of the element x_i to the set A.

For any two intervals $[a,b]$ and $[c,d]$ with $b + d < 1$ belonging to $D[0,1]$, let $t_A(x) = [a,b]$, $f_A(x) = [c,d]$, so an interval-valued intuitionistic fuzzy set whose value is denoted by $A = \{\langle x, [a,b], [c,d]\rangle \, |x \in X\}$.
Fuzzy data is a data type with imprecision or with a source of uncertainty not caused by randomness, but due to ambiguity. Examples of fuzzy data types can easily be found in natural language. It is generally more convenient and useful in describing fuzzy data to use LR-type trapezoidal fuzzy numbers [42]. Zimmermann [68, Subsubsection 5.3.2] defined the LR-type trapezoidal fuzzy numbers as follows:

Definition 2.6 Let L (and R) be decreasing, shape functions from $\Re^+ = [0, \infty)$ to $[0,1]$ with $L(0) = 1$; $L(x) < 1$ for all $x > 0$; $L(x) > 0$ for all $x < 1$; $L(1) = 0$ or ($L(x) > 0$ for all x and $L(+\infty) = 0$). An LR-type trapezoidal fuzzy number

(TFN) X has the following membership function

$$\mu_X(x) = \begin{cases} 0 & \textit{for } x \leq m_1 \textit{ and } x \geq m_4, \\ L\left(\frac{m_2-x}{\alpha}\right) & \textit{for } m_1 < x \leq m_2, \\ 1 & \textit{for } m_2 \leq x \leq m_3, \\ R\left(\frac{x-m_3}{\beta}\right) & \textit{for } m_4 > x \geq m_3, \end{cases}$$

where $m_1 < m_2 < m_3 < m_4$ and $\alpha = m_2 - m_1 > 0$ and $\beta = m_4 - m_3 > 0$ are called the left and right spread, respectively. Symbolically, X is denoted by $(m_1, m_2, m_3, m_4)_{LR}$.

The LR-type TFN is very general and allows one to represent the different types of information. For example, the LR-type TFN $X = (m, m, m, m)_{LR}$ with $m \in \Re = (-\infty, \infty)$ is used to denote a real number m and the LR-type TFN $X = (a, a, b, b)_{LR}$ with $a, b \in \Re$ and $a < b$ is used to denote an interval $[a, b]$.

Definition 2.7 For an LR-type TFN $X = (m_1, m_2, m_3, m_4)_{LR}$, if $L(x) = R(x) = 1 - x$ then X is called a TFN, denoted by $X = (m_1, m_2, m_3, m_4)_T$, i.e.

$$\mu_X(x) = \begin{cases} 0 & \textit{for } x \leq m_1 \textit{ and } x \geq m_4, \\ 1 - \frac{m_2-x}{\alpha} & \textit{for } m_1 < x \leq m_2 (\alpha > 0), \\ 1 & \textit{for } m_2 \leq x \leq m_3, \\ 1 - \frac{x-m_3}{\beta} & \textit{for } m_4 > x \geq m_3 (\beta > 0). \end{cases}$$

where $m_1 < m_2 < m_3 < m_4$ and $\alpha = m_2 - m_1 > 0$ and $\beta = m_4 - m_3 > 0$ are called the left and right spread, respectively.

Definition 2.8 Let $X = (m_1, m_2, m_3, m_4)_{LR}$ be an LR-type trapezoidal fuzzy number then

$$\sup(X) = m_4.$$

Similarly,

$$\sup(X) = m_4$$

for trapezoidal fuzzy number $X = (m_1, m_2, m_3, m_4)_T$.

In LR-type TFNs, the TFNs are most commonly used. In the rest of paper we use TFN and denoted by $X = (m_1, m_2, m_3, m_4)$ instead of $X = (m_1, m_2, m_3, m_4)_T$.

3 Trapezoidal-valued intuitionistic fuzzy sets

Motivated by the IVIFS in [7], we define trapezoidal-valued intuitionistic fuzzy set (TVIFS). The fundamental characteristic of the TVIFS is that the values of its membership function and non-membership function are trapezoidal fuzzy number rather than exact numbers or interval-valued. The set of all trapezoidal fuzzy numbers on $[0, 1]$ is denoted by $Trap[0, 1]$, in which $m_1 \geq 0$ and $m_4 \leq 1$ for all trapezoidal fuzzy numbers (see Definitions 2.7 and **??**).

Definition 3.1 Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse. A trapezoidal-valued intuitionistic fuzzy set A in X is an expression given by $A = \{(x_i, t_A(x_i), f_A(x_i))|x_i \in X\}$, where $t_A : X \to Trap[0, 1]$, $f_A : X \to Trap[0, 1]$ with the condition: $0 \leq \sup t_A(x_i) + \sup f_A(x_i) \leq 1$, for all x_i in X. The trapezoidal fuzzy numbers $t_A(x_i)$ and $f_A(x_i)$ denote, respectively, the degree of belongingness and the degree of non-belongingness of the element x_i to the set A.

For any two trapezoidal fuzzy numbers $(x_{1i}, x_{2i}, x_{3i}, x_{4i})$ and $(x'_{1i}, x'_{2i}, x'_{3i}, x'_{4i})$ with $x_{4i} + x'_{4i} \leq 1$ belonging to $Trap[0,1]$, let $t_A(x_i) = (x_{1i}, x_{2i}, x_{3i}, x_{4i})$, $f_A(x_i) = (x'_{1i}, x'_{2i}, x'_{3i},\ x'_{4i})$, so a trapezoidal-valued intuitionistic fuzzy set whose value is denoted by

$$A = \{\langle x_i, ((x_{1i}, x_{2i}, x_{3i}, x_{4i}), (x'_{1i}, x'_{2i}, x'_{3i}, x'_{4i}))\rangle\, |x_i \in X\}.$$

We call $((x_1, x_2, x_3, x_4), (x'_1, x'_2, x'_3, x'_4))$ a trapezoidal-valued intuitionistic fuzzy value. For convenience, let Ω be the set of all trapezoidal-valued intuitionistic fuzzy values. Obviously, according to Definition 3.1, we know that $((1,1,1,1), (0,0,0,0))$ and $((0,0,0,0),(1,1,1,1))$ are the largest and smallest trapezoidal-valued intuitionistic fuzzy values, respectively.

In the following, we define a distance measure between trapezoidal valued intuitionistic fuzzy values.

Definition 3.2 Let $X = \{x_1, \ldots, x_n\}$ be a universe of discourse, $\tilde{a} = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ and $\tilde{b} = ((b_{1i}, b_{2i}, b_{3i}, b_{4i}), (b'_{1i}, b'_{2i}, b'_{3i}, b'_{4i}))$ $(i = 1, 2, \ldots, n)$ be two trapezoidal-valued intuitionistic fuzzy values on X, then

$$\begin{aligned} d(\tilde{a}, \tilde{b}) &= \frac{1}{8}\sum_{i=1}^{n}\Big(l|a_{1i} - b_{1i}| + |a_{2i} - b_{2i}| + |a_{3i} - b_{3i}| + r|a_{4i} - b_{4i}| \\ &\quad + l|a'_{1i} - b'_{1i}| + |a'_{2i} - b'_{2i}| + |a'_{3i} - b'_{3i}| + r|a'_{4i} - b'_{4i}|\Big), \end{aligned}$$

is called the distance between $\tilde{a}$ and $\tilde{b}$, where $l = \int_0^1 L^{-1}(w)dw$ and $r = \int_0^1 R^{-1}(w)dw$, obviously $l = r = 1/2$ with the reference L and R in Definition 2.7.

If

$$\begin{aligned} D(\tilde{a},\tilde{b}) &= \frac{1}{8}\sum_{i=1}^{n} w_i\Big(l|a_{1i}-b_{1i}| + |a_{2i}-b_{2i}| + |a_{3i}-b_{3i}| + r|a_{4i}-b_{4i}| \\ &\quad +l|a'_{1i}-b'_{1i}| + |a'_{2i}-b'_{2i}| + |a'_{3i}-b'_{3i}| + r|a'_{4i}-b'_{4i}|\Big), \end{aligned}$$

where $w=(w_1,\ldots,w_n)$ is the weight vector of x_j such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i$=1, then $D(\tilde{a},\tilde{b})$ is called the weighted distance between $\tilde{a}$ and $\tilde{b}$.

Definition 3.3 We defined the following expressions for any two trapezoidal-valued intuitionistic fuzzy values,

$\tilde{a}=((a_1,a_2,a_3,a_4),(a'_1,a'_2,a'_3,a'_4))$ and $\tilde{b}=((b_1,b_2,b_3,b_4),(b'_1,b'_2,b'_3,b'_4))$;

1. $\tilde{a} \leq \tilde{b}$ *if and only if* $a_1 \leq b_1$ *and* $a_2 \leq b_2$ *and* $a_3 \leq b_3$ *and* $a_4 \leq b_4$ *and* $a'_1 \geq b'_1$ *and* $a'_2 \geq b'_2$ *and* $a'_3 \geq b'_3$ *and* $a'_4 \geq b'_4$.

2. $\tilde{a} = \tilde{b}$ *if and only if* $a_1 = b_1$ *and* $a_2 = b_2$ *and* $a_3 = b_3$ *and* $a_4 = b_4$ *and* $a'_1 = b'_1$ *and* $a'_2 = b'_2$ *and* $a'_3 = b'_3$ *and* $a'_4 = b'_4$.

Since Definition 3.3 is not satisfied in many cases, thus it cannot be used to compare intuitionistic fuzzy values. We define a score function and an accuracy function of trapezoidal-valued intuitionistic fuzzy values for the comparison between two trapezoidal-valued intuitionistic fuzzy values.

Definition 3.4 Let $\tilde{a}=((a_1,a_2,a_3,a_4),(a'_1,a'_2,a'_3,a'_4))$ be a trapezoidal-valued intuitionistic fuzzy values, if $S(\tilde{a})=(a_1+a_2+a_3+a_4-a'_1-a'_2-a'_3-a'_4)/4$, then $S(\tilde{a})$ is called a score function of $\tilde{a}$, where $S(\tilde{a}) \in [-1,1]$.

Definition 3.5 Let $\tilde{a} = ((a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4))$ be a trapezoidal-valued intuitionistic fuzzy values, if $H(\tilde{a}) = (a_1 + a_2 + a_3 + a_4 + a'_1 + a'_2 + a'_3 + a'_4)/4$, then $H(\tilde{a})$ is called an accuracy function of $\tilde{a}$, where $H(\tilde{a}) \in [0, 1]$.

The score function S and the accuracy function H are, respectively, defined as the difference and the sum of the membership function $t_A(x)$ and the non-membership function $f_A(x)$. Next we define order relation between two trapezoidal-valued intuitionistic fuzzy values.

Definition 3.6 Let $\tilde{a} = ((a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4))$ and $\tilde{b} = ((b_1, b_2, b_3, b_4), (b'_1, b'_2, b'_3, b'_4))$ be any two trapezoidal-valued intuitionistic fuzzy values.

1. If $S(\tilde{a}) < S(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$.

2. If $S(\tilde{a}) = S(\tilde{b})$ and;

   i. if $H(\tilde{a}) < H(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$.

   ii. if $H(\tilde{a}) = H(\tilde{b})$, then $\tilde{a}$ and $\tilde{b}$ represent the same information, denoted by $\tilde{a} = \tilde{b}$.

Motivated by the operations in ([5], [6], [47], [58]), we define two operational laws of trapezoidal-valued intuitionistic fuzzy values.

Definition 3.7 Let $\tilde{a} = ((a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4))$ and $\tilde{b} = ((b_1, b_2, b_3, b_4), (b'_1, b'_2, b'_3, b'_4))$ be two trapezoidal-valued intuitionistic fuzzy values, then

1. $\tilde{a}.\tilde{b} = ((a_1b_1, a_2b_2, a_3b_3, a_4b_4), (a'_1 + b'_1 - a'_1b'_1, a'_2 + b'_2 - a'_2b'_2, a'_3 + b'_3 - a'_3b'_3, a'_4 + b'_4 - a'_4b'_4))$;

2. $\tilde{a}^{\lambda} = ((a_1^{\lambda}, a_2^{\lambda}, a_3^{\lambda}, a_4^{\lambda}), (1-(1-a_1')^{\lambda}, 1-(1-a_2')^{\lambda}, 1-(1-a_3')^{\lambda}, 1-(1-a_4')^{\lambda},)), \qquad \lambda > 0.$

For two operational laws of Definition 3.7, it is easy to obtain the following propositions.

Proposition 3.8 Let $\tilde{a} = ((a_1, a_2, a_3, a_4), (a_1', a_2', a_3', a_4'))$ and $\tilde{b} = ((b_1, b_2, b_3, b_4), (b_1', b_2', b_3', b_4'))$ be two trapezoidal-valued intuitionistic fuzzy values, and let $\tilde{c} = \tilde{a}.\tilde{b}$ and $\tilde{d} = \tilde{a}^{\lambda}$, then both $\tilde{c}$ and $\tilde{d}$ are also trapezoidal-valued intuitionistic fuzzy values.

Proposition 3.9 Let $\tilde{a} = ((a_1, a_2, a_3, a_4), (a_1', a_2', a_3', a_4'))$ and $\tilde{b} = ((b_1, b_2, b_3, b_4), (b_1', b_2', b_3', b_4'))$ be two trapezoidal-valued intuitionistic fuzzy values. Then we have:

1. $\tilde{a}.\tilde{b} = \tilde{b}.\tilde{a}$;
2. $(\tilde{a}.\tilde{b})^{\lambda} = \tilde{a}^{\lambda}.\tilde{b}^{\lambda}$;
3. $\tilde{a}^{\lambda_1+\lambda_2} = \tilde{a}^{\lambda_1}.\tilde{a}^{\lambda_2}$,

   for all $\lambda, \lambda_1, \lambda_2 > 0$.

4 Generalized trapezoidal-valued intuitionistic fuzzy geometric aggregation operator

In the following, based on ρ-fuzzy measure, we first give the definition of generalized trapezoidal-valued intuitionistic geometric aggregation operator and then study its properties.

Definition 4.1 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$ be a collection of trapezoidal-valued intuitionistic fuzzy values on X and μ be a ρ-fuzzy measure on X. Based on ρ-fuzzy measure μ, a generalized trapezoidal-valued intuitionistic fuzzy geometric aggregation (GTIFGA) operator of dimension n is a mapping GTIFGA: $\Omega^n \to \Omega$ such that

$$
\begin{aligned}
& GTIFGA_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \\
= \; & (\tilde{a}_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (\tilde{a}_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})} \cdot \cdots \cdot (\tilde{a}_{(n)})^{\mu(A_{(n)})-\mu(A_{(n+1)})},
\end{aligned}
$$

where $_{(\cdot)}$ indicates a permutation on X such that $\tilde{a}_{(1)} \leq \tilde{a}_{(2)} \leq \cdots \leq \tilde{a}_{(n)}$ and $A_{(i)} = ((i), \ldots, (n))$, $A_{(n+1)} = \phi$.

Theorem 4.2 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$ be a collection of trapezoidal-valued intuitionistic fuzzy values on X, and μ be a ρ-fuzzy measure on X. then their aggregated value by using the GTIFGA$_\mu$ operator is also a trapezoidal-valued intuitionistic fuzzy value, and

$$
\begin{aligned}
GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \;=\; & \Big(\Big(\prod_{i=1}^{n}(a_{1(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(a_{2(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \prod_{i=1}^{n}(a_{3(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(a_{4(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big), \\
& \Big(1-\prod_{i=1}^{n}(1-(a'_{1(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& 1-\prod_{i=1}^{n}(1-(a'_{2(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& 1-\prod_{i=1}^{n}(1-(a'_{3(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& 1-\prod_{i=1}^{n}(1-(a'_{4(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big)\Big) \qquad (4.1)
\end{aligned}
$$

where $_{(\cdot)}$ indicates a permutation on X such that $\tilde{a}_{(1)} \leq \tilde{a}_{(2)} \leq \cdots \leq \tilde{a}_{(n)}$ and $A_{(i)} = ((i), \dots, (n))$, $A_{(n+1)} = \phi$.

Proof. The first result follows immediately from Definition 4.1 and Proposition 3.8. Next we prove Eq. (4.1) by using mathematical induction on n.

By the operational laws of Definition 3.7, we have

$$
\begin{aligned}
(\tilde{a}_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \;=\; & \Big(\Big((a_{1(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}, (a_{2(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}, (a_{3(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}, \\
& (a_{4(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}\Big), \Big(1-(1-(a'_{1(1)}))^{\mu(A_{(1)})-\mu(A_{(2)})}, 1- \\
& (1-(a'_{2(1)}))^{\mu(A_{(1)})-\mu(A_{(2)})}, 1-(1-(a'_{3(1)}))^{\mu(A_{(1)})-\mu(A_{(2)})}, \\
& 1-(1-(a'_{4(1)}))^{\mu(A_{(1)})-\mu(A_{(2)})}\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{a}_{(2)})^{\mu(A_{(1)})-\mu(A_{(2)})} &= \Big(\Big((a_{1(2)})^{\mu(A_{(1)})-\mu(A_{(2)})}, (a_{2(2)})^{\mu(A_{(1)})-\mu(A_{(2)})}, (a_{3(2)})^{\mu(A_{(1)})-\mu(A_{(2)})}, \\
&(a_{4(2)})^{\mu(A_{(1)})-\mu(A_{(2)})}\Big), \Big(1-(1-(a'_{1(2)}))^{\mu(A_{(1)})-\mu(A_{(2)})}, \\
&1-(1-(a'_{2(2)}))^{\mu(A_{(1)})-\mu(A_{(2)})}, 1-(1-(a'_{3(2)}))^{\mu(A_{(1)})-\mu(A_{(2)})}, \\
&1-(1-(a'_{4(2)}))^{\mu(A_{(1)})-\mu(A_{(2)})}\Big)\Big).
\end{aligned}
$$

Also

$$
\begin{aligned}
\tilde{a}_1.\tilde{a}_2 &= ((a_{1(1)}a_{1(2)}, a_{2(1)}a_{2(2)}, a_{3(1)}a_{3(2)}, a_{4(1)}a_{4(2)}), (a'_{1(1)} + a'_{1(2)} - a'_{1(1)}a'_{1(2)}, a'_{2(1)} \\
&+a'_{2(2)} - a'_{2(1)}a'_{2(2)}, a'_{3(1)} + a'_{3(2)} - a'_{3(1)}a'_{3(2)}, a'_{4(1)} + a'_{4(2)} - a'_{4(1)}a'_{4(2)}])
\end{aligned}
$$

$$
\begin{aligned}
\tilde{a}_1.\tilde{a}_2 &= ((a_{1(1)}a_{1(2)}, a_{2(1)}a_{2(2)}, a_{3(1)}a_{3(2)}, a_{4(1)}a_{4(2)}), (1-(1-a'_{1(1)})(1-a'_{1(2)}), 1- \\
&(1-a'_{2(1)})(1-a'_{2(2)}), 1-(1-a'_{3(1)})(1-a'_{3(2)}), 1-(1-a'_{4(1)})(1-a'_{4(2)}))).
\end{aligned}
$$

For $n = 2$ in Eq. (4.1), we have

$$\begin{aligned} GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2) &= (\tilde{a}_1)^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (\tilde{a}_2)^{\mu(A_{(2)})-\mu(A_{(3)})} \\ &= \Big(\Big((a_{1(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (a_{1(2)})^{\mu(A_{(2)})-\mu(A_{(3)})}, \\ &\quad (a_{2(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (a_{2(2)})^{\mu(A_{(2)})-\mu(A_{(3)})}, \\ &\quad (a_{3(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (a_{3(2)})^{\mu(A_{(2)})-\mu(A_{(3)})}, \\ &\quad (a_{4(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (a_{4(2)})^{\mu(A_{(2)})-\mu(A_{(3)})}\Big), \\ &\quad \Big(1-(1-(a'_{1(1)}))^{\mu(A_{(1)})-\mu(A_{(2)})}(1-(a'_{1(2)}))^{\mu(A_{(2)})-\mu(A_{(3)})}, \\ &\quad 1-(1-(a'_{2(1)}))^{\mu(A_{(1)})-\mu(A_{(2)})}(1-(a'_{2(2)}))^{\mu(A_{(2)})-\mu(A_{(3)})}, \\ &\quad 1-(1-(a'_{3(1)}))^{\mu(A_{(1)})-\mu(A_{(2)})}(1-(a'_{3(2)}))^{\mu(A_{(2)})-\mu(A_{(3)})}, \\ &\quad 1-(1-(a'_{4(1)}))^{\mu(A_{(1)})-\mu(A_{(2)})}(1-(a'_{4(2)}))^{\mu(A_{(2)})-\mu(A_{(3)})}\Big)\Big). \end{aligned}$$

That is, for $n = 2$, Eq. (4.1) holds.

Suppose that for $n = k$, Eq. (4.1) holds, i.e.,

$$\begin{aligned} GTIFGA_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) &= \Bigg(\Bigg(\prod_{i=1}^{k}(a_{1(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{k}(a_{2(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\ &\quad \prod_{i=1}^{k}(a_{3(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{k}(a_{4(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg), \\ &\quad \Bigg(1 - \prod_{i=1}^{k}(1 - (a'_{1(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\ &\quad 1 - \prod_{i=1}^{k}(1 - (a'_{2(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\ &\quad 1 - \prod_{i=1}^{k}(1 - (a'_{3(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\ &\quad 1 - \prod_{i=1}^{k}(1 - (a'_{4(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg)\Bigg). \end{aligned}$$

Then, for $n = k + 1$, according to Definition 3.1, we have

$$
\begin{aligned}
& GTIFGA_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{k+1}) \\
= & \Bigg(\Big((a_{1(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})} \prod_{i=1}^{k} (a_{1(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad (a_{2(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})} \prod_{i=1}^{k} (a_{2(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad (a_{3(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})} \prod_{i=1}^{k} (a_{3(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad (a_{4(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})} \prod_{i=1}^{k} (a_{4(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big), \\
& \Big(1 - (1 - (a'_{1(k+1)}))^{\mu(A_{(k+1)})-\mu(A_{(k+2)})} \prod_{i=1}^{k} (1 - (a'_{1(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1 - (1 - (a'_{2(k+1)}))^{\mu(A_{(k+1)})-\mu(A_{(k+2)})} \prod_{i=1}^{k} (1 - (a'_{2(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1 - (1 - (a'_{3(k+1)}))^{\mu(A_{(k+1)})-\mu(A_{(k+2)})} \prod_{i=1}^{k} (1 - (a'_{3(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1 - (1 - (a'_{4(k+1)}))^{\mu(A_{(k+1)})-\mu(A_{(k+2)})} \prod_{i=1}^{k} (1 - (a'_{4(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= & \Bigg(\Big(\prod_{i=1}^{k+1} (a_{1(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{k+1} (a_{2(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad \prod_{i=1}^{k+1} (a_{3(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{k+1} (a_{4(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big), \\
& \Big(1 - \prod_{i=1}^{k+1} (1 - (a'_{1(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1 - \prod_{i=1}^{k+1} (1 - (a'_{2(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1 - \prod_{i=1}^{k+1} (1 - (a'_{3(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1 - \prod_{i=1}^{k+1} (1 - (a'_{4(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big)\Bigg).
\end{aligned}
$$

That is, for $n = k + 1$, Eq. (4.1) still holds.

Therefore, for all n, the Eq. (4.1) holds. ■

Remark 4.3 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ and $\tilde{b}_i = ((b_{1i}, b_{2i}, b_{3i}, b_{4i}), (b'_{1i}, b'_{2i}, b'_{3i}, b'_{4i}))$ $(i = 1, 2, \ldots, n)$ be two collections of trapezoidal-valued intuitionistic fuzzy values on X. Since $a_{ji}, a'_{ji}, b_{ji}, b'_{ji} \in [0, 1]$ for any i and $j = 1, 2, 3, 4$. If we assume that $T_P(a'_{ji}, b'_{ji}) = a'_{ji} \cdot b'_{ji}$, $S_P(a_{ji}, b_{ji}) = a_{ji} + b_{ji} - a_{ji} \cdot b_{ji}$, then $T_P(a'_{ji}, b'_{ji})$ is one of the basic t-norms, called the product, which is satisfying the axioms of definition 2.2. $S_P(a_{ji}, b_{ji})$ is one of the basic t-conorms, called the probabilistic sum [29], and S_P is also called the dual t-conorm of T_P, which is satisfying the axioms of definition 2.3. The associativity of t-norms and t-conorms allows us to extend the product T_P and probabilistic sum S_P in unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1, x_2, ..., x_n) \in [0, 1]^n$ and $(y_1, y_2, ..., y_n) \in [0, 1]^n$, respectively:

$$T_P(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} x_i,$$

$$S_P(y_1, y_2, ..., y_n) = 1 - \prod_{i=1}^{n} (1 - y_i).$$

Assume that $x'_{1i} = 1-(1-(a'_{1(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $x'_{2i} = 1-(1-(a'_{2(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $x'_{3i} = 1 - (1 - (a'_{3(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $x'_{4i} = 1 - (1 - (a'_{4(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $x_{1i} = (a_{1(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $x_{2i} = (a_{2(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $x_{3i} = (a_{3(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $x_{4i} =$

$(a_{4(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$. Theorem 4.2 further implies

$$
\begin{aligned}
GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \Big(\Big(T_P(x_{11}, x_{12}, \ldots, x_{1n}), T_P(x_{21}, x_{22}, \ldots, x_{2n}), \\
&\qquad T_P(x_{31}, x_{32}, \ldots, x_{3n}), T_P(x_{41}, x_{42}, \ldots, x_{4n})\Big), \\
&\quad \Big(S_P(x'_{11}, x'_{12}, \ldots, x'_{1n}), S_P(x'_{21}, x'_{22}, \ldots, x'_{2n}) \\
&\qquad S_P(x'_{31}, x'_{32}, \ldots, x'_{3n}), S_P(x'_{41}, x'_{42}, \ldots, x'_{4n})\Big)\Big).
\end{aligned}
$$

Thus the generalized interval-valued intuitionistic fuzzy geometric aggregation operator can be represented by one of the basic t-norms T_P and t-conorms S_P.

Corollary 4.4 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$ be a collection of trapezoidal-valued intuitionistic fuzzy values on X. If all $\tilde{a}_i$ are equal $(i = 1, 2, \ldots, n)$ that is, for all i, $\tilde{a}_i = \tilde{a} = ((a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4))$, then

$$GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}.$$

Proof. By Theorem 4.2, if for all i $(i = 1, 2, \ldots, n)$, $\tilde{a}_i = \tilde{a}$, then

$$
\begin{aligned}
GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \Big(\Big(a_1^{\sum_{i=1}^{n} \mu(A_{(i)})-\mu(A_{(i+1)})}, a_2^{\sum_{i=1}^{n} \mu(A_{(i)})-\mu(A_{(i+1)})}, \\
&\quad a_3^{\sum_{i=1}^{n} \mu(A_{(i)})-\mu(A_{(i+1)})}, a_4^{\sum_{i=1}^{n} \mu(A_{(i)})-\mu(A_{(i+1)})}\Big), \\
&\quad \Big(1-(1-b_1)^{\sum_{i=1}^{n} \mu(A_{(i)})-\mu(A_{(i+1)})}, \\
&\quad 1-(1-b_2)^{\sum_{i=1}^{n} \mu(A_{(i)})-\mu(A_{(i+1)})}, \\
&\quad 1-(1-b_3)^{\sum_{i=1}^{n} \mu(A_{(i)})-\mu(A_{(i+1)})}, \\
&\quad 1-(1-b_4)^{\sum_{i=1}^{n} \mu(A_{(i)})-\mu(A_{(i+1)})}\Big)\Big)
\end{aligned}
$$

Since

$$\sum_{i=1}^{n} \mu(A_{(i)}) - \mu(A_{(i+1)}) = 1.$$

Thus

$GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = ((a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4)) = \tilde{a}.$ ■

Corollary 4.5 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ and $\tilde{b}_i = ((b_{1i}, b_{2i}, b_{3i}, b_{4i}),$ $(b'_{1i}, b'_{2i}, b'_{3i}, b'_{4i}))$ $(i = 1, 2, \ldots, n)$ be two collections of trapezoidal-valued intuitionistic fuzzy values on X, and μ be a ρ-fuzzy measure on X. $_{(\cdot)}$ indicates a permutation on X such that $\tilde{a}_{(1)} \leq \cdots \leq \tilde{a}_{(n)}$ and $\tilde{b}_{(1)} \leq \cdots \leq \tilde{b}_{(n)}$. If $a_{ji} \leq b_{ji}$ and $a'_{ji} \geq b'_{ji}$ for all i and $j = 1, 2, 3, 4$, that is, $\tilde{a}_{(i)} \leq \tilde{b}_{(i)}$, then

$$GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq GTIFGA_{\mu}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n).$$

Proof. Since $A_{(i+1)} \subseteq A_{(i)}$, therefore $\mu(A_{(i)}) - \mu(A_{(i+1)}) \geq 0$. For all i and $j = 1, 2, 3, 4$, $a_{ji} \leq b_{ji}$ and $a'_{ji} \geq b'_{ji}$, thus we have

$$\begin{aligned}
\prod_{i=1}^{n} (a_{ji})^{\mu(A_{(i)})-\mu(A_{(i+1)})} &\leq \prod_{i=1}^{n} (b_{ji})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
1 - \prod_{i=1}^{n} \left(1 - a'_{ji}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} &\geq \prod_{i=1}^{n} \left(b'_{ji}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}.
\end{aligned}$$

Using Theorem 4.2 and Definition 3.3, we have

$GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq GTIFGA_{\mu}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n).$ ■

Corollary 4.6 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$ be a collection of trapezoidal-valued intuitionistic fuzzy values on X and μ be a ρ-fuzzy measure on X. If

$$\tilde{a}^- = \left(\left(\min_i a_{1i}, \min_i a_{2i}, \min_i a_{3i}, \min_i a_{4i}\right), \left(\max_i a'_{1i}, \max_i a'_{2i}, \max_i a'_{3i}, \max_i a'_{4i}\right)\right)$$

$$\tilde{a}^+ = \left(\left(\max_i a_{1i}, \max_i a_{2i}, \max_i a_{3i}, \max_i a_{4i}\right), \left(\min_i a'_{1i}, \min_i a'_{2i}, \min_i a'_{3i}, \min_i a'_{4i}\right)\right)$$

then

$$\tilde{a}^- \leq GTIFGA_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}^+.$$

Proof. For any $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$, $\tilde{a}^-$ and $\tilde{a}^+$ are trapezoidal valued intuitionistic fuzzy values.

Since $A_{(i+1)} \subseteq A_{(i)}$, therefore $\mu(A_{(i)}) - \mu(A_{(i+1)}) \geq 0$.

Let $_{(\cdot)}$ indicates a permutation on X such that $\tilde{a}_{(1)} \leq \cdots \leq \tilde{a}_{(n)}$, we have

$$\min_i a_{ji} \leq a_{j(i)} \leq \max_i a_{ji}, \text{ and } \min_i a'_{ji} \leq a'_{j(i)} \leq \max_i a_{ji}.$$

Thus

$$\prod_{i=1}^{n} \left(\min_i a_{ji}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \prod_{i=1}^{n} \left(a_{j(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \prod_{i=1}^{n} \left(\max_i a_{ji}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}$$

and

$$\begin{aligned} 1 - \prod_{i=1}^{n} \left(1 - \min_i a'_{ji}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} &\leq 1 - \prod_{i=1}^{n} \left(1 - a'_{j(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \\ &\leq 1 - \prod_{i=1}^{n} \left(1 - \max_i a'_{ji}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}. \end{aligned}$$

i.e.,

$$\left(\min_i a_{ji}\right)^{\sum\limits_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \prod_{i=1}^{n}\left(a_{j(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \left(\max_i a_{ji}\right)^{\sum\limits_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}$$

and

$$\begin{aligned} 1-\left(1-\min_i a'_{ji}\right)^{\sum\limits_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} &\leq 1-\prod_{i=1}^{n}\left(1-a'_{j(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \\ &\leq 1-\left(1-\max_i a'_{ji}\right)^{\sum\limits_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}. \end{aligned}$$

Since

$$\sum_{i=1}^{n}\left(\mu(A_{(i)})-\mu(A_{(i+1)})\right)=1.$$

So we have

$$\min_i a_{ji} \leq \prod_{i=1}^{n}\left(a_{j(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \max_i a_{ji}$$

and

$$\min_i a'_{ji} \leq 1-\prod_{i=1}^{n}\left(1-a'_{j(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leq \max_i a'_{ji}.$$

Using Theorem 4.2 and Definition 3.3, we have

$$\begin{aligned} &\left(\left(\min_i a_{1i}, \min_i a_{2i}, \min_i a_{3i}, \min_i a_{4i}\right), \left(\max_i a'_{1i}, \max_i a'_{2i}, \max_i a'_{3i}, \max_i a'_{4i}\right)\right) \\ \leq\ & GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \\ &\left(\left(\max_i a_{1i}, \max_i a_{2i}, \max_i a_{3i}, \max_i a_{4i}\right), \left(\min_i a'_{1i}, \min_i a'_{2i}, \min_i a'_{3i}, \min_i a'_{4i}\right)\right) \end{aligned}$$

that is,

$\tilde{a}^{-} \leq GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}^{+}$. ■

Corollary 4.7 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$ be a collection of trapezoidal-valued intuitionistic fuzzy values on X and μ be a ρ-fuzzy measure on X. If $\tilde{s} = ((a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4))$ is a trapezoidal valued intuitionistic fuzzy value on X, then

$$GTIFGA_\mu(\tilde{a}_1 \cdot \tilde{s}, \tilde{a}_2 \cdot \tilde{s}, \ldots, \tilde{a}_n \cdot \tilde{s}) = GTIFGA_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \cdot \tilde{s}.$$

Proof. Since for any $i(i = 1, 2, \ldots, n)$

$$\begin{aligned}
\tilde{a}_i \cdot \tilde{s} &= ((a_{1i} \cdot a_1, a_{2i} \cdot a_2, a_{3i} \cdot a_3, a_{4i} \cdot a_4), (a'_{1i} + a'_1 - a'_{1i} \cdot a'_1, \\
&\quad a'_{2i} + a'_2 - a'_{2i} \cdot a'_2, a'_{3i} + a'_3 - a'_{3i} \cdot a'_3, a'_{4i} + a'_4 - a'_{4i} \cdot a'_4)) \\
&= ((a_{1i} \cdot a_1, a_{2i} \cdot a_2, a_{3i} \cdot a_3, a_{4i} \cdot a_4), (1 - (1 - a'_{1i})(1 - a'_1), \\
&\quad 1 - (1 - a'_{2i})(1 - a'_2), 1 - (1 - a'_{3i})(1 - a'_3), 1 - (1 - a'_{4i})(1 - a'_4))).
\end{aligned}$$

By Theorem 4.2, we have

$$\begin{aligned}
&GTIFGA_\mu(\tilde{a}_1 \cdot \tilde{s}, \tilde{a}_2 \cdot \tilde{s}, \ldots, \tilde{a}_n \cdot \tilde{s}) = \\
&\Big(\Big(\prod_{i=1}^{n}(a_{1(i)}a_1)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(a_{2(i)}a_2)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(a_{3(i)}a_3)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
&\prod_{i=1}^{n}(a_{4(i)}a_4)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big), \Big(1 - \prod_{i=1}^{n}((1 - (a'_{1(i)}))(1 - (a'_1)))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
&1 - \prod_{i=1}^{n}((1 - (a'_{2(i)}))(1 - (a'_2)))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
&1 - \prod_{i=1}^{n}((1 - (a'_{3(i)}))(1 - (a'_3)))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
&1 - \prod_{i=1}^{n}((1 - (a'_{4(i)}))(1 - (a'_4)))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \Bigg(\Big(a_1 \prod_{i=1}^{n} (a_{1(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, a_2 \prod_{i=1}^{n} (a_{2(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& a_3 \prod_{i=1}^{n} (a_{3(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, a_4 \prod_{i=1}^{n} (a_{4(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big), \\
& \Big(1-(1-(a'_1)) \prod_{i=1}^{n} (1-(a'_{1(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1-(1-(a'_2)) \prod_{i=1}^{n} (1-(a'_{2(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1-(1-(a'_3)) \prod_{i=1}^{n} (1-(a'_{3(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1-(1-(a'_4)) \prod_{i=1}^{n} (1-(a'_{4(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big)\Bigg)
\end{aligned}
$$

According to Eq. (4.1), we have

$$
\begin{aligned}
& GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \cdot \tilde{s} \\
= \quad & \Bigg(\Big(a_1 \prod_{i=1}^{n} (a_{1(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, a_2 \prod_{i=1}^{n} (a_{2(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& a_3 \prod_{i=1}^{n} (a_{3(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, a_4 \prod_{i=1}^{n} (a_{4(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big), \\
& \Big(1-(1-(a'_1)) \prod_{i=1}^{n} (1-(a'_{1(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1-(1-(a'_2)) \prod_{i=1}^{n} (1-(a'_{2(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1-(1-(a'_3)) \prod_{i=1}^{n} (1-(a'_{3(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \quad 1-(1-(a'_4)) \prod_{i=1}^{n} (1-(a'_{4(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big)\Bigg)
\end{aligned}
$$

Thus,

$$GTIFGA_{\mu}(\tilde{a}_1 \cdot \tilde{s}, \tilde{a}_2 \cdot \tilde{s}, \ldots, \tilde{a}_n \cdot \tilde{s}) = GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \cdot \tilde{s}. \quad \blacksquare$$

Corollary 4.8 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$ be a collection of trapezoidal-valued intuitionistic fuzzy values on X and μ be a ρ-fuzzy measure on X. If $v > 0$, then

$$GTIFGA_\mu((\tilde{a}_1)^v, (\tilde{a}_2)^v, \ldots, (\tilde{a}_n)^v) = (GTIFGA_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n))^v.$$

Proof. Due to Definition 3.7, for any $i(i = 1, 2, \ldots, n)$ and $v > 0$ we have.

By Theorem 4.2, we have

$$\begin{aligned}
& GTIFGA_\mu((\tilde{a}_1)^v, (\tilde{a}_2)^v, \ldots, (\tilde{a}_n)^v) \\
= \; & \Big(\Big(\prod_{i=1}^{n}((a_{1(i)})^v)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}((a_{2(i)})^v)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \prod_{i=1}^{n}((a_{3(i)})^v)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}((a_{4(i)})^v)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big), \\
& \Big(1 - \prod_{i=1}^{n}((1-(a'_{1(i)}))^v)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1 - \prod_{i=1}^{n}((1-(a'_{2(i)}))^v)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \\
& \; 1 - \prod_{i=1}^{n}((1-(a'_{3(i)}))^v)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1 - \prod_{i=1}^{n}((1-(a'_{4(i)}))^v)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big)\Big)
\end{aligned}$$

$$\begin{aligned}
= \; & \Big(\Big(\prod_{i=1}^{n}(a_{1(i)})^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \prod_{i=1}^{n}(a_{2(i)})^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \prod_{i=1}^{n}(a_{3(i)})^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \\
& \prod_{i=1}^{n}(a_{4(i)})^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}\Big), \Big(1 - \prod_{i=1}^{n}(1-(a'_{1(i)}))^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \\
& 1 - \prod_{i=1}^{n}(1-(a'_{2(i)}))^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, 1 - \prod_{i=1}^{n}(1-(a'_{3(i)}))^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \\
& 1 - \prod_{i=1}^{n}(1-(a'_{4(i)}))^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}\Big)\Big)
\end{aligned}$$

Since

$$
\begin{aligned}
& (GTIFGA_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n))^v \\
= & \left(\left(\left(\prod_{i=1}^{n} (a_{1(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^v, \left(\prod_{i=1}^{n} (a_{2(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^v,\right.\right. \\
& \left.\left(\prod_{i=1}^{n} (a_{3(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^v, \left(\prod_{i=1}^{n} (a_{4(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^v\right), \\
& \left(1-\left(\prod_{i=1}^{n} (1-(a'_{1(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^v, 1-\left(\prod_{i=1}^{n} (1-(a'_{2(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^v,\right. \\
& \left.\left. 1-\left(\prod_{i=1}^{n} (1-(a'_{3(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^v, 1-\left(\prod_{i=1}^{n} (1-(a'_{4(i)}))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^v\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \left(\left(\prod_{i=1}^{n} (a_{1(i)})^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \prod_{i=1}^{n} (a_{2(i)})^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \prod_{i=1}^{n} (a_{3(i)})^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))},\right.\right. \\
& \left.\prod_{i=1}^{n} (a_{4(i)})^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}\right), \left(1-\prod_{i=1}^{n} (1-(a'_{1(i)}))^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))},\right. \\
& 1-\prod_{i=1}^{n} (1-(a'_{2(i)}))^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, 1-\prod_{i=1}^{n} (1-(a'_{3(i)}))^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}, \\
& \left.\left. 1-\prod_{i=1}^{n} (1-(a'_{4(i)}))^{v(\mu(A_{(i)})-\mu(A_{(i+1)}))}\right)\right).
\end{aligned}
$$

Thus,

$$GTIFGA_\mu((\tilde{a}_1)^v, (\tilde{a}_2)^v, \ldots, (\tilde{a}_n)^v) = (GTIFGA_\mu(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n))^v. \ \blacksquare$$

Due to Corollaries 4.7 and 4.8, we can obtain the following corollary.

Corollary 4.9 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$ be a collection of trapezoidal-valued intuitionistic fuzzy values on X. and μ be a ρ-fuzzy measure on X. If $v > 0$ and $\tilde{s} = ((a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4))$ is a trapezoidal valued intuitionistic fuzzy value on X, then

$$GTIFGA_{\mu}((\tilde{a}_1)^v \cdot \tilde{s}, (\tilde{a}_2)^v \cdot \tilde{s}, \ldots, (\tilde{a}_n)^v \cdot \tilde{s}) = (GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n))^v \cdot \tilde{s}.$$

Corollary 4.10 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ $(i = 1, 2, \ldots, n)$ be a collection of trapezoidal-valued intuitionistic fuzzy values on X. and μ be a ρ-fuzzy measure on X.

1. *If* $\mu(A) = 1$ *for any* $A \in P(X)$, *then*

$$GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \max(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}_{(n)}.$$

2. *If* $\mu(A) = 0$ *for any* $A \in P(X)$ *and* $A \neq X$, *then*

$$GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \min(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}_{(1)}.$$

3. *For any* $A, B \in P(X)$ *such that* $|A| = |B|$, *if* $\mu(A) = \mu(B)$ *and* $\mu\{(i), \ldots, (n)\} = \frac{n-i+1}{n}$, $1 \leq i \leq n$, *then*

$$\begin{aligned} & GTIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \\ = \; & \Big(\Big(\prod_{i=1}^{n}(a_{1(i)})^{\frac{1}{n}}, \prod_{i=1}^{n}(a_{2(i)})^{\frac{1}{n}}, \prod_{i=1}^{n}(a_{3(i)})^{\frac{1}{n}}, \prod_{i=1}^{n}(a_{4(i)})^{\frac{1}{n}}\Big), \\ & \Big(1 - \prod_{i=1}^{n}(1 - (a'_{1(i)}))^{\frac{1}{n}}, 1 - \prod_{i=1}^{n}(1 - (a'_{2(i)}))^{\frac{1}{n}}, \\ & \; 1 - \prod_{i=1}^{n}(1 - (a'_{3(i)}))^{\frac{1}{n}}, 1 - \prod_{i=1}^{n}(1 - (a'_{4(i)}))^{\frac{1}{n}}\Big)\Big) \end{aligned}$$

Tan [47] proposed a generalized interval-valued intuitionistic fuzzy geometric aggregation (GIIFGA) operator, which is defined as follows.

Definition 4.11 Let $\tilde{a}_i = ([a_{1i}, a_{2i}], [a'_{1i}, a'_{2i}])$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic fuzzy values on X, and μ be a ρ-fuzzy measure on X. Based on ρ-fuzzy measure, a generalized interval-valued intuitionistic fuzzy geometric aggregation (GIIFGA) operator of dimension n is a mapping GIIFGA: $\Omega^n \to \Omega$ such that

$$\begin{aligned} & GIIFGA_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \\ = \; & (\tilde{a}_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (\tilde{a}_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})} \cdot \cdots \cdot (\tilde{a}_{(n)})^{\mu(A_{(n)})-\mu(A_{(n+1)})}, \end{aligned}$$

where $_{(\cdot)}$ indicates a permutation on X such that $\tilde{a}_{(1)} \leq \tilde{a}_{(2)} \leq \cdots \leq \tilde{a}_{(n)}$ and $A_{(i)} = ((i), \ldots, (n))$, $A_{(n+1)} = \phi$.

Remark 4.12 If the trapezoidal number shifted to interval valued by deleting the two terms of trapezoidal number and closed brackets around that number then GIIFGAμ =GTIFGAμ.

5 Trapezoidal valued intuitionistic fuzzy group decision making process

Choquet integral is defined as follows.

Definition 5.1 [22] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, f be a positive real-valued function on X, and μ be a ρ-fuzzy measure on X. The discrete

Choquet integral of f with respective to μ is defined by

$$C_\mu(f) = \sum_{i=1}^{n} f(x_{(i)})[\mu(A_{(i)}) - \mu(A_{(i+1)})],$$

where $_{(\cdot)}$ indicates a permutation on X such that $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n)$. Also $A_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$, $A_{(n+1)} = \phi$.

Choquet integral based distance between two trapezoidal-valued intuitionistic fuzzy values is defined as follows.

Definition 5.2 Let $\tilde{a}_i = ((a_{1i}, a_{2i}, a_{3i}, a_{4i}), (a'_{1i}, a'_{2i}, a'_{3i}, a'_{4i}))$ and $\tilde{b}_i = ((b_{1i}, b_{2i}, b_{3i}, b_{4i}), (b'_{1i}, b'_{2i}, b'_{3i}, b'_{4i}))$ $(i = 1, 2, \ldots, n)$ be two collections of trapezoidal-valued intuitionistic fuzzy values on X, and μ be a ρ-fuzzy measure on X. $C(\tilde{a}, \tilde{b})$ is defined by Choquet integral-based distance as

$$C(\tilde{a}, \tilde{b}) = \sum_{i=1}^{n} d_{(i)}(\tilde{a}, \tilde{b})(\mu(A_{(i)}) - \mu(A_{(i+1)})),$$

where $d_i(\tilde{a}, \tilde{b}) = l|a_{1i} - b_{1i}| + |a_{2i} - b_{2i}| + |a_{3i} - b_{3i}| + r|a_{4i} - b_{4i}| + l|a'_{1i} - b'_{1i}| + |a'_{2i} - b'_{2i}| + |a'_{3i} - b'_{3i}| + r|a'_{4i} - b'_{4i}|$, so that $d_{(1)}(\tilde{a}, \tilde{b}) \leq d_{(2)}(\tilde{a}, \tilde{b}) \leq \cdots \leq d_{(n)}(\tilde{a}, \tilde{b})$, $A_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$, $A_{(n+1)} = \phi$.

In general, multi-criteria group decision making problem includes uncertain and imprecise data and information. We consider the multi-criteria group decision making problems where all the criteria values are expressed in trapezoidal-valued intuitionistic fuzzy values, and interactions phenomena among the decision making criteria or preference of decision makers are taken into account. The following notations are used to depict the considered problems:

$E = \{e_1, e_2, \ldots, e_r\}$ is the set of the experts involved in the decision process;

$A = \{a_1, a_2, \ldots, a_m\}$ is the set of the considered alternatives;

$C = \{c_1, c_2, \ldots, c_n\}$ is the set of the criterias used for evaluating the alternatives.

In group decision making problems, aggregation of expert opinions is very important to appropriately perform evaluation process. In the following, according to Choquet integral-based distance, Choquet integral-based TOPSIS is proposed for multi-criteria trapezoidal valued intuitionistic fuzzy group decision making where expert opinions are aggregated by the generalized trapezoidal-valued intuitionistic fuzzy geometric aggregation operator, which involves the following steps:

Step 1. As for every alternative a_i $(i = 1, 2, \ldots, m)$, each expert e_k $(k = 1, 2, \ldots, r)$, is invited to express their individual evaluation or preference according to each criteria c_j $(j = 1, 2, \ldots, n)$, by a trapezoidal valued intuitionistic fuzzy value $\tilde{a}_{ij}^k = ((a_{1ij^k}, a_{2ij^k}, a_{3ij^k}, a_{4ij^k}), (a'_{1ij^k}, a'_{2ij^k}, a'_{3ij^k}, a'_{4ij^k}))$ $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, n;$ $k = 1, 2, \ldots, r)$, where $(a_{1ij^k}, a_{2ij^k}, a_{3ij^k}, a_{4ij^k})$ indicates the uncertain degree that expert e_k considers what the alternative a_i should satisfy the criteria c_j, $(a'_{1ij^k}, a'_{2ij^k}, a'_{3ij^k}, a'_{4ij^k})$ indicates the uncertain degree that expert e_k considers what the alternative a_i should not satisfy the criteria c_j. Then we can obtain a decision making matrix as follow:

$$R^k = \begin{pmatrix} \tilde{a}_{11}^k & \tilde{a}_{12}^k & \cdots & \tilde{a}_{1n}^k \\ \tilde{a}_{21}^k & \tilde{a}_{22}^k & \cdots & \tilde{a}_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}^k & \tilde{a}_{m2}^k & \cdots & \tilde{a}_{mn}^k \end{pmatrix}$$

Step 2. Confirm the fuzzy density $\mu_i = \mu(e_i)$ of each expert. According to Eq. (2.3), parameter ρ_1 of expert can be determined.

Step 3. By Definition 3.3 or Definition 3.6, $\tilde{a}_{ij}^k$ is reordered such that $\tilde{a}_{ij}^{(k)} \leq \tilde{a}_{ij}^{(k+1)}$. Utilize the trapezoidal valued intuitionistic fuzzy Choquet integral operator

$$\begin{aligned} \tilde{a}_{ij} &= GTIFGA_{\mu}(\tilde{a}_{ij}^1, \tilde{a}_{ij}^2, \ldots, \tilde{a}_{ij}^r) = \Big(\Big(\prod_{k=1}^{r}(a_{1ij^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\ &\prod_{k=1}^{r}(a_{2ij^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \prod_{k=1}^{r}(a_{3ij^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\ &\prod_{i=1}^{n}(a_{4ij^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}\Big), \Big(1 - \prod_{i=1}^{n}(1-(a'_{1ij^{(k)}}))^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\ &1 - \prod_{i=1}^{n}(1-(a'_{2ij^{(k)}}))^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\ &1 - \prod_{i=1}^{n}(1-(a'_{3ij^{(k)}}))^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\ &1 - \prod_{i=1}^{n}(1-(a'_{4ij^{(k)}}))^{\mu(A_{(k)})-\mu(A_{(k+1)})}\Big)\Big) \end{aligned}$$

to aggregate all the trapezoidal valued intuitionistic fuzzy decision matrices $R^k = (\tilde{a}_{ij}^k)_{m\times n}$ $(k = 1, 2, \ldots, r)$ into a complex trapezoidal valued intuitionistic fuzzy decision matrix $R^k = (\tilde{a}_{ij}^k)_{m\times n}$, where $\tilde{a}_{ij} = ((a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}), (a'_{1ij}, a'_{2ij}, a'_{3ij}, a'_{4ij}))$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, $A_{(k)} = \{e_{(k)}, \ldots, e_{(r)}\}$, $A_{(r+1)} = \phi$, and $\mu(A_{(k)})$ can be calculated by Eq. (2.2).

Step 4. Let J_1 be a collection of benefit criteria (i.e., the larger c_j, the greater preference) and J_2 be a collection of cost criteria (i.e., the smaller c_j, the greater preference). The trapezoidal valued intuitionistic fuzzy positive-ideal solution (TV-IFPIS), denoted as $\tilde{\alpha}^+$, and the trapezoidal valued intuitionistic fuzzy negative-ideal solution (TV-IFNIS), denoted as $\tilde{\alpha}^- = (\tilde{\alpha}_1^- \ \tilde{\alpha}_2^- \ \ldots \ \tilde{\alpha}_n^-)$, are defined as follows:

$$
\begin{aligned}
\tilde{\alpha}^+ = \Bigg(\bigg(\Big(& \max_i a_{1ij}, \max_i a_{2ij}, \max_i a_{3ij}, \max_i a_{4ij} \Big) | j \in J_1, \\
& \Big(\min_i a_{1ij}, \min_i a_{2ij}, \min_i a_{3ij}, \min_i a_{4ij} \Big) | j \in J_2 \bigg), \\
& \bigg(\Big(\min_i a_{1ij}, \min_i a_{2ij}, \min_i a_{3ij}, \min_i a_{4ij} \Big) | j \in J_1, \\
& \Big(\max_i a_{1ij}, \max_i a_{2ij}, \max_i a_{3ij}, \max_i a_{4ij} \Big) | j \in J_2 \bigg) \Bigg) \\
& i = 1, 2, \ldots, m,
\end{aligned}
$$

$$
\tilde{\alpha}^+ = (\tilde{\alpha}_1^+ \ \ \tilde{\alpha}_2^+ \ \ \ldots \ \ \tilde{\alpha}_n^+)
$$

where $\tilde{\alpha}_j^+ = ((\alpha_{1j}, \alpha_{2j}, \alpha_{3j}, \alpha_{4j}), (\alpha'_{1j}, \alpha'_{2j}, \alpha'_{3j}, \alpha'_{4j}))$ $(j = 1, 2, \ldots, n)$.

$$
\begin{aligned}
\tilde{\alpha}^- = \Bigg(\bigg(\Big(& \min_i a_{1ij}, \min_i a_{2ij}, \min_i a_{3ij}, \min_i a_{4ij} \Big) | j \in J_1, \\
& \Big(\max_i a_{1ij}, \max_i a_{2ij}, \max_i a_{3ij}, \max_i a_{4ij} \Big) | j \in J_2 \bigg), \\
& \bigg(\Big(\max_i a_{1ij}, \max_i a_{2ij}, \max_i a_{3ij}, \max_i a_{4ij} \Big) | j \in J_1, \\
& \Big(\min_i a_{1ij}, \min_i a_{2ij}, \min_i a_{3ij}, \min_i a_{4ij} \Big) | j \in J_2 \bigg) \Bigg) \\
& i = 1, 2, \ldots, m,
\end{aligned}
$$

$$
\tilde{\alpha}^- = (\tilde{\alpha}_1^- \ \ \tilde{\alpha}_2^- \ \ \ldots \ \ \tilde{\alpha}_n^-)
$$

where $\tilde{\alpha}_j^- = ((\alpha_{1j}, \alpha_{2j}, \alpha_{3j}, \alpha_{4j}), (\alpha'_{1j}, \alpha'_{2j}, \alpha'_{3j}, \alpha'_{4j}))$ $(j = 1, 2, \ldots, n)$.

Moreover, we denote the alternatives a_i $(i = 1, 2, \ldots, m)$ by $x_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in})$.

Step 5. Confirm the fuzzy density $\mu_i = \mu(c_i)$ of each criteria. According to Eq. (2.3), parameter ρ_2 of criteria can be determined.

Step 6. According to Choquet integral based distance, calculate the distance between the alternative x_i and the IV-IFPIS $\tilde{\alpha}^+$ and the distance between the alternative x_i and the IV-IFNIS $\tilde{\alpha}^-$, respectively:

$$d_i(x_i, \tilde{\alpha}^+) = \sum_{j=1}^{n} d_{i(j)}(\tilde{a}_{ij}, \tilde{\alpha}_j^+)(\mu(A_{(j)}) - \mu(A_{(j+1)})), \tag{5.1}$$

where

$d_{ij}(\tilde{a}_{ij}, \tilde{\alpha}_j^+) = |\alpha_{1j} - a_{1ij}| + |\alpha_{2j} - a_{2ij}| + |\alpha_{3j} - a_{3ij}| + |\alpha_{4j} - a_{4ij}| + |\alpha'_{1j} - a'_{1ij}| + |\alpha'_{2j} - a'_{2ij}| + |\alpha'_{3j} - a'_{3ij}| + |\alpha'_{4j} - a'_{4ij}|$,

so that

$d_{i(1)}(\tilde{a}_{ij}, \tilde{\alpha}_j^+) \leq d_{i(2)}(\tilde{a}_{ij}, \tilde{\alpha}_j^+) \leq \cdots \leq d_{i(n)}(\tilde{a}_{ij}, \tilde{\alpha}_j^+)$, $A_{(j)} = \{c_{(j)}, \ldots, c_{(n)}\}$, $A_{(n+1)} = \phi$, $\mu(A_{(j)})$ can be calculated by Eq. (2.2)

$$d_i(x_i, \tilde{\alpha}^-) = \sum_{j=1}^{n} d_{i(j)}(\tilde{a}_{ij}, \tilde{\alpha}_j^-)(\mu(A_{(j)}) - \mu(A_{(j+1)})), \tag{5.2}$$

where $d_{ij}(\tilde{a}_{ij}, \tilde{\alpha}_j^-) = |\alpha_{1j} - a_{1ij}| + |\alpha_{2j} - a_{2ij}| + |\alpha_{3j} - a_{3ij}| + |\alpha_{4j} - a_{4ij}| + |\alpha'_{1j} - a'_{1ij}| + |\alpha'_{2j} - a'_{2ij}| + |\alpha'_{3j} - a'_{3ij}| + |\alpha'_{4j} - a'_{4ij}|$, so that $d_{i(1)}(\tilde{a}_{ij}, \tilde{\alpha}_j^-) \leq d_{i(2)}(\tilde{a}_{ij}, \tilde{\alpha}_j^-) \leq \cdots \leq d_{i(n)}(\tilde{a}_{ij}, \tilde{\alpha}_j^-)$, $A_{(j)} = \{c_{(j)}, \ldots, c_{(n)}\}$, $A_{(n+1)} = \phi$, $\mu(A_{(j)})$ can be calculated by Eq. (2.2)

Step 7. Calculate the closeness coefficient of each alternative:

$$r(x_i) = \frac{d_i(x_i, \tilde{\alpha}^-)}{d_i(x_i, \tilde{\alpha}^+) + d_i(x_i, \tilde{\alpha}^-)}, i = 1, 2, \ldots, m. \tag{5.3}$$

Step 8. Rank all the alternatives a_i $(i = 1, 2, \ldots, m)$ according to the closeness coefficient $r(x_i)$, the greater the value $r(x_i)$, the better the alternative $a_{i.}$

The main difference between the traditional TOPSIS and Choquet integral based TOPSIS (CITOPSIS) is that the CITOPSIS takes the Choquet integral based distance into account. It is reasonable to employ the Choquet integral in terms of the ρ-fuzzy measure to aggregate the performance values instead of the weighted average method, since the Choquet integral does not assume the independence of one element from another.

6 Illustrative example

In this example, we utilized the proposed method where inter-dependent or interactive characteristics among criteria and preference of decision makers are taken into account to get the most desirable alternative.

Step 1. There is an investment company, which wants to invest money in the best option (adapted from [23]). There is a panel with five possible alternatives in which to invest the money: a_1 is a car industry, a_2 is a food company, a_3 is a computer company, a_4 is an arms company, a_5 is a TV company. The investment company must take a decision according to the following four criteria: c_1 is the risk analysis; c_2 is the growth analysis; c_3 is the social-political impact analysis, c_4 is the environmental impact analysis. The five possible alternatives a_i $(i = 1, 2, 3, 4, 5)$ are to be evaluated using the trapezoidal valued intuitionistic fuzzy information by three decision makers e_k $(k = 1, 2, 3)$, as listed in R^1, R^2 and R^3 matrices.

$$R^1 = \begin{pmatrix} (0.4, 0.45, 0.45, 0.5), (0.3, 0.35, 0.35, 0.4) & (0.4, 0.45, 0.55, 0.6), (0.2, 0.25, 0.35, 0.4) & (0.1, 0.15, 0.25, 0.3), (0.5, 0.55, 0.55, 0.6) & (0.3, 0.35, 0.35, 0.4), (0.3, 0.35, 0.45, 0.5) \\ (0.6, 0.65, 0.65, 0.7), (0.2, 0.25, 0.25, 0.3) & (0.6, 0.65, 0.65, 0.7), (0.2, 0.25, 0.25, 0.3) & (0.4, 0.50, 0.60, 0.7), (0.1, 0.15, 0.15, 0.2) & (0.5, 0.55, 0.55, 0.6), (0.1, 0.15, 0.25, 0.3) \\ (0.6, 0.65, 0.65, 0.7), (0.1, 0.15, 0.15, 0.2) & (0.5, 0.55, 0.55, 0.6), (0.3, 0.35, 0.35, 0.4) & (0.5, 0.55, 0.55, 0.6), (0.1, 0.15, 0.25, 0.3) & (0.4, 0.45, 0.45, 0.5), (0.2, 0.25, 0.35, 0.4) \\ (0.3, 0.35, 0.35, 0.4), (0.2, 0.25, 0.25, 0.3) & (0.6, 0.65, 0.65, 0.7), (0.1, 0.15, 0.25, 0.3) & (0.3, 0.35, 0.35, 0.4), (0.1, 0.15, 0.15, 0.2) & (0.3, 0.40, 0.60, 0.7), (0.1, 0.15, 0.15, 0.2) \\ (0.7, 0.75, 0.75, 0.8), (0.1, 0.15, 0.15, 0.2) & (0.3, 0.35, 0.45, 0.5), (0.1, 0.15, 0.25, 0.3) & (0.5, 0.55, 0.55, 0.6), (0.2, 0.25, 0.25, 0.3) & (0.3, 0.35, 0.35, 0.4), (0.5, 0.55, 0.55, 0.6) \end{pmatrix}$$

$$R^2 = \left(\begin{array}{l} (0.3, 0.35, 0.35, 0.4), (0.4, 0.45, 0.45, 0.5) \\ (0.3, 0.40, 0.50, 0.6), (0.3, 0.35, 0.35, 0.4) \\ (0.6, 0.65, 0.75, 0.8), (0.1, 0.15, 0.15, 0.2) \\ (0.4, 0.45, 0.45, 0.5), (0.3, 0.35, 0.45, 0.5) \\ (0.6, 0.65, 0.65, 0.7), (0.2, 0.25, 0.25, 0.3) \end{array} \right.$$

$$\begin{array}{l} (0.5, 0.55, 0.55, 0.6), (0.1, 0.15, 0.25, 0.3) \\ (0.4, 0.50, 0.60, 0.7), (0.1, 0.15, 0.15, 0.2) \\ (0.5, 0.55, 0.55, 0.6), (0.1, 0.15, 0.15, 0.2) \\ (0.5, 0.60, 0.70, 0.8), (0.1, 0.15, 0.15, 0.2) \\ (0.6, 0.65, 0.65, 0.7), (0.1, 0.15, 0.15, 0.2) \end{array}$$

$$\begin{array}{l} (0.4, 0.45, 0.45, 0.5), (0.3, 0.35, 0.35, 0.4) \\ (0.5, 0.55, 0.55, 0.6), (0.2, 0.25, 0.25, 0.3) \\ (0.5, 0.55, 0.65, 0.7), (0.2, 0.25, 0.25, 0.3) \\ (0.2, 0.30, 0.40, 0.5), (0.3, 0.35, 0.35, 0.4) \\ (0.5, 0.55, 0.65, 0.7), (0.2, 0.25, 0.25, 0.3) \end{array}$$

$$\left. \begin{array}{l} (0.4, 0.45, 0.55, 0.6), (0.2, 0.25, 0.35, 0.4) \\ (0.6, 0.65, 0.65, 0.7), (0.2, 0.25, 0.25, 0.3) \\ (0.1, 0.15, 0.25, 0.3), (0.5, 0.55, 0.55, 0.6) \\ (0.4, 0.50, 0.60, 0.7), (0.1, 0.15, 0.15, 0.2) \\ (0.6, 0.65, 0.65, 0.7), (0.1, 0.15, 0.25, 0.3) \end{array} \right)$$

$$R^3 = \begin{pmatrix} (0.2, 0.30, 0.40, 0.5), (0.3, 0.35, 0.35, 0.4) & (0.4, 0.45, 0.45, 0.5), (0.1, 0.15, 0.15, 0.2) & (0.3, 0.40, 0.50, 0.6), (0.2, 0.25, 0.25, 0.3) & (0.3, 0.40, 0.60, 0.7), (0.1, 0.15, 0.25, 0.3) \\ (0.2, 0.30, 0.60, 0.7), (0.2, 0.25, 0.25, 0.3) & (0.3, 0.40, 0.50, 0.6), (0.2, 0.25, 0.35, 0.4) & (0.4, 0.50, 0.60, 0.7), (0.1, 0.15, 0.15, 0.2) & (0.5, 0.60, 0.70, 0.8), (0.1, 0.15, 0.15, 0.2) \\ (0.5, 0.55, 0.55, 0.6), (0.3, 0.35, 0.35, 0.4) & (0.7, 0.75, 0.75, 0.8), (0.1, 0.15, 0.15, 0.2) & (0.5, 0.55, 0.55, 0.6), (0.2, 0.25, 0.25, 0.3) & (0.4, 0.45, 0.45, 0.5), (0.3, 0.35, 0.35, 0.4) \\ (0.3, 0.40, 0.50, 0.6), (0.2, 0.25, 0.35, 0.4) & (0.4, 0.45, 0.55, 0.6), (0.2, 0.25, 0.25, 0.3) & (0.1, 0.20, 0.30, 0.4), (0.3, 0.40, 0.50, 0.6) & (0.3, 0.40, 0.60, 0.7), (0.1, 0.15, 0.15, 0.2) \\ (0.6, 0.65, 0.65, 0.7), (0.1, 0.15, 0.25, 0.3) & (0.5, 0.55, 0.55, 0.6), (0.3, 0.35, 0.35, 0.4) & (0.5, 0.55, 0.55, 0.6), (0.2, 0.25, 0.25, 0.3) & (0.5, 0.55, 0.55, 0.6), (0.2, 0.25, 0.35, 0.4) \end{pmatrix}$$

Step 2. We firstly determine fuzzy density of each decision maker, and its ρ parameter. Suppose that $\mu(e_1) = 0.4$, $\mu(e_2) = 0.4$, $\mu(e_3) = 0.4$, Then ρ of expert can be determined:

$\rho_1 = -0.44$. According to Eq. (2.2), we have $\mu(e_1, e_2) = \mu(e_1, e_3) = \mu(e_2, e_3) = 0.73$, $\mu(e_1, e_2, e_3) = 1$.

Step 3. By Definition 3.3 or Definition 3.6, $\tilde{a}_{ij}^{k}$ is reordered such that $\tilde{a}_{ij}^{(k)} \leq \tilde{a}_{ij}^{(k+1)}$, then utilize the generalized trapezoidal valued intuitionistic fuzzy geometric aggregation operator

$$\begin{aligned}
\tilde{a}_{ij} \;=\; & GTIFGA_{\mu}(\tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{2}, \tilde{a}_{ij}^{3}) = \Big(\Big(\prod_{k=1}^{3}(a_{1ij^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\
& \prod_{k=1}^{3}(a_{2ij^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \prod_{k=1}^{3}(a_{3ij^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\
& \prod_{i=1}^{3}(a_{4ij^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}\Big), \Big(1 - \prod_{i=1}^{3}(1 - (a'_{1ij^{(k)}}))^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\
& 1 - \prod_{i=1}^{3}(1 - (a'_{2ij^{(k)}}))^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\
& 1 - \prod_{i=1}^{3}(1 - (a'_{3ij^{(k)}}))^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \\
& 1 - \prod_{i=1}^{3}(1 - (a'_{4ij^{(k)}}))^{\mu(A_{(k)})-\mu(A_{(k+1)})}\Big)\Big)
\end{aligned}$$

to aggregate all the trapezoidal valued intuitionistic fuzzy decision matrices $R^k = (\tilde{a}_{ij})_{m\times n}$ $(k = 1, 2, 3)$ into a complex trapezoidal valued intuitionistic fuzzy decision matrix $R = (\tilde{a}_{ij})_{m\times n}$ as follows:

$$R = \left(\begin{array}{l} ((0.2944, 0.3678, 0.4044, 0.4708), (0.3285, 0.3787, 0.3787, 0.4288)) \\ ((0.3463, 0.4417, 0.5898, 0.6715), (0.2283, 0.2784, 0.2784, 0.3285)) \\ ((0.5712, 0.6213, 0.6579, 0.7083), (0.1590, 0.2094, 0.2094, 0.2598)) \\ ((0.3242, 0.3951, 0.4320, 0.4996), (0.2283, 0.2784, 0.3486, 0.3990)) \\ ((0.6382, 0.6883, 0.6883, 0.7384), (0.1282, 0.1782, 0.2115, 0.2616)) \end{array} \right.$$

$$\begin{array}{l} ((0.4373, 0.4876, 0.5148, 0.5650), (0.1282, 0.1782, 0.2480, 0.2983)) \\ ((0.4231, 0.5133, 0.5865, 0.6715), (0.1614, 0.2115, 0.2414, 0.3268)) \\ ((0.5720, 0.6226, 0.6226, 0.6732), (0.1590, 0.2094, 0.2094, 0.2598)) \\ ((0.5000, 0.5700, 0.6400, 0.7083), (0.1282, 0.1782, 0.2115, 0.2616)) \\ ((0.4685, 0.5205, 0.5570, 0.6075), (0.1716, 0.2220, 0.2479, 0.2982)) \end{array}$$

$$\begin{array}{l} ((0.2452, 0.3191, 0.4005, 0.4685), (0.3257, 0.3768, 0.3768, 0.4280)) \\ ((0.4248, 0.5130, 0.5861, 0.6715), (0.1282, 0.1782, 0.1782, 0.2283)) \\ ((0.5000, 0.5500, 0.5880, 0.6382), (0.1683, 0.2184, 0.2500, 0.3000)) \\ ((0.1951, 0.2860, 0.3509, 0.4306), (0.2260, 0.2918, 0.3259, 0.3966)) \\ ((0.5000, 0.5500, 0.5880, 0.6382), (0.2000, 0.2500, 0.2500, 0.3000)) \end{array}$$

$$\left. \begin{array}{l} ((0.3299, 0.4011, 0.5041, 0.5720), (0.1911, 0.2414, 0.3421, 0.3925)) \\ ((0.5310, 0.6018, 0.6400, 0.7083), (0.1343, 0.1844, 0.2115, 0.2616)) \\ ((0.2751, 0.3345, 0.3840, 0.4356), (0.3257, 0.3768, 0.4114, 0.4622)) \\ ((0.3366, 0.4373, 0.6000, 0.7000), (0.1000, 0.1500, 0.1500, 0.2000)) \\ ((0.4685, 0.5205, 0.5205, 0.5720), (0.2614, 0.3131, 0.3768, 0.4280)) \end{array} \right)$$

Step 4. Since $((1,1,1,1),(0,0,0,0))$ and $((0,0,0,0),(1,1,1,1))$ are the largest and smallest trapezoidal valued intuitionistic fuzzy values, respectively. For cost criteria c_1, c_4 and benefit criteria c_2, c_3 TV-IFPIS $\tilde{\alpha}^+$ and TV-IFNIS $\tilde{\alpha}^-$ can be simply denoted as follows:

$$\tilde{\alpha}^+ = \begin{pmatrix} ((0,0,0,0),(1,1,1,1)) & ((1,1,1,1),(0,0,0,0)) \\ ((1,1,1,1),(0,0,0,0)) & ((0,0,0,0),(1,1,1,1)) \end{pmatrix}$$

$$\tilde{\alpha}^- = \begin{pmatrix} ((1,1,1,1),(0,0,0,0)) & ((0,0,0,0),(1,1,1,1)) \\ ((0,0,0,0),(1,1,1,1)) & ((1,1,1,1),(0,0,0,0)) \end{pmatrix}$$

Denote the alternatives a_i $(i = 1,2,\ldots,5)$ by $x_i = \begin{pmatrix} \tilde{a}_{i1} & \tilde{a}_{i2} & \tilde{a}_{i3} & \tilde{a}_{i4} \end{pmatrix}$:

Step 5. We determine fuzzy density of each criterion, and its parameter. Suppose that $\mu(c_1) = 0.4$, $\mu(c_2) = 0.25$, $\mu(c_3) = 0.37$, $\mu(c_4) = 0.20$, according to Eq. (2.3), the ρ of criteria can be determined: $\rho_2 = -0.44$. By Eq. (2.2), we have $\mu(c_1,c_2) = 0.6$, $\mu(c_1,c_3) = 0.7$, $\mu(c_1,c_4) = 0.56$, $\mu(c_2,c_3) = 0.68$, $\mu(c_2,c_4) = 0.43$, $\mu(c_3,c_4) = 0.54$, $\mu(c_1,c_2,c_3) = 0.88$, $\mu(c_1,c_2,c_4) = 0.75$, $\mu(c_2,c_3,c_4) = 0.73$, $\mu(c_1,c_3,c_4) = 0.84$, $\mu(c_1,c_2,c_3,c_4) = 1$.

Step 6. According to Eqs. (5.1) and (5.2), respectively, we calculate that

$d_1(x_1,\tilde{\alpha}^+) = 3.04068$, $d_1(x_1,\tilde{\alpha}^-) = 3.07328$,

$d_2(x_2,\tilde{\alpha}^+) = 3.36406$, $d_2(x_2,\tilde{\alpha}^-) = 3.11366$,

$d_3(x_3,\tilde{\alpha}^+) = 2.89115$, $d_3(x_3,\tilde{\alpha}^-) = 3.48042$,

$d_4(x_4,\tilde{\alpha}^+) = 3.25302$, $d_4(x_4,\tilde{\alpha}^-) = 2.99290$,

$d_5(x_5,\tilde{\alpha}^+) = 3.40700$, $d_5(x_5,\tilde{\alpha}^-) = 3.05841$.

Step 7. According to Eq. (5.3), we calculate the closeness coefficient of each alternative as follows:

$r(x_1) = 0.5026$, $r(x_2) = 0.4806$, $r(x_3) = 0.5462$, $r(x_4) = 0.4792$, $r(x_5) = 0.473$.

Step 8. Rank all the alternatives a_i $(i = 1, 2, \ldots, 5)$ according to the closeness coefficient $r(x_i)$:

$a_3 \succ a_1 \succ a_2 \succ a_4 \succ a_5$.

Thus the most desirable alternative is a_3.

7 Conclusion

We have studied the situation that the attributes in the decision making problem are interactive or inter-dependent and the evaluation values are trapezoidal fuzzy numbers. We have defined some new aggregation operators with Choquet integral for trapezoidal valued intuitionistic fuzzy group decision making process based TOPSIS, where the inter-dependent of attributes is considered. The trapezoidal valued intuitionistic fuzzy sets is the best way to deal with uncertainty. GTIFGA operator is used to aggregate the values of decision makers. trapezoidal valued intuitionistic fuzzy positive and negative ideal solution calculated by using distance based on Choquet integral. The relative closeness coefficient is used to rank alternatives. The properties of these operators are studied, such as idempotency, commutativity, boundedness and monotonicity. We have applied this operator to the multi-criteria trapezoidal valued intuitionistic fuzzy group decision making with Choquet integral based TOPSIS. Finally, an example has been provided to compare our method with some other to show the feasibility of our proposed decision making method. The proposed method is different from all the previous techniques for group decision making due to the fact that the proposed method use trapezoidal valued intuitionistic fuzzy set theory rather than intuitionistic fuzzy set or fuzzy set theory, which will not cause any loss of information in the process. So it is efficient and feasible for real-world decision making applications. In future, we shall continue working in the extension and application of the developed multi-criteria trapezoidal valued intuitionistic fuzzy group decision making with Choquet integral based TOPSIS to other domains.

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