Existence and Uniqueness of Equilibrium Points for Concave N-Person Games

by

J. B. Rosen*

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Abstract

A constrained n-person game is considered in which the constraints for each player, as well as his payoff function, may depend on the strategy of every player. The existence of an equilibrium point for such a game is shown. By requiring appropriate concavity in the payoff functions a concave game is defined. It is proved that there is a unique equilibrium point for every strictly concave game. A dynamic model for nonequilibrium situations is proposed. This model consists of a system of differential equations which specify the rate of change of each player's strategy. It is shown that for a strictly concave game the system is globally asymptotically stable with respect to the unique equilibrium point of the game. Finally, it is shown how a gradient method suitable for a concave mathematical programming problem can be used to find the equilibrium point for a concave game.

Author

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1. Introduction.

The concept of an equilibrium point for an n-person game was introduced by Nash [10, 11] and the existence of such points proved under certain assumptions on each player's strategy space and corresponding payoff function. He showed that if each player is restricted to a simplex in his own strategy space and if the payoff functions are bilinear functions of the strategies, then an equilibrium point exists. This result has been generalized to an abstract economy by Arrow and Debreu [1] and McKenzie [9], where each player's strategy space may depend on the strategy of the other players.

This more general problem is considered here. Specifically, it is only required that every joint strategy, represented by a point in the product space of the individual strategy spaces, lie in a convex, closed and bounded region R in the product space and that each player's payoff function φ_i, $i = 1, \dots n$, be concave in his own strategy. The existence of an equilibrium point for this concave n-person game is shown in Theorem 1, using a mapping of R into R and the Kakutani fixed point theorem [4].

One of the difficulties which has limited the usefulness of the concept of an equilibrium point for an n-person game is the lack of uniqueness of such points, as shown by the fact that many games possess an infinite number of equilibrium points (for example, see Shapley [12]). This difficulty is overcome by requiring that the payoff functions satisfy an additional concavity requirement which is called diagonal strict concavity. With this additional requirement it is shown in Theorems 2, 3 and 4 that every concave n-person game has a unique equilibrium point. Theorem 2 shows uniqueness for a game with orthogonal constraint sets, that is, where R is the direct product of the individual player's strategy spaces. In Theorems 3 the more general case of coupled constraints is considered. A normalized equilibrium point is defined in terms of a specified positive constant r_i for each player, which determines the value of the dual variables for the $i^{\underline{th}}$ player. Theorems 3 and 4 show that a unique normalized equilibrium point exists for each specified value of the parameters r_i. The monotone behavior at the equilibrium point of the payoff function φ_i with respect to r_i is shown in Theorem 5. Section 3 is completed by giving a sufficient condition for diagonal strict concavity in terms of certain Hessian matrices of the φ_i. The interesting case where each φ_i is bilinear in the strategies is discussed to illustrate this condition. The bimatrix game [7, 8] is a special case of this bilinear payoff function.

In Section 4 we consider a reasonable dynamic model of the n-person concave game. It is assumed that if the game is not at equilibrium each player will attempt to change his own strategy so as to obtain the maximum rate of change of his own payoff function with respect to a change in his own strategy. It is shown that the system of differential equations obtained in this way has the property that every solution starting in R remains in R (Theorem 7). The stability of the system is considered in Theorems 8 and 9. It is shown that when concavity conditions sufficient for uniqueness are satisfied the system of differential equations is globally asymptotically stable. Furthermore, starting at any feasible point in the strategy space R, the system of differential equations will always converge to the unique equilibrium point of the original n-person concave game. Thus the dynamic model and the concave game have the same unique equilibrium point. The stability proof uses the square of the norm of the right hand side of the differential equations as a Liapunov function to show that the norm approaches zero. The stability of a different dynamic model of a competitive equilibrium represented by a system of differential equations has previously been investigated [13, 19].

In Section 5 it is shown that the unique equilibrium point to the concave game can be found computationally by using a gradient method suitable for a concave mathematical programming problem [17, 18]. This may be considered as a generalization of the well known relationship between the two-person zero-sum game and linear programming [15]. It should also be noted that the general concave constrained maximization problem is obtained for the case $n = 1$, so that such a problem may be considered as a special case of the n-person concave game. For this special case of $n = 1$, the results of Sections 2 and 3 reduce to known results. However, the results of Section 4, in particular Theorem 7, appear to be new even for $n = 1$.

2. Formulation and Existence of Equilibrium Point.

The concave n-person game to be considered is described in terms of the individual strategy vector for each of the n players. The strategy of the $i^{\underline{th}}$ player is represented by the vector x_i in the Euclidian space E^{m_i}, $i = 1, \ldots n$. The vector $x \in E^m$ then denotes the simultaneous strategies of all players, where E^m is the product space $E^{m_1} \times E^{m_2} \ldots \times E^{m_n}$ and $m = \sum_{i=1}^{n} m_i$. The allowed strategies will be limited by the requirement that x be selected from a convex, closed and bounded set $R \subset E^m$. If we denote by P_i the projection of R on E^{m_i}, we will also consider the convex, closed and bounded product set $S \supseteq R$, given by $S = P_1 \times P_2 \ldots \times P_n$. This is illustrated in Fig. 1 for $n = 2$.

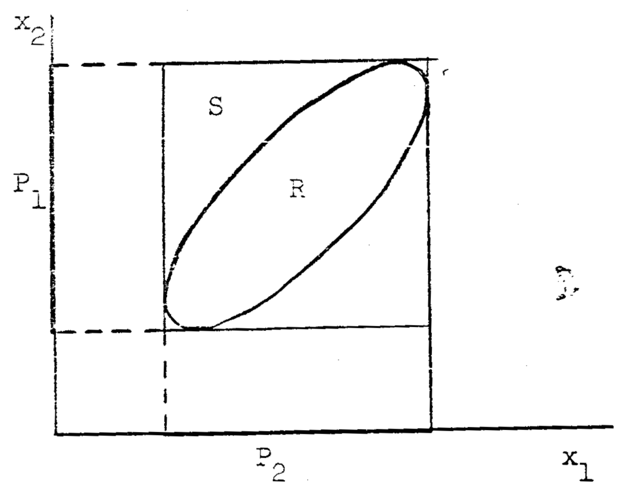


Figure 1

In most game theory papers consideration is limited to the case where each player's strategy x_i is restricted to a convex set $R_i \subset E^{m_i}$ in his own strategy space. For example, in Nash [10,11] the set R_i is the simplex in E^{m_i}. In this special case where the constraint sets are orthogonal we have $P_i = R_i$, so that $R = S = R_1 \times R_2 \ldots \times R_n$. In the general case where $R \subset S$ we will say that R is a coupled constraint set.

The payoff function for the $i^{\underline{th}}$ player depends on the strategies of all the other players as well as his own strategy, and is given by the function $\varphi_i(x) = \varphi_i(x_1, \ldots x_i, \ldots x_n)$. It will be assumed that for $x \in S$, $\varphi_i(x)$ is continuous in x and is concave in x_i for each fixed value of $(x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_n)$. With this formulation an equilibrium point of the n-person concave game is given by a point $x^\circ \in R$ such that

$$\varphi_i(x^\circ) = \max_{y_i} \{\varphi_i(x_1^\circ, \ldots y_i, \ldots x_n^\circ) \mid (x_1^\circ, \ldots y_i, \ldots x_n^\circ) \in R\} \qquad i = 1, \ldots n \tag{2.1}$$

At such a point no player can increase his payoff by a unilateral change in his strategy.

The results to follow make use of the function $\rho(x,y)$ defined for $(x,y) \in R \times R$ by

$$\rho(x,y) \equiv \sum_{i=1}^{n} \varphi_i(x_1, \ldots y_i, \ldots x_n) \tag{2.2}$$

We observe that for $(x,y) \in R \times R$ we have $(x_1, \ldots y_i, \ldots x_n) \in S$, $i = 1, \ldots n$, so that $\rho(x,y)$ is continuous in x and y and is concave in y for every fixed x, for $(x,y) \in R \times R$. We now prove the existence theorem for the concave n-person game.

<u>Theorem 1</u>

<u>An equilibrium point exists for every concave n-person game.</u>

Proof:

Consider the point to set mapping $x \in R \to \Gamma x \subset R$, given by

$$\Gamma x = \{y \mid \rho(x,y) = \max_{z \in R} \rho(x,z)\} \tag{2.3}$$

It follows from the continuity of $\rho(x,z)$ and the concavity in z of $\rho(x,z)$ for fixed x, that Γ is an upper semicontinuous mapping which maps each point of the convex, compact set R into a closed convex subset of R. Then by the Kakutani fixed point theorem [4,5] there exists a point $x^\circ \in R$ such that $x^\circ \in \Gamma x^\circ$, or

$$\rho(x^\circ,x^\circ) = \max_{z \in R} \rho(x^\circ,z) \tag{2.4}$$

The fixed point x° is an equilibrium point satisfying (2.1). For suppose that it were not. Then, say for $i = \ell$, there would be a point $x_\ell = \overline{x}_\ell$ such that $\overline{x} = (x_1^\circ, \ldots \overline{x}_\ell, \ldots x_n^\circ) \in R$ and $\varphi_\ell(\overline{x}) > \varphi_\ell(x^\circ)$. But then we have $\rho(x^\circ,\overline{x}) > \rho(x^\circ,x^\circ)$ which contradicts (2.4)

3. Uniqueness of Equilibrium Point.

In order to discuss the uniqueness of an equilibrium point we must describe the convex set R more explicitly. For the general coupled constraint set where $R \subset S$, we will describe R by means of the mapping $h(x)$ of $E^m \to E^k$, where each component $h_j(x)$, $j = 1, \ldots k$ of $h(x)$ is a concave function of x. It is assumed that

$$R = \{x \mid h(x) \geq 0\} \tag{3.1}$$

is nonvoid and bounded. It follows from the concavity of the $h_j(x)$ that the closed set R is convex. For the orthogonal constraint set $R = S = R_1 \times R_2 \ldots \times R_n$, we consider the nonvoid and bounded sets

$$R_i = \{x_i \mid h_i(x_i) \geq 0\} \;, \; i = 1, \ldots n \tag{3.2}$$

where each component $h_{ij}(x_i)$, $j = 1, \ldots k_i$, is a concave function of x_i, so that R_i is a convex, closed and bounded set in E^{m_i}. We will also assume that the set R contains a point which is strictly interior to every nonlinear constraint, that is, $\exists\, \bar{x} \in R$, such that $h_j(\bar{x}) > 0$ for every nonlinear constraint $h_j(x)$. This is a sufficient condition for the satisfaction of the Kuhn-Tucker constraint qualification [2].

We wish to use the differential form of the necessary and sufficient Kuhn-Tucker conditions for a constrained maximum [6]. We therefore make the additional assumption that the $h_j(x)$ possess continuous first derivatives for $x \in R$. We also assume that for $x \in R$ the payoff function $\varphi_i(x)$ for the $i^{\underline{th}}$ player possesses continuous first derivatives with respect to the components of x_i. For any scalar function $\varphi(x)$ we denote by $\nabla_i \varphi(x)$ the gradient with respect to x_i of $\varphi(x)$. Thus $\nabla_i \varphi(x) \in E^{m_i}$.

The Kuhn-Tucker conditions equivalent to (2.1) with R given by (3.1) can now be stated as follows:

$$h(x^\circ) \geq 0 \qquad (3.3)$$

and for $i = 1,\ldots n$, $\exists\ u_i^\circ \geq 0$, $u_i^\circ \in E^k$, such that

$$u_i^{\circ\prime} h(x^\circ) = 0 \qquad (3.4)$$

and

$$\varphi_i(x^\circ) \geq \varphi_i(x_1^\circ,\ldots y_i,\ldots x_n^\circ) + u_i^{\circ\prime} h(x_1^\circ,\ldots y_i,\ldots x_n^\circ) \qquad (3.5)$$

Since $\varphi_i(x)$ and $h_j(x)$ are concave and differentiable, the inequality (3.5) is equivalent to

$$\nabla_i \varphi_i(x^\circ) + \sum_{j=1}^{k} u_{ij}^\circ \nabla_i h_j(x^\circ) = 0, \quad i = 1,\ldots n \qquad (3.6)$$

We will also use the following relation which holds as a result of the concavity of $h_j(x)$. For every $x^\circ, x' \in R$ we have

$$h_j(x') - h_j(x^\circ) \leq (x'-x^\circ)'\nabla h_j(x^\circ) = \sum_{i=1}^{n} (x_i'-x_i^\circ)'\nabla_i h_j(x^\circ) \qquad (3.7)$$

A weighted nonnegative sum of the functions $\varphi_i(x)$ is given by

$$\sigma(x,r) = \sum_{i=1}^{n} r_i \varphi_i(x), \quad r_i \geq 0 \qquad (3.8)$$

for each nonnegative vector $r \in E^n$. For each fixed r, a related mapping $g(x,r)$ of E^m into itself is defined in terms of the gradients $\nabla_i \varphi_i(x)$ by

$$g(x,r) = \begin{pmatrix} r_1\nabla_1\varphi_1(x) \\ r_2\nabla_2\varphi_2(x) \\ \vdots \\ r_n\nabla_n\varphi_n(x) \end{pmatrix} \tag{3.9}$$

An important property of $\sigma(x,r)$ is given by the following

<u>Definition</u>

The function $\sigma(x,r)$ will be called <u>diagonally strictly concave</u> for $x \in R$ and fixed $r \geq 0$, if for every x°, $x' \in R$ we have

$$(x'-x^\circ)' g(x^\circ,r) + (x^\circ-x')' g(x',r) > 0 \tag{3.10}$$

As shown later, a sufficient condition that $\sigma(x,r)$ be diagonally strictly concave is that the symmetric matrix $[G(x,r) + G'(x,r)]$ is negative definite for $x \in R$, where $G(x,r)$ is the Jacobian with respect to x of $g(x,r)$.

We first give the uniqueness theorem for orthogonal constraint sets where $R = S$.

<u>Theorem 2</u>

<u>If $\sigma(x,r)$ is diagonally strictly concave for some $r = \bar{r} > 0$, then the equilibrium point x° satisfying (2.1) is unique.</u>

Proof:

Assume there are two distinct equilibrium points x° and $x' \in R$, each of which satisfies (2.1). Then by the necessity of the Kuhn-Tucker conditions we have for $\ell = 0, 1$ and $i = 1, \ldots, n$,

$$h_i(x_i^\ell) \geq 0 \tag{3.11}$$

$$\exists\, u_i^\ell \geq 0,\ u_i^\ell \in E^{k_i}, \text{ such that}$$

$$u_i^{\ell\prime} h_i(x_i^\ell) = 0 \tag{3.12}$$

$$\nabla_i \varphi_i(x^\ell) + \sum_{j=1}^{k_i} u_{ij}^\ell \nabla_i h_{ij}(x_i^\ell) = 0 \tag{3.13}$$

We multiply (3.13) by $\bar{r}_i(x_i^1-x_i^\circ)'$ for $\ell = 0$ and by $\bar{r}_i(x_i^\circ-x_i^1)'$ for $\ell = 1$, and sum on i. This gives

$$\beta + \gamma = 0 \tag{3.14}$$

where

$$\beta = (x^1-x^\circ)' g(x^\circ,\bar{r}) + (x^\circ-x^1)' g(x^1,\bar{r}) \tag{3.15}$$

and

$$\begin{aligned}\gamma &= \sum_{i=1}^{n} \sum_{j=1}^{k_i} \bar{r}_i \{u_{ij}^\circ (x_i^1-x_i^\circ)' \nabla_i h_{ij}(x_i^\circ) + u_{ij}^1 (x_i^\circ-x_i^1)' \nabla_i h_{ij}(x_i^1)\} \\ &\geq \sum_{i=1}^{n} \sum_{j=1}^{k_i} \bar{r}_i \{u_{ij}^\circ [h_{ij}(x_i^1)-h_{ij}(x_i^\circ)] + u_{ij}^1 [h_{ij}(x_i^\circ)-h_{ij}(x_i^1)]\} \\ &= \sum_{i=1}^{n} \bar{r}_i \{u_i^{\circ\prime} h_i(x_i^1) + u_i^{1\prime} h_i(x_i^\circ)\}\end{aligned} \tag{3.16}$$

The inequality follows from the concavity of the $h_{ij}(x)$ and (3.7), and the last relation from (3.12). Then from (3.11) we have that $\gamma \geq 0$. Since $\sigma(x,\bar{r})$ is diagonally strictly concave it follows from (3.10) that $\beta > 0$. But this contradicts (3.14), so that we cannot have two distinct equilibrium points and therefore x° is unique.

We now consider the general case where R is a coupled constraint set and is given by (3.1). The values of the nonnegative multipliers u_i°, $i=1,\ldots n$ given by the Kuhn-Tucker conditions at an equilibrium point will, in general, not be related to each other. We will consider a special kind of equilibrium point such that each u_i° is given by

$$u_i^\circ = u^\circ/r_i, \quad i = 1,\dots n \tag{3.17}$$

for some $r > 0$, and $u^\circ \geq 0$. We will call this a <u>normalized equilibrium point</u>.

<u>Theorem 3</u>

<u>There exists a normalized equilibrium point to a concave n-person game for every specified</u> $r > 0$.

Proof:

For a fixed value $r = \bar{r} > 0$, let

$$\rho(x,y,\bar{r}) = \sum_{i=1}^{n} \bar{r}_i\varphi_i(x_1,\dots y_i \dots x_n) \tag{3.18}$$

Using the fixed point theorem as in Theorem 1, there exists a point x° such that

$$\rho(x^\circ,x^\circ,\bar{r}) = \max_y \{\rho(x^\circ,y,\bar{r}) \mid h(y) \geq 0\} \tag{3.19}$$

Then by the necessity of the Kuhn-Tucker conditions, $h(x^\circ) \geq 0$, and $\exists\, u^\circ \geq 0$, such that $u^{\circ\prime} h(x^\circ) = 0$ and

$$\bar{r}_i\nabla_i\varphi_i(x^\circ) + \sum_{j=1}^{k} u_j^\circ \nabla_i h_j(x^\circ) = 0, \quad i = 1,\dots.n \tag{3.20}$$

But these are just the conditions (3.3), (3.4) and (3.6), with $u_{ij}^\circ = u_j^\circ/\bar{r}_i$, or $u_i^\circ = u^\circ/\bar{r}_i$, which are sufficient to insure that x° satisfies (2.1). x° is therefore a normalized equilibrium point for the specified value of $r = \bar{r}$.

Theorem 4

Let $\sigma(x,r)$ be diagonally strictly concave for every $r \in Q$, where Q is a convex subset of the positive orthant of E^n. Then for each $r \in Q$ there is a unique normalized equilibrium point.

Proof:

Assume that for some $r = \bar{r} \in Q$ we have two normalized equilibrium points x° and x'. Then we have for $\ell = 0, 1$ and $i = 1,\ldots,n$,

$$h(x^\ell) \geq 0 \tag{3.21}$$

$$\exists\, u^\ell \geq 0,\ u^\ell \in E^k, \quad \text{such that}$$

$$u^{\ell\prime} h(x^\ell) = 0 \tag{3.22}$$

$$\bar{r}_i \nabla_i \varphi_i(x^\ell) + \sum_{j=1}^{k} u_j^\ell \nabla_i h(x^\ell) = 0 \tag{3.23}$$

We multiply (3.23) by $(x_i^\prime - x_i^\circ)^\prime$ for $\ell = 0$ and by $(x_i^\circ - x_i^\prime)$ for $\ell = 1$, and sum on i. As in the proof of Theorem 2 this gives $\beta + \gamma = 0$, where β is given by (3.15) and

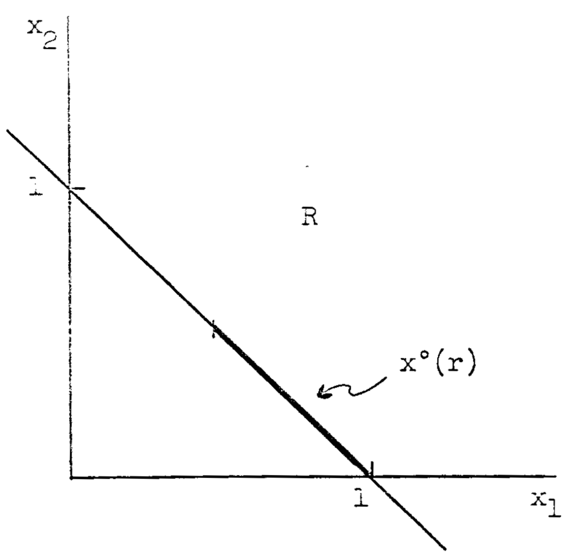
$$\gamma = \sum_{j=1}^{k} \sum_{i=1}^{n} \{u_j^\circ (x_i^\prime - x_i^\circ)^\prime \nabla_i h(x^\circ) + u_j^\prime (x_i^\circ - x_i^\prime)^\prime \nabla_i h(x^\prime)\}$$

$$\geq u^{\circ\prime}[h(x') - h(x^\circ)] + u^{\prime\prime}[h(x^\circ) - h(x')] \tag{3.24}$$

$$= u^{\circ\prime} h(x') + u^{\prime\prime} h(x^\circ) \geq 0$$

Then since $\sigma(x,\bar{r})$ is diagonally strictly concave we have $\beta > 0$, which contradicts $\beta + \gamma = 0$ and proves the theorem.

We will now investigate the dependence of the normalized equilibrium point on the value of r for the general case where R is a coupled constraint set. For an orthogonal constraint set it follows from Theorem 2 that if $\sigma(x,r)$ is diagonally strictly concave for some $r = \bar{r} > 0$, the equilibrium point x° is independent of r. On the other hand it is not difficult to construct a simple example with a coupled constraint set (see Fig. 2) where the equilibrium point x° does depend on r.



$$\varphi_1(x) = -\frac{1}{2}x_1^2 + x_1x_2$$

$$\varphi_2(x) = -x_2^2 - x_1x_2$$

$$h_1(x) = x_1 \geq 0$$

$$h_2(x) = x_2 \geq 0$$

$$h_3(x) = x_1 + x_2 - 1 \geq 0$$

$$\varphi_1(x^\circ) = \max_{x_1}\left\{ \varphi_1(x_1,x_2^\circ) \,\middle|\, h(x_1,x_2^\circ) \geq 0 \right\} = x_1^\circ(1\cdot$$

$$\varphi_2(x^\circ) = \max_{x_2}\left\{ \varphi_2(x_1^\circ,x_2) \,\middle|\, h(x_1^\circ,x_2) \geq 0 \right\} = x_1^\circ -$$

$$x_1^\circ = \begin{cases} 1 & , \quad r_1 \leq r_2 \\ \dfrac{r_1 + 2r_2}{2r_1 + r_2} & , \quad r_1 > r_2 \end{cases} , \qquad x_2^\circ = 1 - x_1^\circ$$

Figure 2

In such a case we will now show that in a certain sense the equilibrium value of φ_i is a monotone increasing function of r_i.

Theorem 5

Let $\sigma(x,r)$ be diagonally strictly concave for $r \in Q$. Let $r^\circ, r' \in Q$ be such that $r_i' = r_i^\circ$, $i \neq q$ and $r_q' > r_q^\circ$. Let x° and x', with $x' \neq x^\circ$, be the corresponding unique normalized equilibrium points. Then the directional derivative of $\varphi_q(x^\circ)$ along the ray $(x_q'-x_q^\circ)$ is positive.

Proof:

Let u° and u' be the multipliers corresponding to the normalized equilibrium points x° and x'. Then for $\ell = 1$ and $i = 1,\ldots,m$, and for $\ell = 0$ and $i \neq q$, the relations (3.21), (3.22) and (3.23) are satisfied with $\bar{r}_i = r_i'$. For $\ell = 0$ and $i = q$, we have

$$(r_q^\circ - r_q')\nabla_q \varphi_q(x^\circ) + r_q' \nabla_q \varphi_q(x^\circ) + \sum_{j=1}^{k} u_j^\circ \nabla_q h(x^\circ) = 0 \qquad (3.25)$$

Multiplying by $(x_i'-x_i^\circ)'$ for $\ell = 0$ and $(x_i^\circ - x_i')'$ for $\ell = 1$, and summing now gives

$$(r_q^\circ - r_q')(x_q'-x_q^\circ)'\nabla_q \varphi_q(x^\circ) = -(\beta + \gamma) < 0 \qquad (3.26)$$

or since $r_q' > r_q^\circ$,

$$(x_q'-x_q^\circ)'\nabla_q \varphi_q(x^\circ) > 0 \qquad (3.27)$$

But this is just the directional derivative of $\varphi_q(x^\circ)$ along the ray $(x_q'-x_q^\circ)$.

A useful interpretation of Theorem 5 is obtained by observing that if $\varphi_q(x)$ has bounded 2^{nd} partial derivatives and $\|x_q'-x_q^\circ\|$ is sufficiently small then it follows from (3.27) that $\varphi_q(\bar{x}) > \varphi_q(x^\circ)$, where $\bar{x} = (x_1^\circ,\ldots x_q',\ldots x_n^\circ)$. Since x° is an equilibrium point $\bar{x}$ cannot be a feasible point, and the value of $\varphi_q(x)$ may decrease as x goes from the infeasible point $\bar{x}$ to the new (feasible) equilibrium point x^1, as illustrated in Fig. 3. Because of the diagonal concavity property of $\varphi_i(x)$, the dependence of $\varphi_q(x)$ on x_q will usually dominate its dependence on x_i, $i \neq q$. Therefore, it will usually be true that $\varphi_q(x') > \varphi_q(x^\circ)$. This is illustrated by the example of Fig. 2, where it is easy to show that both $\frac{\partial\varphi_1}{\partial r_1}$ and $\frac{\partial\varphi_2}{\partial r_2}$ are nonnegative.

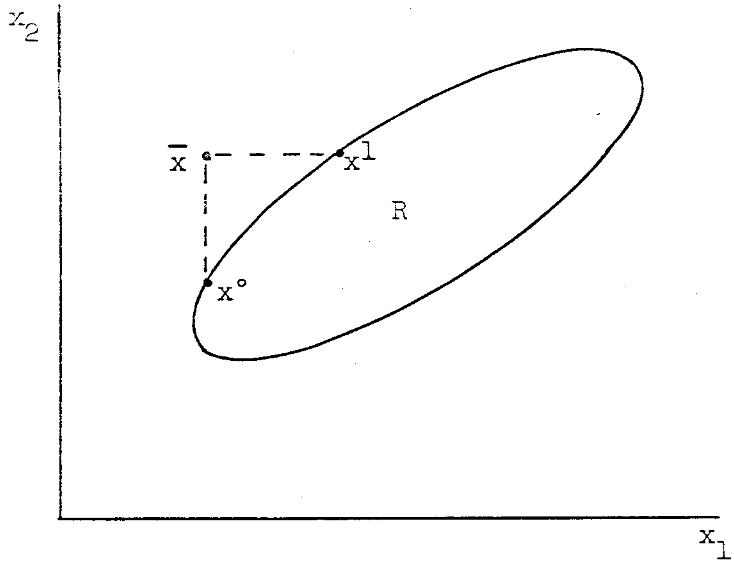


Figure 3

We complete this section by giving a sufficient condition on the functions $\varphi_i(x)$, which insures that $\sigma(x,r)$ is diagonally strictly concave. The condition is given in terms of the $m\times m$ matrix $G(x,r)$ which is the Jacobian of $g(x,r)$ for fixed $r > 0$. That is, the j^{th} column of $G(x,r)$ is $\frac{\partial g(x,r)}{\partial x_j}$, $j = 1,\ldots m$, where $g(x,r)$ is defined by (3.9).

Theorem 6

A sufficient condition that $\sigma(x,r)$ be diagonally strictly concave for $x \in R$ and fixed $r = \bar{r} > 0$, is that the symmetric matrix $[G(x,\bar{r}) + G'(x,\bar{r})]$ be negative definite for $x \in R$.

Proof:

Let x°, x' be any two distinct points in R, and let $x(\theta) = \theta x' + (1-\theta)x^\circ$ so that $x(\theta) \in R$ for $0 \le \theta \le 1$. Now since $G(x,\bar{r})$ is the Jacobian of $g(x,\bar{r})$ we have

$$\frac{dg(x(\theta),\bar{r})}{d\theta} = G(x(\theta),\bar{r})\,\frac{dx(\theta)}{d\theta} = G(x(\theta),\bar{r})(x'-x^\circ) \qquad (3.28)$$

or

$$g(x',\bar{r}) - g(x^\circ,\bar{r}) = \int_\circ^1 G(x(\theta), \bar{r})(x'-x^\circ)\, d\,\theta \qquad (3.29)$$

Multiplying both sides by $(x^\circ-x')'$ gives

$$(x^\circ-x')' g(x',\bar{r}) + (x'-x^\circ)' g(x^\circ,\bar{r}) = - \int_\circ^1 (x'-x^\circ)' G(x(\theta),\bar{r})(x'-x^\circ)\, d\,\theta$$

$$= - \frac{1}{2} \int_\circ^1 (x'-x^\circ)' [G(x(\theta),\bar{r}) + G'(x(\theta),\bar{r})]\, (x'-x^\circ)\, d\,\theta > 0 \quad (3.30)$$

which shows that (3.10) is satisfied.

The interesting case where $\varphi_i(x)$ is bilinear in the strategies x_j emphasizes an important relation between this condition and a stability matrix. We let

$$\varphi_i(x) = \sum_{j=1}^{n} [e'_{ij} + x'_i C_{ij}]\, x_j, \quad i = 1,\dots n \qquad (3.31)$$

where e_{ij} is a constant vector in E^{m_j} and C_{ij} is an $m_i \times m_j$ constant matrix. The bimatrix game [7,8] is a special case of (3.31) with $n = 2$, $e_{ij} = 0$, $C_{11} = C_{22} = 0$ and $C_{12} \neq 0$, $C_{21} \neq 0$. The two-person zero-sum game is a further specialization with $C_{21} = -C'_{12}$.

From the definition (3.9) of $g(x,r)$ and $G(x,r)$ as its Jacobian matrix, we obtain

$$G(x,r) = D\, C \qquad (3.32)$$

where C is the $m \times m$ constant matrix

$$C = \begin{pmatrix} 2C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & 2C_{22} & & \\ \vdots & & & \\ C_{n1} & & & 2C_{nn} \end{pmatrix} \qquad (3.33)$$

and D is the diagonal positive definite matrix $D = \mathrm{diag}\{r_i\}$. For this bilinear case it follows from Theorems 2 and 6 that we have uniqueness if there exists some $\bar{r} > 0$ such that

$$\bar{D}\, C + C'\bar{D} = - I \tag{3.34}$$

where $\bar{D} = \mathrm{diag.}\{\bar{r}_i\}$. But this is just the condition which ensures that every eigenvalue of C has a negative real part (see, for example, Bellman [3]). Thus the same condition which guarantees uniqueness also implies that C is a stability matrix.

A case which might be considered as a generalization of the two-person zero-sum game is the n-person "skew-symmetric" game where $C_{ji} = - C'_{ij}$, $i, j = 1, \ldots n$. For such a game we will have $[C + C']$ negative definite if $[C_{ii} + C'_{ii}]$ is negative definite for $i = 1, \ldots n$.

4. Global Stability of Equilibrium Point.

We will now consider a reasonable dynamic model of a concave n-person game in which each player changes his own strategy in such a way that the joint strategy remains in R and his own payoff function would increase if all other players held to their current strategy. That is, each player changes his strategy at a rate proportional to the gradient with respect to his strategy of his payoff function, subject to the constraints. If we let the proportionality constant for the $i^{\underline{th}}$ player be r_i, we obtain the following system of differential equations for the strategies x_i,

$$\frac{dx_i}{dt} = \dot{x}_i = r_i \nabla_i \varphi_i(x) + \sum_{j=1}^{k} u_j \nabla_i h_j(x) \;, \quad i = 1, \ldots n \tag{4.1}$$

where the vector u lies in a bounded subset $U(x)$ of the positive orthant of E^k. The effect of the summation term, with the appropriate choice of u, is to ensure that starting with any $x \in R$, the solution to (4.1) remains in R. In fact, the right hand side of (4.1) is just the projection of the gradient $\nabla_i \varphi_i(x)$ on the manifold formed by the active constraints at x. If we define an $m \times k$ matrix $H(x)$, whose $j^{\underline{th}}$ column is $\nabla h_j(x)$,

$$H(x) = [\nabla h_1(x) \quad \nabla h_2(x) - - - \nabla h_k(x)] \tag{4.2}$$

and use the definition (3.9) of $g(x,r)$, we can define the mapping $f(x,u,r)$ of $E^{m+k} \to E^m$ for each fixed $\bar{r} > 0$, as follows

$$f(x,u,\bar{r}) = g(x,\bar{r}) + H(x)u \tag{4.3}$$

Then the system (4.1) can be written

$$\dot{x} = f(x,u,\bar{r}) \;, \quad u \in U(x) \tag{4.4}$$

The set $U(x) \subset E^k$ is determined as follows

$$U(x) = \left\{ u \;\middle|\; \|f(x,u,\bar{r})\| = \min_{\substack{v_j \geq 0,\, j \in J \\ v_j = 0,\, j \notin J}} \|f(x,v,\bar{r})\| \right\} \tag{4.5}$$

where

$$J = J(x) = \{ j \mid h_j(x) \leq 0 \} \tag{4.6}$$

Note that for every interior point x of R the set $J(x)$ is empty and $U(x) = 0$, so that $f(x,u,\bar{r}) = g(x,\bar{r})$.

We will assume that $g(x,r)$ and $H(x)$ are continuous in x for all $x \in \bar{R}$, where $\bar{R} \supset R$ is a compact set such that every point of the compact set R is interior to $\bar{R}$.

<u>Theorem 7</u>

<u>Starting at any point</u> $x \in R$ <u>a continuous solution</u> $x(t)$ <u>to</u> (4.4) <u>exists, such that</u> $x(t)$ <u>remains in</u> R <u>for all</u> $t > 0$.

Proof:

Because of the continuity in x, and assuming only that u is measurable in t, we have from the Carathéodory existence theory [14,16] that a continuous solution $x(t)$ exists for $x(t)$ in $\bar{R}$, which satisfies (4.4) almost everywhere. Now suppose that for some point $x' \in \bar{R}$ on the trajectory $x(t)$ we have $h_\ell(x') < 0$. Then by the continuity of $x(t)$ there must be an earlier point $\bar{x}$ on the trajectory, such that $h_\ell(\bar{x}) = 0$ and $\dot{h}_\ell(\bar{x}) < 0$. But from the latter and (4.4) we have

$$\dot{h}_\ell(\bar{x}) = \nabla h'_\ell(\bar{x})\, \dot{x} = \nabla h'_\ell(\bar{x})\, f < 0 \tag{4.7}$$

We let the corresponding value of u be $\bar{u} \in U(\bar{x})$. From the definition (4.3) we have

$$\|f\|^2 = g'g + 2\,\bar{u}'H'g + \bar{u}'H'H\,\bar{u} \tag{4.8}$$

or

$$\frac{\partial\|f\|^2}{\partial u_\ell} = 2 \quad h_\ell'(\bar{x})[g + H\bar{u}] = 2\nabla h_\ell'(\bar{x})\ f < 0 \tag{4.9}$$

According to (4.9) we could decrease the norm $\|f\|$ by increasing $\bar{u}_\ell > 0$. But since $h_\ell(\bar{x}) = 0$, we have $\ell \in J(\bar{x})$ by (4.6) and therefore $\bar{u}$ cannot satisfy (4.5) so that $\bar{u} \notin U(\bar{x})$. This contradiction shows that there is no point x' on the trajectory such that $h_i(x') < 0$, for any i, which proves the theorem.

By a direct application of the necessity of the Kuhn-Tucker conditions for the constrained minimization problem in (4.5) it is not difficult to demonstrate the following

Lemma

The nonzero elements of every vector $u \in U(x)$ are given by a vector $\bar{u} \in E^{\bar{k}}$, $\bar{k} \le k$, where

$$\bar{u} = -(\bar{H}'\bar{H})^{-1}\,\bar{H}'g(x,\bar{r}) \ge 0 \tag{4.10}$$

The $m \times \bar{k}$ matrix $\bar{H} = \bar{H}(x)$ consists of $\bar{k}$ linearly independent columns of $H(x)$ selected from $\nabla h_j(x)$ for $j \in J$.

We now consider an equilibrium point $\bar{x}$ of the system of differential equations (4.4). That is, for a fixed $r = \bar{r}$, we will call $\bar{x}$ an equilibrium point of (4.4) if

$$f(\bar{x},u,\bar{r}) = 0, \quad u \in U(\bar{x}) \tag{4.11}$$

The system (4.4) will be called asymptotically stable in R if for every initial point $x \in R$, the solution $x(t)$ to (4.4) converges to an equilibrium point $\bar{x} \in R$ as $t \to \infty$.

Theorem 8

If R is given by (3.1) and $[G + G']$ is negative definite for $x \in R$, where G is the Jacobian of $g(x,\bar{r})$, then the system (4.4) is asymptotically stable in R.

Proof:

The proof consists of showing that for x and u satisfying (4.4), the rate of change of $\|f(x,u,\bar{r})\|^2$ is always negative for $f(x,u,\bar{r}) \neq 0$. We first consider the situation when the selection of columns in $\bar{H}(x)$ remains unchanged. Then since all elements of u are zero except those given by $\bar{u} \geq 0$, we have from (4.3)

$$f = g + \bar{H}\,\bar{u} = g + \sum \bar{u}_j \nabla h_j \tag{4.12}$$

and

$$\dot{f} = G\,\dot{x} + \sum \bar{u}_j\, Q_j\, \dot{x} + \bar{H}\,\dot{\bar{u}} \tag{4.13}$$

where Q_j is the Jacobian of $\nabla h_j(x)$ (or its equivalent, the Hessian of $h_j(x)$) and is therefore negative semidefinite from the concavity of $h_j(x)$. Now using (4.13) and (4.4) we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2 = \frac{1}{2}\frac{d}{dt}(f'f) = f'\dot{f} = f'G\,f + \sum \bar{u}_j\, f'Q_j\, f + f'\bar{H}\,\dot{\bar{u}} \tag{4.14}$$

We consider the last term and make use of (4.12) and (4.10) to show that

$$f'\bar{H}\,\dot{\bar{u}} = [g'\bar{H} + \bar{u}'\bar{H}'\bar{H}]\,\dot{\bar{u}} = [g'\bar{H} - g'\bar{H}]\,\dot{\bar{u}} = 0 \tag{4.15}$$

Then since $[G + G']$ is negative definite and the Q_j are negative semidefinite we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2 = \frac{1}{2} f'[G + G'] f + \sum \bar{u}_j f'Q_j f \leq -\epsilon \|f\|^2 \qquad (4.16)$$

for some $\epsilon > 0$.

A change in the columns selected for $\bar{H}(x)$ can never increase the value of $\|f\|$ since the selection as determined by (4.5) will always minimize $\|f\|$. It therefore follows from (4.16) that $\lim_{t\to\infty} \|f\| = 0$, so that $x(t) \to \bar{x}$, where $\bar{x}$ is an equilibrium point which satisfies (4.11). By Theorem 7, we have that $\bar{x} \in R$, so that (4.4) is asymptotically stable in R.

An equilibrium point $x^\circ \in R$ will be called globally asymptotically stable in R if for every starting point $x \in R$ the solution $x(t)$ to (4.4) converges to x°. We will now show that with the appropriate concavity conditions the unique equilibrium point x° of (2.1) is also globally asymptotically stable in R.

<u>Theorem 9</u>

<u>Let R be given by (3.1) and G be the Jacobian of $g(x,r)$ for some fixed $r = \bar{r} > 0$. Then if $[G + G']$ is negative definite for $x \in R$, the normalized equilibrium point $x^\circ(\bar{r})$ is globally asymptotically stable in R.</u>

Proof:

Since $[G + G']$ is negative definite, $\sigma(x,\bar{r})$ is diagonally strictly concave by Theorem 6. Then by Theorem 4 there is a unique normalized equilibrium point $x^\circ = x^\circ(\bar{r})$, which satisfies (3.21), (3.22) and (3.23). But an equilibrium point $\bar{x}$ of (4.4) also satisfies these three relations. The first relation is satisfied since $\bar{x} \in R$, while (4.11) is equivalent to (3.22) and (3.23). Therefore we must have $\bar{x} = x^\circ$. By Theorem 8, the system (4.4) is asymptotically stable in R. Since $\overline{x} = x^\circ$ is unique the solution to (4.4) will converge to x° from every starting point in R, and the system is globally asymptotically stable.

5. Determination of Equilibrium Point.

The global stability of the equilibrium point permits us to determine the unique equilibrium point for any concave game by appropriate mathematical programming computational methods. In particular, gradient methods for the concave nonlinear programming problem [17,18] can be modified to find the equilibrium point for a concave game. Such methods take finite steps in the direction of the gradient of the function to be maximized taking account of the constraints by projection, or appropriate penalties, in order to remain in the feasible region R. The essential idea in applying one of these gradient methods to the concave game problem is to use the vector $g(x,r)$, given by (3.9), as if it were the gradient of a function of x, where the function is to be maximized for $x \in R$. The solution to this "maximization" problem will give a point $x^\circ \in R$ where the Kuhn-Tucker conditions (3.21), (3.22) and (3.23) are satisfied. But as has been shown such a point is the unique equilibrium point for the concave game. Note that the optimality conditions involve only the gradient $g(x,r)$ and do not require that the function itself be known. The gradient projection method can be considered as a finite difference approximation to the system (4.4), where the solution is obtained by a sequence of finite steps in the direction of the projected gradient $f(x,u,\bar{r})$. The only practical difference between this and a true maximization problem is that in the latter case we choose the step length so as to give a maximum of the true function value along the chosen ray, whereas for the equilibrium point problem we choose the step length so as to minimize the norm of f.

To show how this is done we consider the finite difference approximation to (4.4) given by

$$x^{j+1} = x^j + \tau^j f(x^j,u^j,\bar{r}) , \quad u^j \in U(x^j) \tag{5.1}$$

where τ^j is the step length to be selected.

Theorem 10.

If the assumptions of Theorem 8 are satisfied then a finite step length τ^j can be chosen so that $\|f^{j+1}\| < \|f^j\|$, for $f^j \neq 0$, where $f^j = f(x^j,u^j,\bar{r})$.

Proof:

For $u = u^j$ held fixed we have

$$\bar{f}^{j+1} = f(x^{j+1},u^j,\bar{r}) = f^j + \bar{F}(x^{j+1} - x^j) \tag{5.2}$$

where $\bar{F}$ is a mean value of the Jacobian of f, so that $f'\bar{F} f < 0$, for $f \neq 0$. Then from (5.1) we have

$$\bar{f}^{j+1} = (I + \tau^j \bar{F})f^j \tag{5.3}$$

The norm of $\bar{f}^{j+1}$ is minimized by the choice

$$\tau^j = - \|\bar{F}f^j\|^2/f^{j\prime}\bar{F}f^j > 0 \tag{5.4}$$

which gives

$$\|\bar{f}^{j+1}\|^2 = \|f^j\|^2 + (\tau^j)^2 f^{j\prime}\bar{F}f^j < \|f^j\|^2 \tag{5.5}$$

Finally since $f^{j+1} = f(x^{j+1},u^{j+1},\bar{r})$, where $u^{j+1} \in U(x^{j+1})$, it follows from (4.5) and (5.2) that $\|f^{j+1}\| \leq \|\bar{f}^{j+1}\| < \|f^j\|$.

The convergence of this finite difference procedure to the unique equilibrium point x° can be shown as in Theorem 8.

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