PLANAR EMBEDDING OF PLANAR GRAPHS

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ABSTRACT

Planar embedding with minimal area of graphs on an integer grid is an interesting problem in VLSI theory. Valiant [12] gave an algorithm to construct a planar embedding for trees in linear area; he also proved that there are planar graphs that require quadratic area.

We fill in a spectrum between Valiant's results by showing that an N-node planar graph has a planar embedding with area $O(NF)$, where F is a bound on the path length from any node to the exterior face. In particular, an outerplanar graph can be embedded without crossings in linear area. This bound is tight, up to constant factors. For any N and F, there exist graphs requiring $\Omega(NF)$ area for planar embedding. Furthermore, those graphs need that much area even if $o(N)$ crossing are allowed.

Also, finding a minimal embedding area is shown to be NP-complete for forests and, hence, for more general types of graphs.

Advances in Computing Research, vol. 2 pages 147–161.

ISBN: 0-89232-461-9

1. INTRODUCTION

VLSI design motivates the following class of problems.

> *Given a graph, map its vertices onto a plane and its edges onto paths in that plane between the corresponding mapped vertices.*

Normally there are some restrictions that the mappings must obey, such as a minimum distance between mapped vertices. The maps give a *layout*, and the problem is to find a layout with a small cost. The mapping restrictions and the cost function together specify a particular member of the class of layout problems.

Embedding of graphs has been extensively studied during the last few years ([1], [2], [3], [5], [6], [7], [10], [11], [12]). Here we shall consider the layout problem when the layouts are rectilinear embeddings *without crossings* and the cost is the area of a box bounding the layout. To avoid complications, we assume that graphs are restricted to have vertices of degree 4 or less.

In the VLSI application, an embedding with crossings means that separate electrical layers need to be used for edges that cross, to avoid an unintended connection. There are usually several layers available, so this has not been seen as much of a problem. However, several factors make it desirable to know something about crossing-free layouts. For one thing, a *via* connection between different layers in VLSI circuits has a significant cost in area, performance, and reliability. Another important fact is that VLSI layers differ greatly in their electrical characteristics, so that some layers are much more desirable to use than others. Embeddings with a small number of crossings can be used as a wiring scheme using a high-performance layer for most of the connections, switching to another for edges that cross others. If the basic embedding is good, this may give a better implementation than some common two-layer wiring methods, which use one layer for horizontal wires and another for vertical wires.

Sometimes only one layer is available for wiring. This is the case for one-sided printed circuit boards. In VLSI, the power and ground networks should be wired totally in metal because of the large amount of current involved. This means a planar embedding is required for a graph like Figure 1. Our layout method is useful in this case.

In [12], Valiant looked at the layout problem for rectilinear embeddings (both with and without crossings), using the bounding box area cost. He proved that a tree of vertices with maximum degree 4 can be laid out without crossings in an area that is linear in the number of edges (or vertices). He also showed how to get such an embedding for any planar graph using quadratic area, and he proved that there are planar graphs requiring quadratic area.

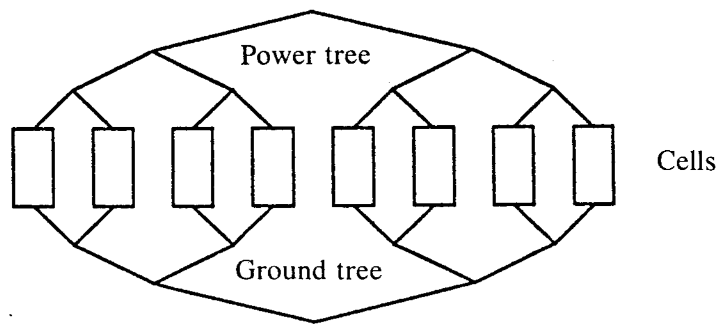

Figure 1. Network requiring planar embedding.

Definition. A planar graph has *width F* is there is a planar embedding of the graph such that every node of the graph is linked to the external face of the embedding by a path of at most F vertices.

We shall show that any N-node planar graph of width F can be laid out in $O(NF)$ area. Special cases of this include linear area embeddings for trees and outerplanar graphs and quadratic area embeddings for graphs of width $O(N)$. Furthermore, the area bound is tight up to a constant factor, even if $o(N)$ crossings are allowed. This fills in a spectrum between Valiant's results.

We shall also show that finding an optimal embedding for a forest is NP-complete.

2. A PLANAR GRAPH SEPARATOR

The layout method is basically that used by Valiant [12] and Leiserson [7] to get embeddings with crossings allowed. The graph is split into two by removing edges, each subpart is recursively laid out, and then the subproblem layouts are "married" by embedding the edges that were removed.

The key to methods like these are separator theorems, which guarantee that one can always split up a graph as needed without having to remove too many edges. Lipton and Tarjan [8] investigated planar graph separators and showed that any planar graph of N nodes can be split into approximately equal-size parts by removing $O(\sqrt{N})$ edges. One of the three simple splits shown in Figure 2 can always be found [9].

Unfortunately, the split in Figure 2c will not work with our layout method. We want to keep the same topology, so using a split like Figure 2c would mean that G_1 would have to be embedded with a hole of just

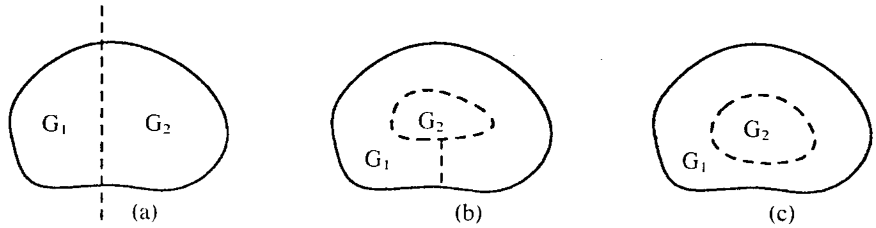

Figure 2. Separation schemes.

the right shape to hold G_2 and the removed edges. It appears to be difficult to manufacture such embeddings. When the split is like Figure 2a or 2b (call such a split *good*), then the pieces can be laid out in relative isolation and embedded side by side. The area of a layout resulting from this divide-and-conquer method depends critically on the number of edges cut in the separation. Our key result is the following theorem, which places an upper bound on the size of "good" separators.

THEOREM 1. *A planar graph with $N > 2$ vertices of degree at most 4 and width F can be separated into two subgraphs by removing $O(F)$ edges, such that each subgraph has at least one-third of the vertices. Given a planar drawing of the graph with width F, the separation can be made as shown in Figure 2a or 2b, rather than Figure 2c.*

PROOF. If necessary, add dummy edges to the graph until the given drawing has a simple cycle as the outer face and there are only triangles as interior faces. This can always be done, keeping the graph planar and without increasing the width. Call this graph G. Nodes in G may now have degree greater than 4, but this does not matter because the dummy edges will not actually be embedded.

Define the *distance* of a vertex in G to be the number of nodes in the shortest path from the vertex to the outer face. Let a *separating path* in G be a path from a vertex on the outer face to another one or the same one, such that the distances of the vertices on the path go like $1, 2, \ldots, k-1, k, k, k-1, \ldots, 2, 1$ or $1, 2, \ldots, k-1, k, k-1, \ldots, 2, 1$. We want the path to divide the graph into *regions*, as the dotted lines do in Figure 2a and 2b, which means there is a further restriction on separating paths: All vertices should be distinct, except that the final vertices are allowed to trace the initial ones in reverse. In other words, only simple paths and paths of the form $x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_{i+p}, x_i, \ldots, x_2, x_1$ are allowed, where the x's are distinct.

We will find a separating path with $k \leq F$ such that no more than two thirds of the vertices of G are in either of the two plane regions that the path divides the graph into. Then G can be separated as required in the thorem statement by removing some of the edges incident on the vertices of the separating path. The vertices on the path itself can be divided between the two regions so that neither ends up with more than two-thirds of G. At most $4 \times 2k$ nondummy edges are removed in this operation because of the restriction on the original graph that it have only vertices of degree 4 or less.

Start with any outer-face edge as the separating path. We will make incremental modifications to the path, with each modification reducing the number of vertices or edges in the bigger region. Assume, in general, that we have a situation with A vertices in one region, B vertices in the other, and $N - A - B$ vertices on the path itself. If $A \le \frac{2}{3}N$ and $B \le \frac{2}{3}N$ then we are done, so assume that $B > \frac{2}{3}N$.

The cases that arise are shown in Figure 3 (where vertex distances are shown after colons). In all cases the current separating path is the leftmost path shown inside the perimeter. Though several vertices are marked on the perimeter, these may in fact all be the same one (when the current separating path is like Figure 2b).

Figures 3i, 3ii, and 3iii are degenerate cases that are handled by using the right b–c edge in place of the current b-to-c path section. Each of these path modifications reduces the size of the larger region by one.

In Figure 3iv, vertex f is not the same as b or d. Such a vertex must exist in a triangulated graph if case (i) does not hold. Also, if f is not already on the outer face, there must be a vertex g $\neq$ c as shown, because

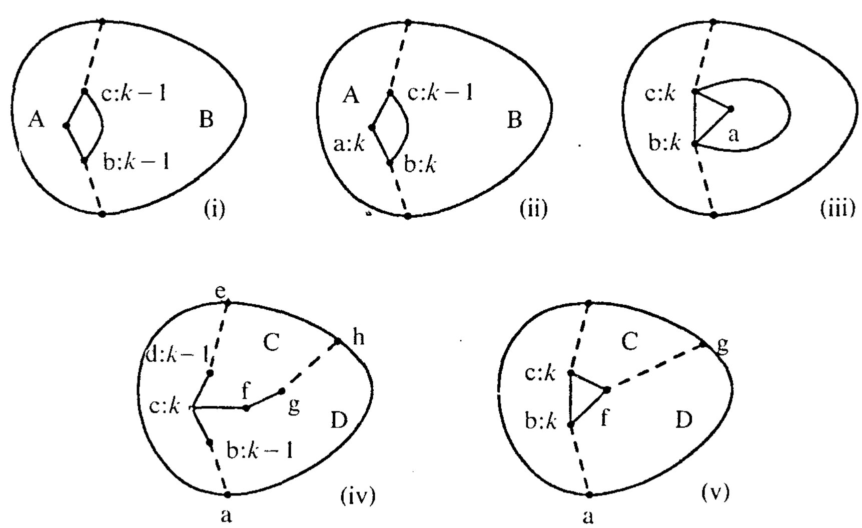

Figure 3. Cases for separator theorem.

vertices have degree greater than one. By the definition of distance of a vertex, there is an exit path, g- . . . -h, with vertices of distances $j, j - 1, \ldots, 1$ or $j, j + 1, \ldots, m - 1, m, m - 1, \ldots, 1$ where $j \leq k + 2$ and $m \leq F$. This exit path may coincide wholly or in part with d- . . . -e or b- . . . -a, but it never need cross over them because it can merge with the rest of whichever path it touches. Also, the path should not go back through f; this can always be avoided in a triangulated graph.

Most of the B vertices that were in the bigger region (on the right side of the original separating path in the figure) are now divided into pieces of sizes C and D. Assuming $D \geq C$, the new separating path is a- . . . -b-c-f-g- . . . -h. Clearly, this new path is of the required form. If $D \leq \frac{2}{3}N$, then we are done, otherwise repeat the process. The vertices f and g are part of the B vertices of the original bigger region, so we must have $D < B$. This means that progress has been made toward the stopping condition, since we have decreased the number of vertices in the big region of the graph. Note that the new separating path may have a greater maximum distance, but this is irrelevant as far as progress toward stopping is concerned.

The situation of Figure 3v, where the path has two vertices of maximum distance in the middle, is handled similarly. If $D \geq C$, the new separating path a- . . . -b-f- . . . -g is of the required form. Progress toward the stopping condition has been made because we will have lost vertex c at the very least.

The preceding operations can be repeated until a separating path has been found with no more than $\frac{2}{3}N$ vertices on either side, proving the theorem. □

So the width of a graph seems to have something to do with the size of good planar separators. Unfortunately, it is not a complete characterization, since there are many graphs with such separators much smaller than the width. For example, the power and ground graph in Figure 1 has width $O(\log N)$, but there are always straight line separators of size 2. (This means that the method of the next section will yield a linear area layout for that network.)

3. PLANAR EMBEDDING ALGORITHM

The layout method used in [12] and [7] for embedding with crossings almost works for planar embeddings. The difference is in the marrying step.

For the layout method to work recursively, it has to be able to embed a graph so that it is topologically equivalent to a given planar drawing.

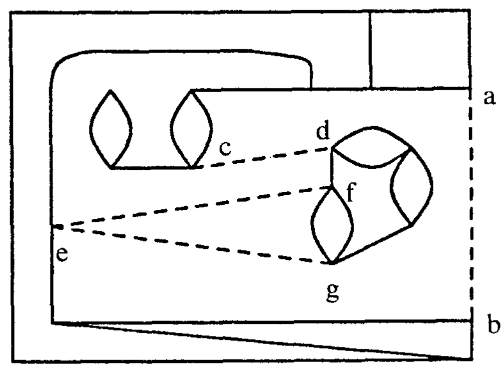

Figure 4. Graph for recursive embedding.

Suppose G is separated into G_1 and G_2 using the separator theorem of the previous section, and then the subparts are embedded, respecting topology. Then the removed edges can be drawn in the plane without crossings because they are attached to vertices that are still on the outer faces of G_1 and G_2, in the same order. For example, see Figure 4, where the separating edges are shown dotted. This might be embedded and married as shown in Figure 5.

To turn such a drawing into a grid embedding, insert a new grid line for every dotted straight line segment. For the diagonal lines making the connections, at most two new horizontal and two new vertical grid lines may be needed. (The existing edges may have to be shifted so they make

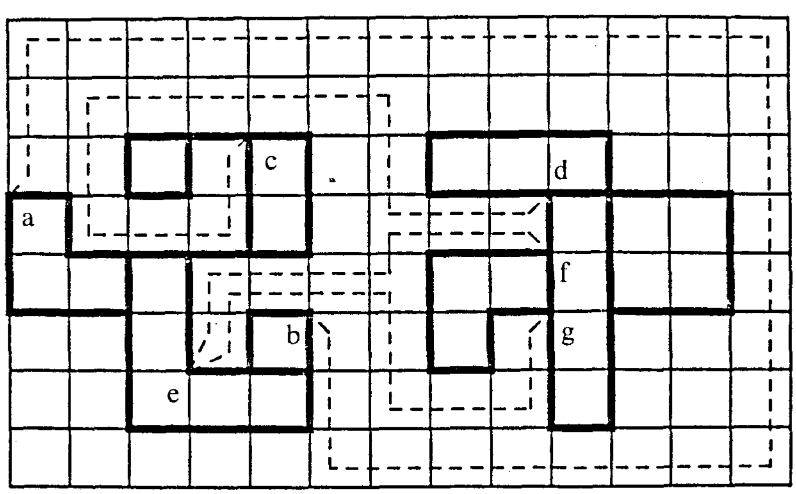

Figure 5. Marrying two embeddings.

their final approach from a different direction.) Let K be the number of "kinks," i.e., horizontal and vertical grid lines that need to be added to connect any exterior face vertex of a given embedding to somewhere completely outside. It is easy to see that K increases only by $O(1)$ at each marrying step because the added edges need not wrap around the layout more than once. Thus, if i is the maximum of the number of marrying stages involved in laying out G_1 and G_2, then they can be married by added $O(iF)$ horizontal and vertical grid lines to embed the $O(F)$ separating edges.

THEOREM 2. *Any planar graph G with N vertices of degree at most 4, and width at most F, has a planar embedding in a grid of area $A(N) = O(FN)$.*

PROOF. Other than the separation and marrying methods, the layout algorithm is the same as the one in [12]. It has to be able to produce an embedding in any $H \times W$ grid, as long as $\frac{1}{3} \le H/W \le 3$, and HW is sufficiently large. Suppose by induction that $A(N)$ is sufficient area for an N-vertex graph. Also, suppose that $K(N)$ is a bound on the number of kinks.

G is separated into G_1 and G_2 by removing $O(F)$ edges, with $|G_1| = x|G|$, $\frac{1}{3} \le x \le \frac{2}{3}$. Then an

$$(H - cFK(N)) \times (W - cFK(N))$$

grid is divided in two by a cut parallel to the shorter side in the ratio $x:(1 - x)$. By an elementary theorem in [12], the aspect ratios of the two pieces will be in the range $[\frac{1}{3}, \frac{2}{3}]$. If G_1 and G_2 can be laid out in these pieces, then the embedding can be completed as described above, inserting at most $cFK(N)$ horizontal and vertical grid lines, for some constant c. So the theorem is true if

$$x\,(H - cFK(N))\,(W - cFK(N)) \ge A(xN)$$

for all x, $1/3 \le x \le 2/3$. Using $HW \ge A(N)$ and the easily derived fact that $(H + W)/\sqrt{A(N)} \ge 4/\sqrt{3}$, this will be true if

$$x\,(A(N) - (4/\sqrt{3})\,\sqrt{A(N)} \ge cFK(N)) \ge A(xN)$$

for all x, $1/3 \le x \le 2/3$.

After $\log_{2/3} N/F$ separation steps, the graph pieces are no larger than F, so if we stop the recursion at that point we have $K(N) = O(\log N/$

F). It is easily verified by substitution that

$$A(N) = \alpha NF - \beta N^{1/2}F^{3/2}\log(N/F)$$

satisfies the recurrence, for some α and β independent of N and F. In the base case, with $N = F$, an $O(N^2)$ embedding (see [12]) can be used. One has to be careful to get an embedding that preserves the topology of a given planar drawing, but it is easy to see how to do this. □

We could calculate an actual α that would work as the multiplicative constant in the area formula, but this would not be too instructive. In practice, the method should work quite a bit better than the analysis would indicate, for several reasons. First, the separators will probably get smaller than F as the recursion proceeds. And second, the number of extra grid lines that need to be added during marriage will often be a lot smaller than the worst-case allotment. For example, added grid lines may be useful for more than one line segment, and all the lines allowed for the "final approach" are rarely needed.

4. LOWER BOUNDS

The upper bound of $O(NF)$ area for an N-mode planar graph of width F is matched by a lower bound, to within a constant factor.

THEOREM 3. *For any N and any F, there exist graphs with N nodes and width F requiring $\Omega(NF)$ area for any planar embedding.*

PROOF. The *nested triangles* graph of N nodes, shown in Figure 6, requires $\Omega(N^2)$ area for a planar embedding (see [12]). A nested triangles graph of $3F$ nodes has width F, so a graph consisting of $N/3F$ such components has width F and needs $\Omega(NF)$ area for an embedding without crossings. □

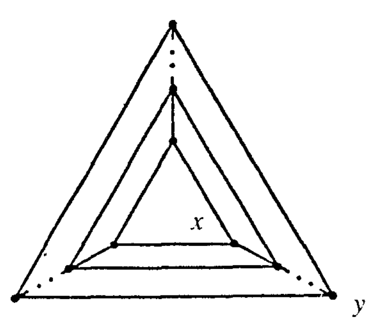

Figure 6. Nested triangles.

Because the upper bound of the previous section matches this lower bound to within a constant factor, it might seem that we have completely characterized the growth rate of planar embedding area. This is not so, unfortunately, since the lower bound is only existential, and graphs like Figure 1 show that it cannot be universal.

Now we turn to a different but related problem: What happens when a small number of crossings are allowed? If $O(N)$ crossings are permitted, a layout with only $O(N)$ area can be found for the nested triangles graph, rather than the $O(N^2)$ area needed when no crossings are allowed. What about an intermediate number of crossings? In this section we shall present evidence that a few crossings are not much better than none. We shall prove that the example used in the preceding proof requires $\Omega(NF)$ area even if $o(N)$ crossings are allowed.

The first step is to show that the *cycled nested triangles* graph shown in Figure 7 needs a lot of crossings.

LEMMA 4. *The cycled nested triangles graph of N vertices has $\Omega(N)$ crossings in any layout.*

PROOF. Suppose that there is a layout with only $o(N)$ crossings. We shall remove some edges from that layout and then show that they cannot be put back in without producing $\Omega(N)$ crossings, to obtain a contradiction.

Call each edge either a *triangle* or a *spine* edge, depending on whether or not it is part of one of the triangles, or goes between two triangles at different nesting levels. The first step is to remove every edge involved in a crossing in the given layout. If a removed edge is a triangle edge, then remove the other nodes and edges of the triangle. If it is a spine edge, then remove the other two spine edges that go between the same pair of triangles. Note that the total number of removed triangles is $o(N)$.

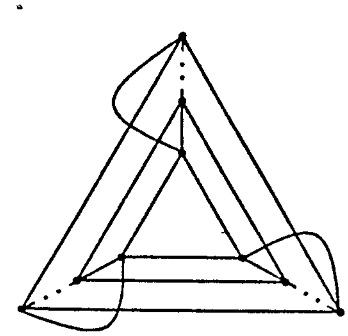

Figure 7. Cycled nested triangles.

Now the layout consists of $o(N)$ components, each a planar embedding of a nested triangles graph. Some components may be inside faces of other components, but this does not matter for what follows.

Consider the possible ways of laying out the nested triangles graph without crossings. One possible embedding is shown in Figure 6; another completely reverses the order of the nesting. Any other embedding has one of the quadrilateral faces as the outer face and looks like Figure 8. A theorem of Whitney [13] says that any 3-connected planar graph can be embedded only one way on a sphere. So once an outer face is chosen, there is only one way to complete the embedding without crossings.

Now consider adding back the removed spine edges. The nodes marked x and y in Figure 8 will be part of one of the three cycles of spine edges in the cycled nested triangles. Thus, somehow another path of spine edges must eventually go from x to y (also making its way through all the other components). There is no way to do this without crossing all but two of the triangles in the component shown. This is also true if the embedding is like Figure 6.

Suppose the edge removal left our layout with r components of n_1, n_2, . . . , n_r triangles, respectively. Then the number of crossings obtained by adding back the spine edges is at least

$$\sum_{i=1}^{r} 3(n_i - 2) = \Omega(N - o(N) - 6r)$$
$$= \Omega(N)$$

since $r = o(N)$. This contradicts the original assumption that there were only $o(N)$ crossings, so the assumption must be wrong. □

Given the above lemma, it is easy to prove the main result of this section.

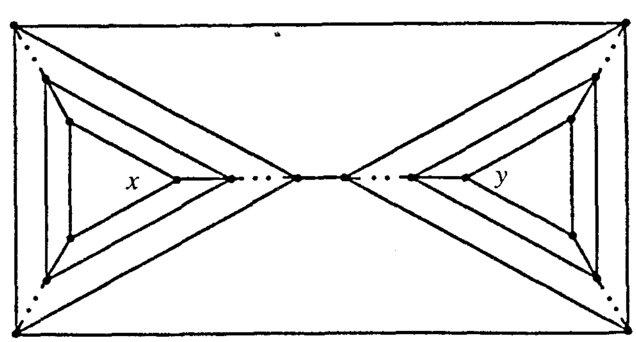

Figure 8. Embedding of nested triangles.

THEOREM 5. *For any N and any F, there exist planar graphs with N nodes and width F requiring $\Omega(NF)$ area for any embedding with $o(N)$ crossings.*

PROOF. The example graphs are the same as those used in Theorem 3: $N/3F$ copies of a nested triangles graph with $3F$ nodes. The result will follow if we can show that the nested triangles graph with N nodes cannot have an embedding with $o(N^2)$ area and $o(N)$ crossings. Assume, for contradiction, that such a layout is given.

If the layout is long and skinny, fold it into a square with $o(N)$ sides and the same topology (see [7]). Now by adding only a constant number of tracks to the layout we can embed the three edges needed to convert the graph into cycled nested triangles. These edges will cross at most $o(N)$ edges, since no distance in the layout is longer than that. This means we now have a layout for cycled nested triangles with $o(N)$ crossings, thereby contradicting the result of the previous lemma. Therefore, the assumption is incorrect, and the theorem is proved. □

5. NP-COMPLETENESS OF OPTIMAL FOREST EMBEDDING

Given a forest and an integer A, the *forest layout problem* is to find whether or not there is a planar rectilinear embedding with area less than or equal to A. In this section we shall show that the forest layout problem is NP-complete. The reduction is from the 3-partition problem.

In the 3-partition problem, there is a set of integers $x_1, \ldots, x_{3m}$ such that

$$\sum_{i=1}^{3m} x_i = mB$$

and $B/4 < x_i < B/2$ for $1 \le i \le 3m$. The question is whether the set can be partitioned into m disjoint sets such that each set sums to B. This problem is known to be strongly NP-complete [4].

Consider the tree in Figure 9a. Call it the *frame tree*. There are vertices at every grid point except for $m = 2n$ holes of size B. (The case for m odd will be considered later; it is just a trivial modification.)

LEMMA 6. *The only embeddings of the frame tree with a bounding box area of $(4n + 3) \times (2B + 3)$ or less and leaving mB free grid points are either exactly like that shown in Figure 9a (possibly after point relabeling), or modifications of that diagram where some of the tops of the vertical spines are changed as in Figure 9b and its various reflections.*

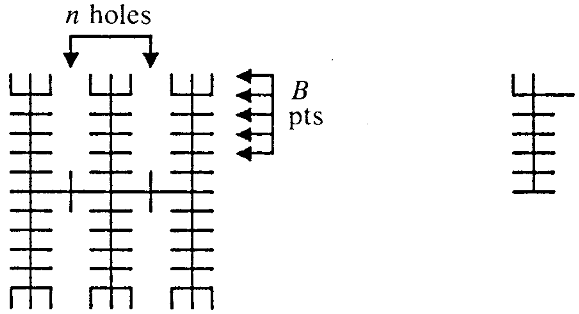

Figure 9. Frame tree for $n = 2$, $B = 5$.

PROOF. The tree has $(4n + 3) \times (2B + 3) - mB$ vertices, so the embedding is required to use every grid point for a vertex or else leave it free. This means that no edge of the tree can be stretched to a path of 2 units, for that would take up a grid point in the middle that is not used for embedding a graph vertex.

Any layout using only unit-length edges must have all of the degree-4 vertices of a vertical spine one on top of the other, as in the diagram. For otherwise there would have to be two degree-4 vertices at opposite corners of a 1-unit square, which is impossible (one of the other corners would have to be shared between two vertices).

Therefore, the only possible changes to the given diagram, other than point renaming, are ones at the degree-2 vertices, such as in Figure 9b. □

Notice that if the frame tree is embedded using an allowed folding near the top of a spine, this cuts a hole into two pieces of sizes 2 and $B - 2$. There cannot be more than one fold into a hole. From now on, use the term *hole* to mean either a B-point vertical slot or one of these $2 + (B - 2)$-point aggregates.

THEOREM 7. *The forest layout problem is NP-complete.*

PROOF. Given an instance of the 3-partition problem, construct the frame tree and add $3m$ other pieces, unconnected to that tree so that for each x_i there is a piece consisting of x_i vertices joined into a line by $x_i - 1$ edges. If m is odd, use the frame graph for the next higher even number and fill in one of the vertical holes.

Now we claim that the 3-partition problem instance has a solution iff there is an embedding of this forest with a bounding box area of $(4n + 3) \times (2B + 3)$. For, by the lemma, if there is such an embedding, then it must be as shown in Figure 9a with the extra pieces filling up the holes. Since all the grid points are to be used, this gives a solution to the 3-partition problem, because the size restrictions on the x's imply that there must be exactly three pieces in each hole. Conversely, given a solution to the 3-partition problem, a suitable embedding can be found by filling the holes in the frame tree with the pieces corresponding to the partitioned sets.

This is not a polynomial reduction, since the frame tree has a number of vertices of the order of the numbers involved in the 3-partition problem, rather than the number of bits required to represent those numbers. This does not matter, however, since the 3-partition problem is *strongly* NP-complete. The layout problem is in NP because one can simply guess an embedding of the graph and then verify that the embedding is legitimate. Therefore, the forest layout problem is NP-complete. □

ACKNOWLEDGMENT

We are indebted to Gary Miller for suggesting the power-ground example in Figure 1 as well as for contributing other helpful ideas. Thanks also to Tim Magee and Ron Rivest for their helpful remarks.

Part of this work was done while Danny Dolev was visiting Stanford University and was supported by a Chaim Weizmann postdoctoral fellowship and DARPA contract MDA 903-80-C-0102. His current address is: Institute of Mathematics and Computer Science, Hebrew University, Givat Ram 91904, Jerusalem, Israel.

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