Moment-of-Fluid on Cartesian grids

Interface representation & reconstruction

Antoine Lemoine [1] Stéphane Glockner [1] Jérôme Breil [2]

[1] I2M [2] CEA/CELIA

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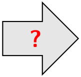
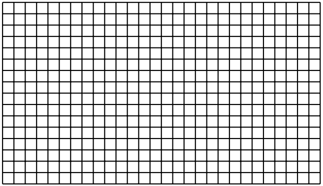
Introduction: representing multiphase flow

How to represent multiphase flow on meshes?

Source: BlueWater by trancedman, DeviantArt





Cartesian grid

Physics

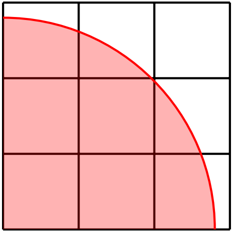
- Multiphase flow
- Immiscible fluids
- Mass conservation
- ...

Numerical representation

- Volume-of-fluid
- Level-set
- Front tracking
- Moment-of-fluid
- ...

Example: Volume-of-Fluid

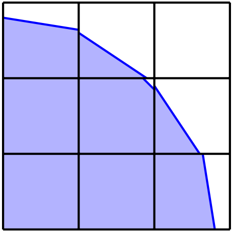
Representation $\neq$ Reconstruction



'True' interface

0.8	0.3	0
1	0.95	0.3
1	1	0.8

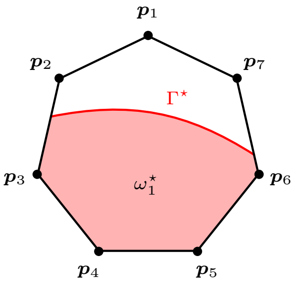
VOF representation



PLIC reconstruction

VOF: Volume-of-Fluid
PLIC: Piecewise Linear Interface Construction

VOF-PLIC formulation



Original data

- Ω polygonal cell of vertices $\{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n\}$
- $\omega_1^\star$ portion of fluid 1 in the cell Ω

Representation

- $|\omega_1^\star|$ volume of fluid 1

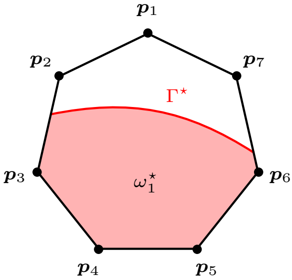
Reconstruction

- $\omega_1^\ell(\phi)$ polygonal approximation of $\omega_1^\star$
- $|\omega_1^\ell(\phi)| = |\omega_1^\star|$ (volume conservation)
- $\Gamma^\ell(\phi) = \partial\Omega \setminus \partial\omega_1^\ell(\phi)$ interface
- $\boldsymbol{n}(\phi)$ interface normal
- ϕ angle with the horizontal axis

Parametrization $\Gamma^\ell = \{\boldsymbol{x} \in \Omega / \boldsymbol{x} \cdot \boldsymbol{n} = d\}$

- $\boldsymbol{n}$ interface normal
- d distance to the origin

VOF-PLIC formulation



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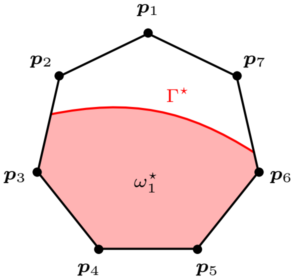

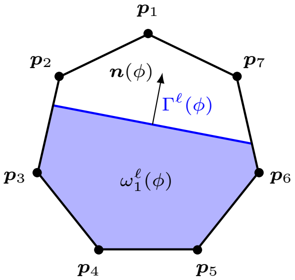

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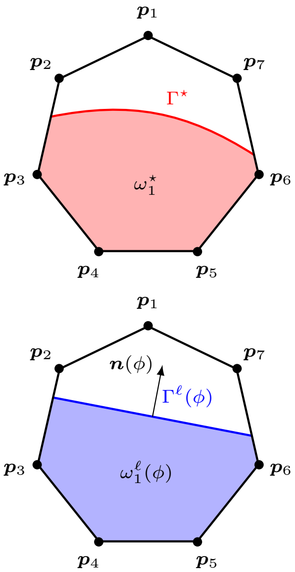

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How to find d and $\boldsymbol{n}$?

What do we want?

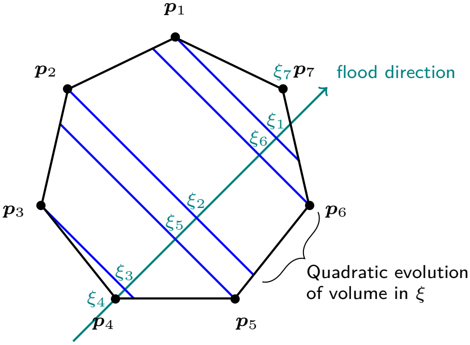
$\rightarrow$ Find a linear interface *as close as possible* of the original interface

- Problem: 1 constraint, 2 unknown
- Hard part: finding $\boldsymbol{n}$

  Classic methods that use information of *surrounding cells*:

  - Gradient of volume fraction
  - Least-square
  - LVIRA, ELVIRA
  - ...
- Easy part: if we know $\boldsymbol{n}$, the distance d can be deduced from the volume constraint $\longrightarrow$ *Flood algorithm*

Flood algorithm

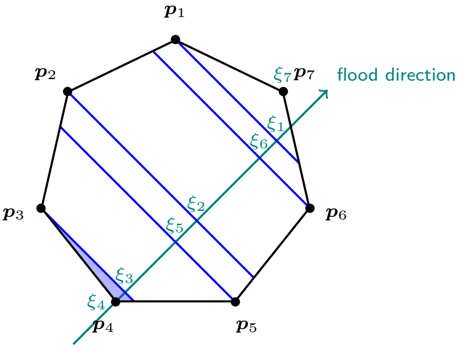


Initial condition

- Flood direction $\boldsymbol{n}$
- Volume of fluid $|\omega^{\star}| = |\Omega|/2$
- $\boldsymbol{p}_4$ first point
- ξ_4 first distance

Reference: Breil, J., Gelera, S., & Maire, P. H. (2011).

Flood algorithm



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Sequential algorithm (find ξ^*)

1. $\xi^* \notin [\xi_4, \xi_3]$
2. $\xi^* \notin [\xi_3, \xi_5]$
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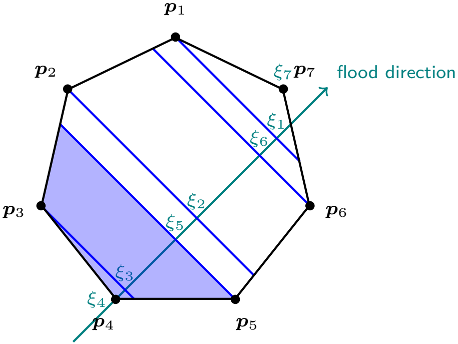

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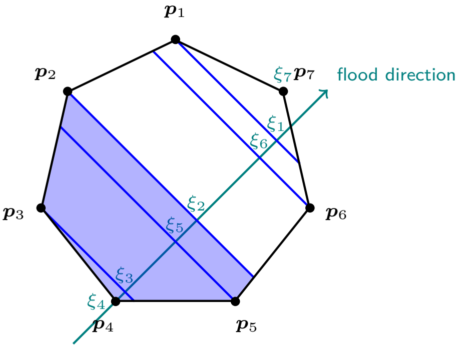

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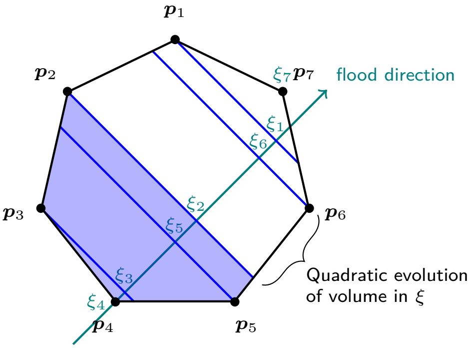

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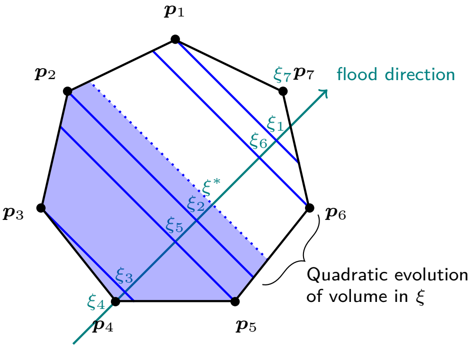

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Result

- ξ^* given by quadratic interpolation

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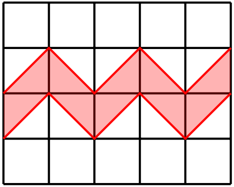
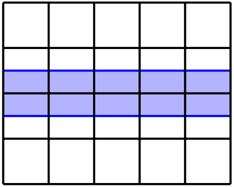
Limitations of VOF methods



Original interface

0	0	0	0	0
0.5	0.5	0.5	0.5	0.5
0.5	0.5	0.5	0.5	0.5
0	0	0	0	0

VOF representation



PLIC reconstruction

- Original interface is piecewise linear in each cell
- Reconstruction should be *exact*!

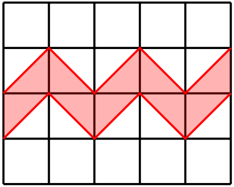
Problem

The volume fraction is insufficient to make a cell-wise reconstruction

Idea

Add information to have a *local* (cell-wise) reconstruction ⟶ Moment of Fluid

Limitations of VOF methods



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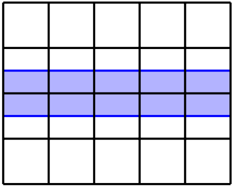

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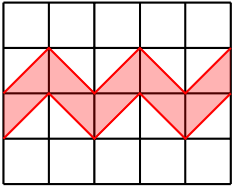

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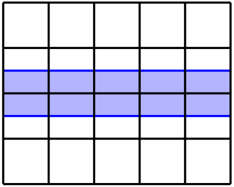

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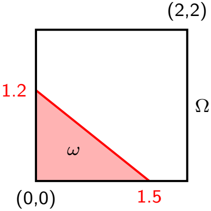
Moment-of-fluid: Moment?

Momentum of order 0 (volume)

$$M_0(\omega) = \int_\omega d\boldsymbol{x} = |\omega|$$

Volume fraction (relative to a cell Ω)

$$\mu(\omega) = \frac{M_0(\omega)}{M_0(\Omega)}$$



- $M_0(\Omega) = 4$
- $M_0(\omega) = 0.9$
- $\mu(\omega) = 0.225$

Momentum of order 1

$$\boldsymbol{M_1}(\omega) = \int_\omega \boldsymbol{x} d\boldsymbol{x}$$

Centroid

$$\boldsymbol{x}_c(\omega) = \frac{\boldsymbol{M_1}(\omega)}{M_0(\omega)}$$

- $M_0(\omega) = 0.9$
- $\boldsymbol{M_1}(\omega) = (0.45, 0.36)$
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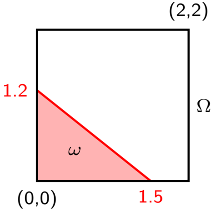

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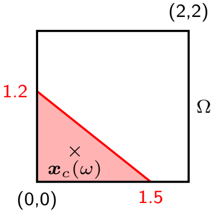

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Moment-of-fluid: Formulation

Data:

- Volume fraction of any fluids μ in each cells
- Centroid of any fluids $\boldsymbol{x}_c$ in each cells

Reconstruction method:

- VOF: $|\omega^\ell| = |\omega^\star|$ for each cell
  $\longrightarrow$ *under-determined* problem!
- MOF: $|\omega^\ell| = |\omega^\star|$ and $\boldsymbol{x}_c(\omega^\ell) = \boldsymbol{x}_c(\omega^\star)$ for each cell
  $\longrightarrow$ *over-determined* problem!

Minimization problem:

- Find $\omega^\ell = \underset{\omega^\ell}{\operatorname{argmin}} \; |\boldsymbol{x}_c(\omega^\ell) - \boldsymbol{x}_c(\omega^\star)|^2$
- Under constraint $|\omega^\ell| = |\omega^\star|$

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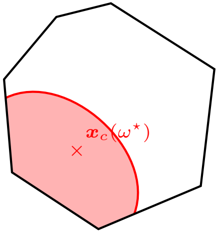
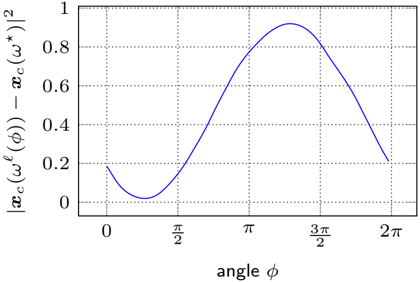
Minimization: Example

- $\mu(\omega^\star) = 0.3$
- $\boldsymbol{x}_c(\omega^\star) = (-0.3, -0.3)$



Objective function:



Solution: $\phi \approx 0.841$

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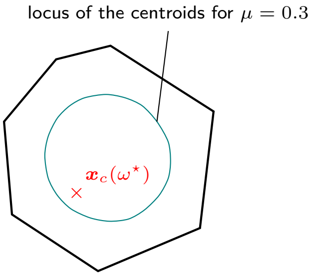

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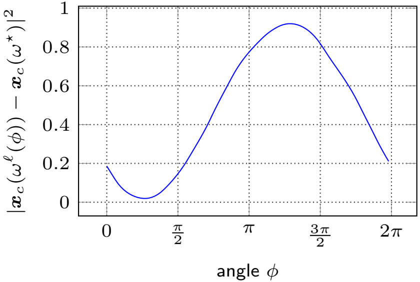

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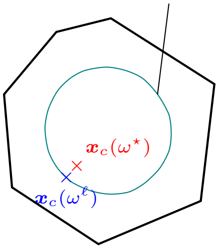
locus of the centroids for $\mu = 0.3$



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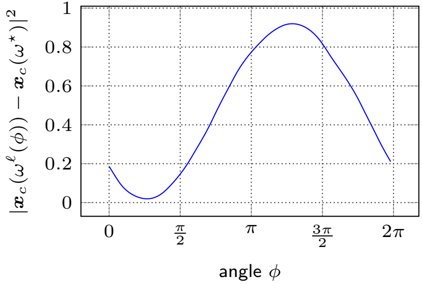

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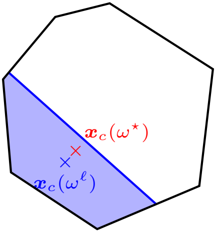

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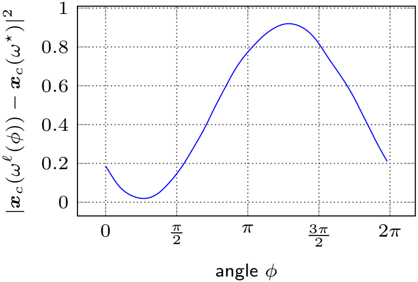

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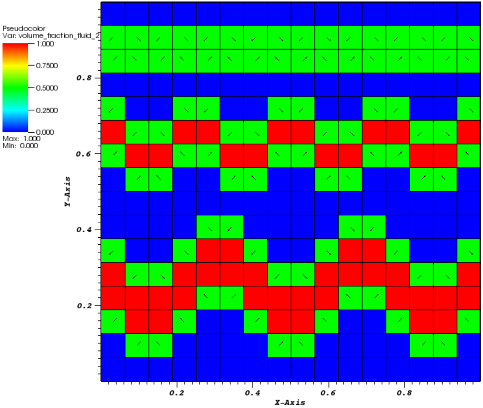
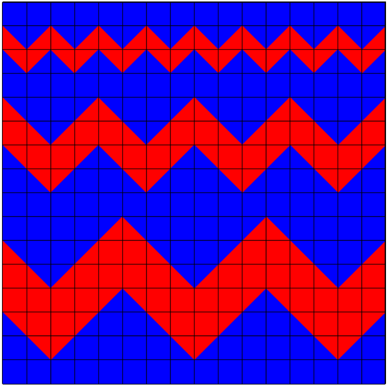
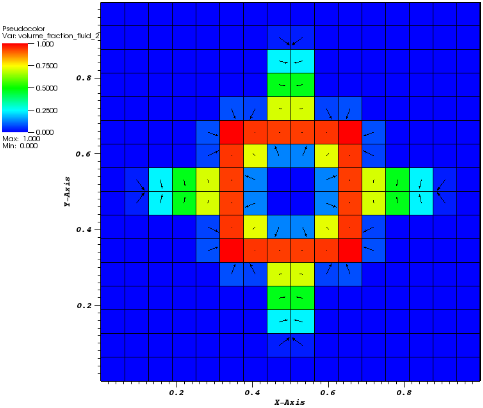
Example of static reconstruction: Zigzags

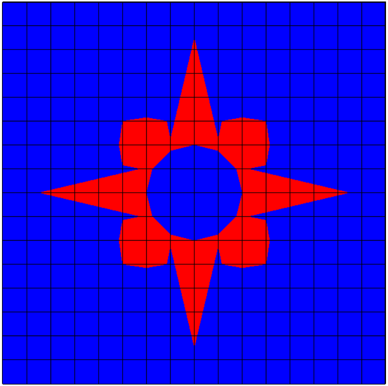




Volume fraction and centroids data from: Dyadechko, V., Shashkov, M. (2007)

Example of static reconstruction: Compass





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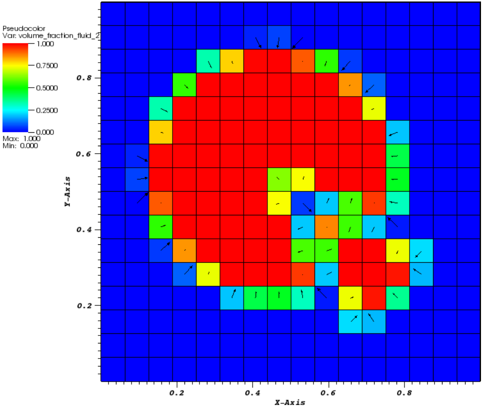
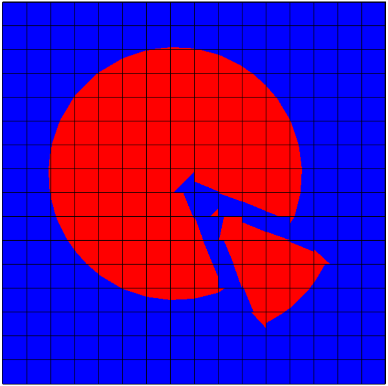
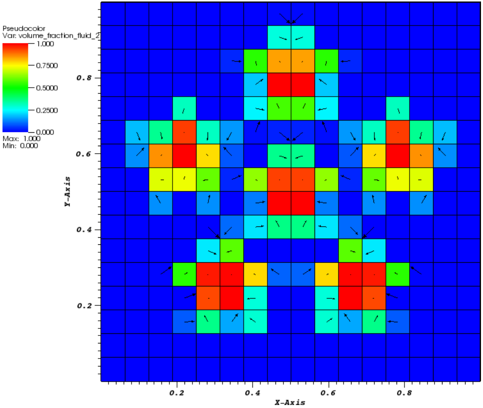
Example of static reconstruction: Pie

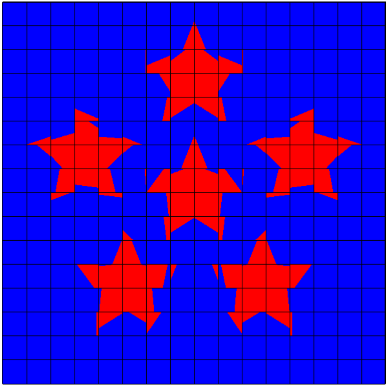




Volume fraction and centroids data from: Dyadechko, V., Shashkov, M. (2007)

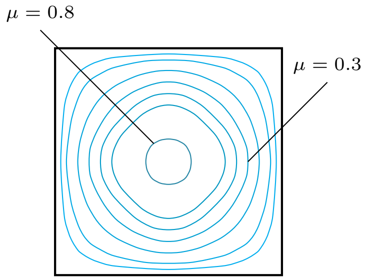
Example of static reconstruction: Stars





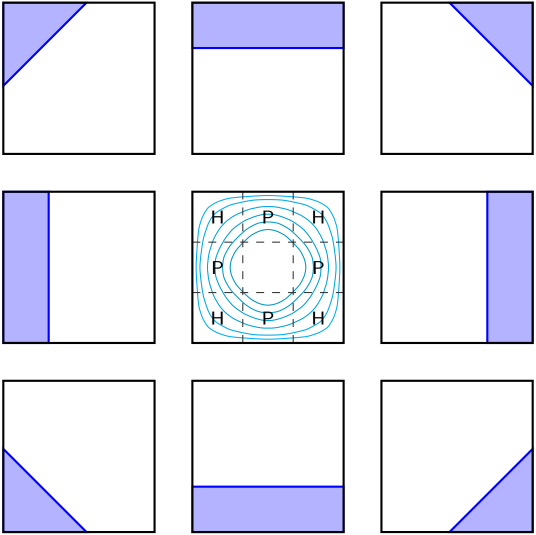
Volume fraction and centroids data from: Dyadechko, V., Shashkov, M. (2007)

Analytic reconstruction: Motivations

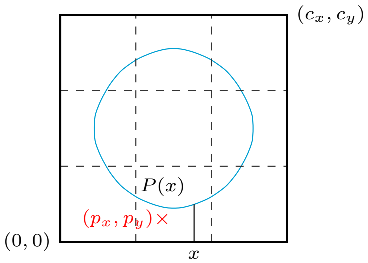


- On Cartesian grids, cells are rectangles
- Rectangle are very simple shapes
- Upgrade a VOF algorithm
- Easier to implement
- No problem with local minima
- Faster?

Analytic reconstruction: possible configurations ($\mu \leq 0.5$)



Analytic solution: Parabola



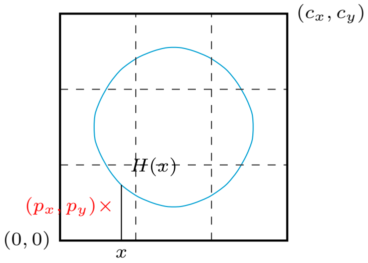
For all $x \in \left[\frac{c_x}{3}, \frac{2c_x}{3}\right]$

$$P(x) = \frac{V}{2c_x} + \frac{6V}{c_x}\left(\frac{1}{2} - \frac{x}{c_x}\right)^2$$

Let $\boldsymbol{p} = (p_x, p_y)$ any point of $\mathbb{R}^2$
The closest point of $\boldsymbol{p}$ to the parabola P is its orthogonal projection
The coordinate x of $\boldsymbol{x}_c(\omega^\ell) = (x, P(x))$ is one of the solution of

$$x - p_x - \frac{12V}{c_x^2}\left(\frac{1}{2} - \frac{x}{c_x}\right)\left(\frac{V}{2c_x} - p_y\right) - \frac{72V^2}{c_x^3}\left(\frac{1}{2} - \frac{x}{c_x}\right)^3 = 0$$

Analytic solution: Hyperbola



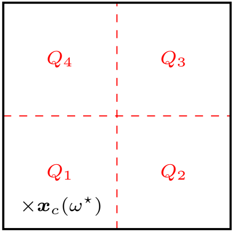
For all $x \in \left[\frac{1}{3}\sqrt{2V\frac{c_x}{c_y}}, \frac{c_x}{3}\right]$

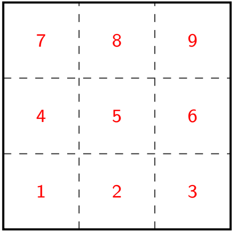
$$H(x) = \frac{9V}{2x}$$

Let $\boldsymbol{p} = (p_x, p_y)$ any point of $\mathbb{R}^2$
The closest point of $\boldsymbol{p}$ to the hyperbola H is its orthogonal projection
The coordinate x of $\boldsymbol{x}_c(\omega^\ell) = (x, H(x))$ is one of the solution of

$$x^4 + p_x x^3 + \frac{2}{9}Vp_y x - \left(\frac{2V}{9}\right)^2 = 0$$

Analytic solution: Algorithm



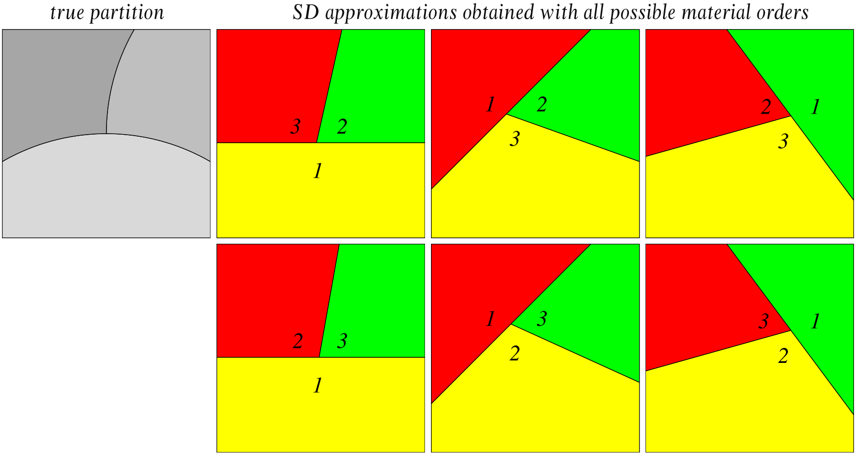


1. If $\mu > 0.5$ solve the dual problem
2. Locate the quadrant where $\boldsymbol{x}_c(\omega^\star)$ is
   - $\boldsymbol{x}_c(\omega^\star) \in Q_1$ try $\{1, 2, 4\}$
   - $\boldsymbol{x}_c(\omega^\star) \in Q_2$ try $\{2, 3, 6\}$
   - $\boldsymbol{x}_c(\omega^\star) \in Q_3$ try $\{6, 8, 9\}$
   - $\boldsymbol{x}_c(\omega^\star) \in Q_4$ try $\{4, 7, 8\}$
3. Solve 2 cubic and 1 quartic
4. Find the closest solution
5. Compute $\boldsymbol{n}$ and d from the solution

Moment of Fluid: Summary

- Stencil reduced to only one cell
- Analytic reconstruction is about 30% faster than minimization
- What about multimaterial?

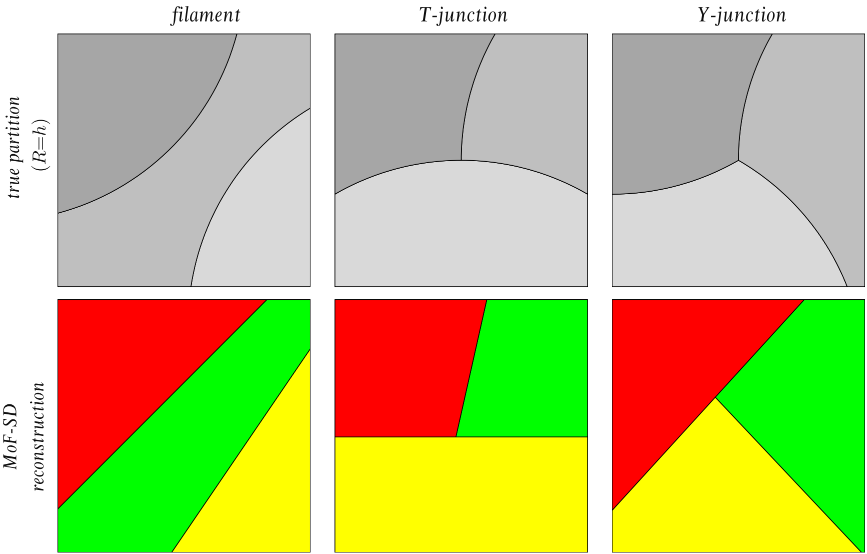
Multiphase reconstruction: Serial dissection



The best solution minimizes the sum of the centroid defects.

Source: Dyadechko, V., Shashkov, M. (2008)

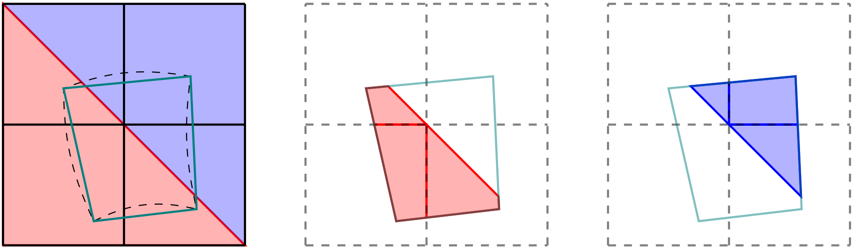
Multiphase reconstruction: Examples



Source: Dyadechko, V., Shashkov, M. (2008)

Advection

Backward advection: Compute the pre-image Ω^{n-1} of Ω^n with a Runge-Kutta 2 method.



Backward advection

Polygonal intersection red fluid

Polygonal intersection blue fluid

Compute the volume of red and blue fluid.

Remark: Only the vertices of the cell are advected. The volume is not exactly preserved.

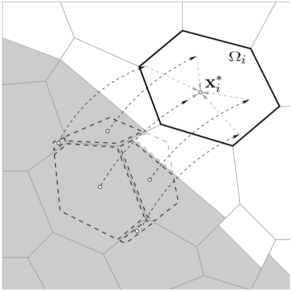
Advection: Centroids

We can show that the centroids almost follow an advection equation:

$$\frac{d}{dt}\boldsymbol{x}_c(\omega) = \boldsymbol{v}(\boldsymbol{x}_c(\omega)) + \mathcal{O}(h^2)$$

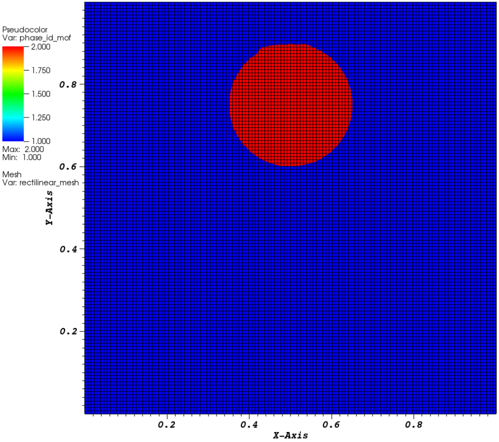
Advection algorithm:

- Backward advection of the cell (RK2)
- Intersection of the fluid polygons
- Compute volume and centroids
- Forward advection of the centroids (RK2)

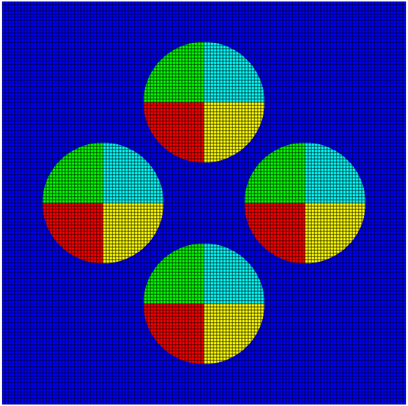


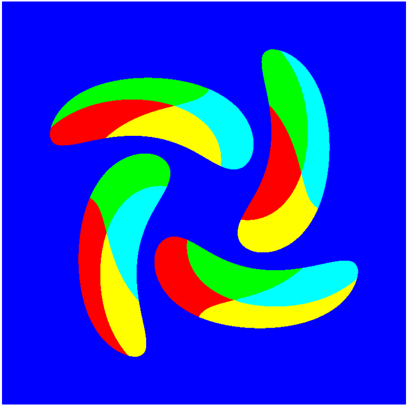
Source: Dyadechko, V., Shashkov, M. (2007)

Advection: 2 fluids on a sheared flow

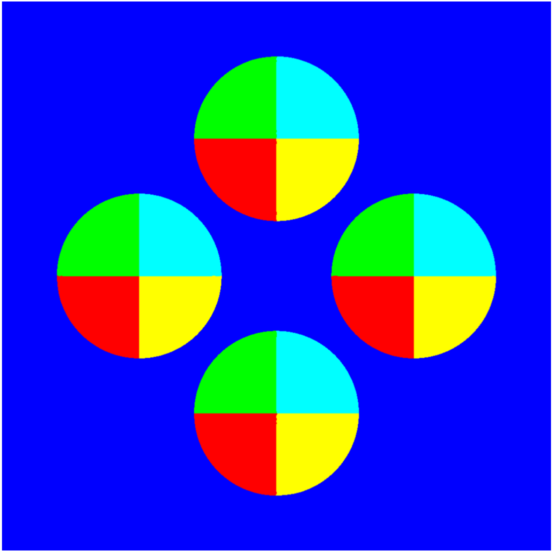


Advection: 5 fluids on a sheared flow

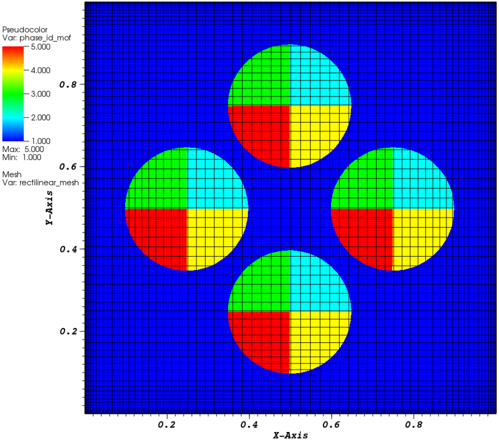




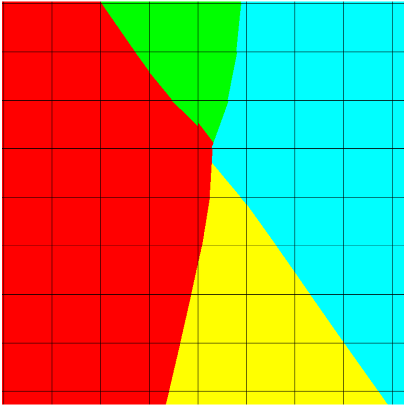
Advection: 5 fluids on a sheared flow



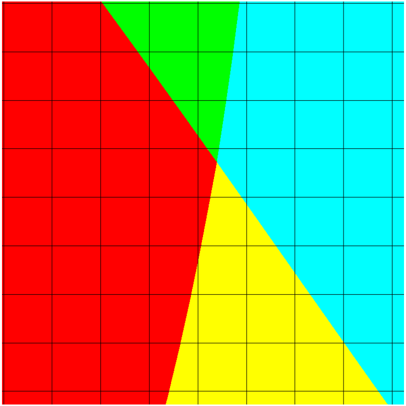
Advection: 5 fluids on a periodic flow



Advection: Focus on the B-tree dissection

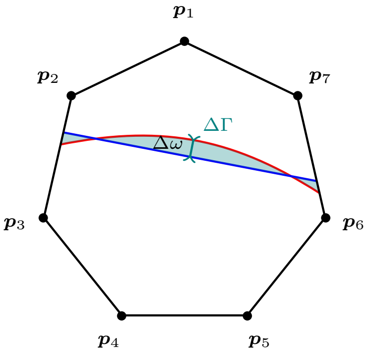


Without B-tree dissection



With B-tree dissection

Numerical results: Error computation



Local errors

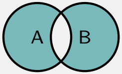
- Distance error

$$\Delta\Gamma = \max_{x^\star \in \Gamma^\star} \min_{x \in \Gamma^\ell} |x - x^\star|$$

- Area of symmetric difference

$$\Delta\omega = |\omega^\ell \triangle \omega^\star|$$

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$



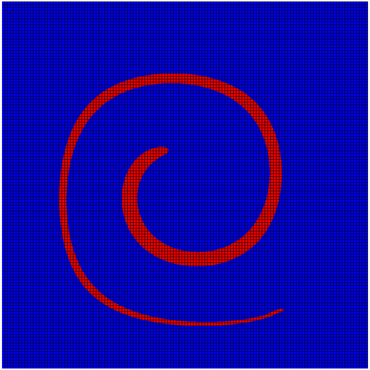
Global error

- Average deviation (equivalent to $\Delta\Gamma$)

$$\Delta\Gamma_{avg} = \frac{1}{|\partial\omega^\star|} \sum_{i=1}^{N} |\omega_i^\ell \triangle \omega_i^\star|$$

Reference: Dyadechko, V., Shashkov, M. (2007)

Numerical results: Sheared flow spatial convergence



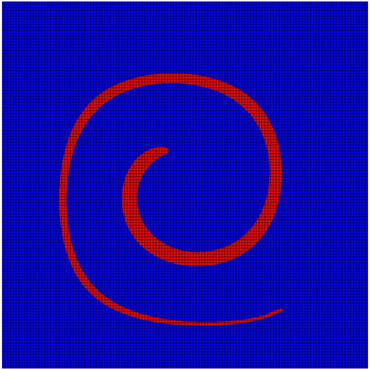
Case description:

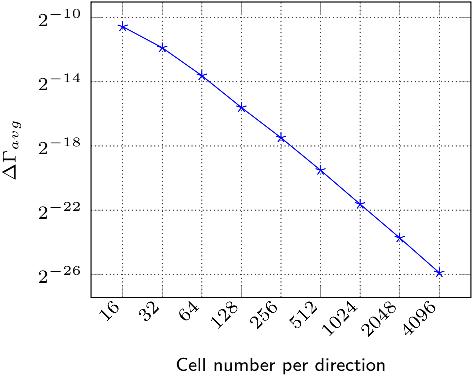
$$\boldsymbol{u}(x,y) = \begin{bmatrix} -2\sin^2(\pi x)\sin(\pi y)\cos(\pi y) \\ 2\sin^2(\pi y)\sin(\pi x)\cos(\pi x) \end{bmatrix} \cos\left(\pi \frac{t}{T}\right)$$

Parameters:

- Iterations: 1000
- Time step: 10^{-4} s
- Mesh: $N \times N$, $N \in \{16, 32, \cdots, 4096\}$

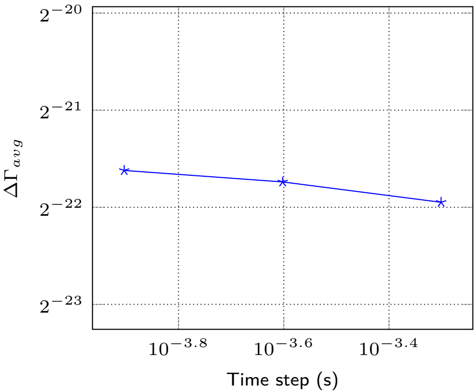
Numerical results: Sheared flow spatial convergence





N	error	order
1 024	$3.09 \cdot 10^{-7}$	2.11
2 048	$7.19 \cdot 10^{-8}$	2.10
4 096	$1.60 \cdot 10^{-8}$	2.17

Numerical results: Sheared flow time convergence

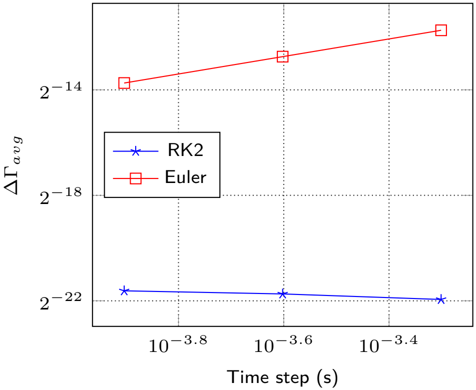


Parameters:

- Total time: 0.5 s
- Time step: $\{5 \cdot 10^{-4}, \cdots, 1.25 \cdot 10^{-4}\}$ s
- Mesh: 1024×1024

Does not converge!
Error about 10^{-7}

Numerical results: Sheared flow time convergence



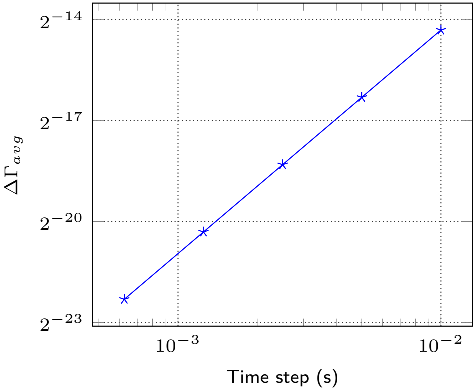
Parameters:

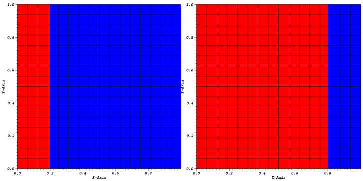
- Total time: 0.5 s
- Time step: $\{5 \cdot 10^{-4}, \cdots, 1.25 \cdot 10^{-4}\}$ s
- Mesh: 1024×1024

Order 1 convergence with Euler.
Error RK2 $<$ Error Euler

Spatial error dominates
$\Rightarrow$ Case without spatial error

Numerical results: Accelerated front flow time convergence



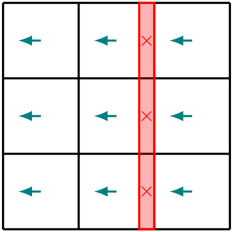


Velocity:

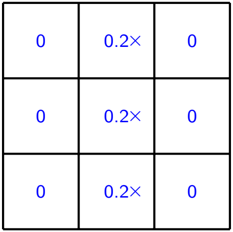
$$u_x(x, y) = 0.3\pi \sin(\pi t)$$

Order 2 with RK2

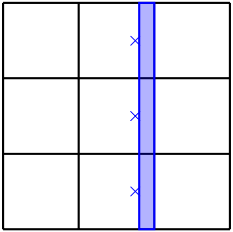
Limitations of MOF: Filaments



Initial configuration



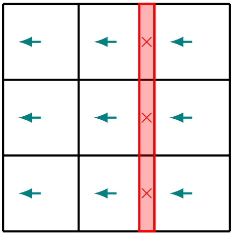
MOF representation after advection



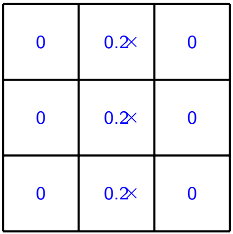
MOF reconstruction

- The filament does not move if the time step is too small!

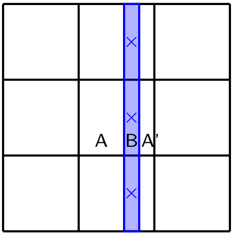
Limitations of MOF: Possible solution



Initial configuration



MOF representation after advection



MOF reconstruction

Virtual fluid A’

- Detection of filament at the advection step
- Introduction of a virtual fluid A’
- 3-fluid reconstruction

Reference: Jemison, M., Sussman, M., Shashkov, M. (2015)

Perspectives

MOF:

- 3D
- Filaments
- Analytic solutions for triangles and quadrangles

Around MOF:

- Coupling with immersed boundaries
- CLS-MOF
- Order 2 with the energy equation and Navier-Stokes

Appendix

Centroid advection

Fluid domain $\omega(t)$. Eulerian velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. $\operatorname{div} \boldsymbol{u} = 0$.

$$
\begin{aligned}
\frac{d}{dt} \int_{\omega(t)} \boldsymbol{x} d\boldsymbol{x} &= \int_{\omega(t)} \left(\frac{\partial}{\partial t} \mathrm{Id}(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla \mathrm{Id}(\boldsymbol{x}) + \mathrm{Id}(\boldsymbol{x}) \operatorname{div} \boldsymbol{u}(\boldsymbol{x}, t) \right) d\boldsymbol{x} \\
&= \int_{\omega(t)} \boldsymbol{u}(\boldsymbol{x}, t) d\boldsymbol{x} \\
&= \int_{\omega(t)} \left(\boldsymbol{u}(\boldsymbol{x}_c, t) + [\nabla \boldsymbol{u}(\boldsymbol{x}_c, t)] (\boldsymbol{x} - \boldsymbol{x}_c) + \mathcal{O}(|\boldsymbol{x} - \boldsymbol{x}_c|^2) \right) d\boldsymbol{x} \\
&= |\omega(t)| \boldsymbol{u}(\boldsymbol{x}_c, t) + \nabla \boldsymbol{u}(\boldsymbol{x}_c, t) \underbrace{\int_{\omega(t)} (\boldsymbol{x} - \boldsymbol{x}_c) d\boldsymbol{x}}_{=\mathbf{0}} + \mathcal{O}(h^2)
\end{aligned}
$$

Thus

$$
\frac{d}{dt} \boldsymbol{x}_c = \boldsymbol{u}(\boldsymbol{x}_c) + \mathcal{O}(h^2)
$$