JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 4, October–December 1963

On the Graphs of Finite Idempotent Boolean Relation Matrices

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(July 16, 1963)

This paper presents a graph-theoretic characterization of idempotent Boolean relation matrices of finite order. A relation-theoretic point of view is adopted in the paper. Idempotent matrices appear in the sequence of powers of any Boolean relation matrix, and are of purely theoretical as well as applied interest in connection with issues of convergence. The results provide a detailed description of the connectivity and cyclic structure of the directed graphs of idempotent matrices. The study is basically motivated by certain connectivity and flow problems which arise in the analysis of large-scale information systems. The formal results are exemplified in an investigation of the asymptotic forms of a recursive model of an information system which affords a conjoint representation of processes of communication and derivation of information. A second principal application is given in a process formulation for the generation of consistent rank orderings. The relation between system design and idempotent forms is exhibited in the two applications.

1. Introduction

In this paper we present a graph-theoretic characterization of idempotent Boolean relation matrices of finite order, that is, matrices S such that $S^2=S$. We adopt an approach which corresponds to that of an earlier study of the graphs and powers of finite relation matrices and stochastic matrices [9].[1] The present results essentially treat issues of detailed matrix structure.

This study is basically motivated by certain connectivity and flow problems which arise in the analysis of the structure of large-scale information systems. The formal results are exemplified in part in an investigation of the asymptotic forms of a recursive model of an information system [8, 10] (sec. 5). Additional applications are noted in the body of the paper, including among others, a Boolean process formulation of consistent rank orderings [2] (sec. 4).

2. Definitions

We adhere essentially to a relation-theoretic point of view in the analysis of the directed graphs of Boolean relation matrices [5]. The terminology given below corresponds in the main to that of some earlier studies [9, 10] and is also consistent with certain current expositions of the theory of graphs [4].

A homogeneous binary or dyadic relation defined on a finite set Σ of elements is construed in classical fashion as any rule ρ which specifies for each ordered couple (a_i, a_j) of elements of Σ that either the relation ρ obtains between a_i and a_j (symbolically $a_i \rho a_j$) or that it does not obtain (symbolically $a_i \bar{\rho} a_j$). The one-one representation of binary relations ρ on Σ by square Boolean matrices $R=\|r_{ij}\|$ $(i,j=1, \ldots, n)$ is defined in the following way: $r_{ij}=1$ if $a_i \rho a_j$; $r_{ij}=0$ if $a_i \bar{\rho} a_j$.

The graph-theoretic representation of binary relations ρ on Σ may be usefully stated in terms of the one-one representation of Boolean relation matrices by finite directed graphs. Given any square Boolean relation matrix R of order n, the *graph of* R, $G(R)$, consists of n objects $\alpha_1, \ldots, \alpha_n$ called *vertices* and the totality of *ordered pairs* of vertices $\overrightarrow{\alpha_i, \alpha_j}$, such that $\overrightarrow{\alpha_i, \alpha_j}$ exists in $G(R)$ if, and only if, $r_{ij}=1$ in R. The ordered pair or *edge* $\overrightarrow{\alpha_i, \alpha_j}$ is represented by an arrowed line directed from α_i to α_j with arrowhead pointing toward α_j; an edge of the reflexive form $\overrightarrow{\alpha_i, \alpha_i}$ is taken to be admissible for any vertex α_i in $G(R)$ and is represented by a simple loop from α_i back to α_i. A *subgraph* of an arbitrary graph G is a subset of the edges and vertices of G containing with each edge its endpoints. Given this one-one representation, it is then useful to consider the Boolean relation matrix $R(G)$ corresponding to any specified directed graph G. Any given subgraph H of G $(H \subset G)$ may then be represented by the submatrix $R(H)$ (in the general sense of subrelation) of the relation matrix $R(G)$ corresponding to G.

We consider next a sequence of definitions which provide the basis for a primary classification of the directed graphs of homogeneous binary relations defined on finite sets of elements. The fundamental notion employed here is that of *connectivity* [4].

A vertex α of graph G is said to be *connected* to a vertex β in a subgraph $H \subset G$ if, and only if, H contains edges $\overrightarrow{\alpha, \gamma_1}$, $\overrightarrow{\gamma_1, \gamma_2}$, $\ldots$, $\overrightarrow{\gamma_{m-1}, \gamma_m}$ and $\gamma_m=\beta$. It is expedient, in this context, to say that in G, β *is*

[1] Figures in brackets indicate the literature references at the end of this paper.

attainable or accessible from α by means of a directed path of length m *steps*. In a graph G with subgraphs H, K, H is said to be *connected* to K, if some vertex of K is attainable from a vertex of H. H is said to be *strongly connected* to K, if every vertex of K is attainable from a single vertex of H.

We consider next the concept of "cyclic net." A subgraph $H \subset G$ is said to be a *cyclic net of order* m if, and only if, H contains $m\,(m>0)$ vertices of G and each vertex of H is connected (in H) to every vertex of H. A cyclic net H of order m in graph G is said to be *simple* if, and only if, no proper subgraph $K \subset H$ is a cyclic net. A simple cyclic net will also be called a *cycle*. A *cyclic traverse* is said to exist from a vertex α to a vertex β in graph G if, and only if, β is accessible from α by means of a directed path with at least one path-vertex contained in a cycle of G. An edge in G is said to be *cyclic* if it is contained in some cycle of G and is otherwise *acyclic*.

A cyclic net H of order m in graph G is said to be *maximal in* G if, and only if, every cyclic net in G is a subgraph of H or contains no vertex in common with H. A cyclic net H of order m in graph G is said to be *closed in* G if, and only if, H is maximal in G and every vertex of G attainable from any vertex in H is contained in H. A cyclic net H of order m in graph G is said to be *universal* if for some positive integer $\hat{q}$ every vertex of H is accessible in $\hat{q}$ steps from some (fixed) vertex α in H. A cyclic net which is universal will be called a *universal net*; a universal net is clearly not necessarily maximal in G. A cyclic net H is thus universal if, and only if, there is a positive integer q_0 such that for all $q \geqslant q_0$, each vertex of H is attainable from every vertex of H in q steps.

Two properties of universal nets which are of interest in the sequel are the following [9]. A cyclic net is universal if, and only if, the greatest common divisor of the orders of all cycles contained therein is unity. A cyclic net is then at once simple and universal if, and only if, it is of unit order.

3. Idempotence

For Boolean relation matrices of finite order, it is clear that idempotence is intrinsically connected with the issue of convergence of matrix powers (cf. [1, 3, 7, 9]). Thus, if R^* is in some sense the unique limit matrix in the sequence of powers of a relation matrix R, R^* is idempotent.

For the purpose of this paper, we employ the following notation: R, S square Boolean relation matrices of order n; *union*, $R \cup S$; *intersection*, $R \cap S$; *inclusion*, $R \leqslant S$; *identity*, $R=S$, for $R \leqslant S$ and $S \leqslant R$; *proper inclusion*, $R<S$, for $R \leqslant S$ and $R \neq S$; *relative product or matrix product*, RS; *converse* (or *transpose*), $\breve{R}$; *negation* (or *complement*), $\bar{R}$. This is also the scheme of notation employed in connection with binary relations and rectangular Boolean relation matrices.

In the following, a Boolean relation matrix R will be said to be *convergent* in its powers if, and only if, there exists in the sequence $\{R^k; k=1,2,\ldots\}$ a power R^m such that $R^m = R^{m+1}$. A relation matrix R will be said to be *oscillatory* or *periodic* in its powers if, and only if, there exists in the sequence $\{R^k; k=1,2,\ldots\}$ a power R^m such that $R^m = R^{m+p}$ where p is the smallest integer for which this holds and $p>1$. Any matrix R' which appears infinitely often in the sequence of powers of a Boolean relation matrix R will be called a *limit matrix* of R.

For Boolean relation matrices of finite order, the following two results are immediate [9]. First, a relation matrix is finitely either convergent or oscillatory in its powers. Second, a convergent relation matrix R of order $n(n \geqslant 2)$ converges to one of the following: (i) The *null matrix* $\Lambda_n = \|r_{ij}\|$, $r_{ij}=0$ for all i, j; (ii) the *universal matrix* $V_n = \|r_{ij}\|$, $r_{ij}=1$ for all i, j; (iii) some idempotent matrix R^* such that $\Lambda_n < R^* < V_n$.

It may be shown that there always exist for any Boolean relation matrix R of order n (periodic or convergent) three idempotent relation matrices which are functions of the powers of R. Let $[R]$ denote the set of limit matrices in the sequence of powers of R and let $p \leqslant n$ denote the cardinal number or period of $[R]$. The elements R^*_α $(\alpha=1,\ldots,p)$ of $[R]$ constitute a group under the operation of Boolean matrix multiplication with some distinguished one of these (say) R^*_1 the group identity [9]. The identity element R^*_1 is clearly idempotent. Let R^{**} denote the union of the group elements, $R^{**} = \bigcup_{\alpha=1}^{p} R^*_\alpha$. R^{**} is idempotent. If R is convergent, then $[R]$ is evidently a unit set. Finally, the *transitive closure* (or "ancestral") of R,

$$R_* = \bigcup_{h=0}^{\infty} R^h = \bigcup_{h=0}^{n-1} R^h$$

is idempotent; here $R^0 \equiv I_n$, the identity relation matrix of order n. The three idempotent relation matrices satisfy the relation $R^*_1 \leqslant R^{**} \leqslant R_*$ and are in general distinct (see Corollary 2a). If R is convergent, there are at most two distinct matrices. If $I_n \leqslant R$ so that R is convergent, the three matrices coincide identically. The preceding idempotent forms are of interest in connection with the recursive model of an information system (sec. 5).

The conditions for the convergence of a Boolean relation matrix of order $n(n \geqslant 2)$ are stated below in a lemma which is given without proof; this result constitutes a reformulation of Theorem 2 of [9].

LEMMA 1: *A Boolean relation matrix* R *of order* n(n $\geqslant$ 2) *converges in its powers if, and only if, either the graph* G(R) *contains no cycles or every maximal cyclic net in* G(R) *is universal*.

In a Boolean relation matrix $R = \|r_{ij}\|$ $(i, j=1, \ldots, n)$ a matrix element r_{ij} will be said to be *1-convergent* (*0-convergent*) in the powers of R if $r^*_{ij(\alpha)}=1$ (0) in all limit matrices R^*_α $(\alpha=1, \ldots, p)$ of R. An element r_{ij} is said to be *oscillatory* in the powers of R if, and only if, it is neither 1-convergent nor 0-convergent. In the general case, a matrix element may be 1-convergent or 0-convergent even though R is periodic in its powers. For convenience, an element r_{ij} which is not 0-convergent will be called *recurrent* in the powers of R. The following lemmas provide a detailed account of these properties; since these results are essentially obvious we omit proofs.

LEMMA 2a: *An element* r_{ij} *of* R (i, j = *1*, . . . , n) *is 0-convergent in the powers of* R *if, and only if, there exists no cyclic traverse from vertex* α_i *to vertex* α_j *in* G(R).

LEMMA 2b: *An element* r_{ij} *of* R (i, j = *1*, . . . , n) *is recurrent in the powers of* R *if, and only if, there exist one or more cyclic traverses from vertex* α_i *to vertex* α_j *in* G(R).

LEMMA 2c: *A recurrent element* r_{ij} *of* R *is 1-convergent in the powers of* R *if, and only if, for some positive integer* q_0 *the cyclic traverses from vertex* α_i *to vertex* α_j *in* G(R) *exhibit paths of all integral lengths* $q \geqslant q_0$.

It may be remarked that an element r_{ij} of R is 1-convergent in the powers of R only if the cyclic traverses from vertex α_i to vertex α_j exhibit paths of all integer lengths $q \leqslant k \pmod{k}$ where k is some cycle order of $G(R)$.

THEOREM 1: *Let* R *be a convergent Boolean relation matrix of order* n. *An element* r_{ij} *of* R *is 1-convergent (0-convergent) if, and only if, there exists a (no) cyclic traverse from vertex* α_i *to vertex* α_j *in the graph* G(R).

PROOF: The result follows directly from Lemmas 1, 2a, and 2c, since 1-convergence coincides with recurrence in this case.

COROLLARY: *Let* R *be a convergent Boolean relation matrix of order* n *with limit matrix* $R^* \neq \Lambda_n$. R^* *is an equivalence relation matrix if, and only if, all vertices and edges of* G(R) *are contained in cycles.*

PROOF: R^* is an equivalence relation matrix if, and only if, it is at once reflexive, symmetric, and transitive. The result then follows directly from the Theorem, since symmetry of R^* requires the mutual inaccessibility of vertices in distinct maximal cyclic nets of $G(R)$.

We consider next a graph-theoretic characterization of the structure of idempotent relation matrices. For this purpose we complete the primary classification of vertices of directed graphs begun earlier. A vertex α in a graph G will be said to be *circuit free in G* if, and only if, α is contained in no cycle of G. We proceed to a classification of circuit free vertices.

A subgraph $J \subset G$ is a *null net of order m* in graph G if, and only if, J contains $m (m > 0)$ vertices circuit free in G and all vertices of J are connected to the same vertices of G. Obviously, no vertex in a null net J is connected to any vertex of J. A null net J of order m in graph G is said to be *maximal in G* if, and only if, every null net in G is a subgraph of J or contains no vertex in common with J. A maximal cyclic net or a maximal null net of a graph G will be called a *maximal net* of G. Every vertex of a graph G is, therefore, uniquely assigned to some maximal net of G.

A maximal null net $N \subset G$ is said to be an *articulated net in G* if, and only if, the following condition obtains: for every vertex α accessible from N in G there exists a cyclic traverse from N to α in G. In the graph of a strong ordering relation matrix, for example, the terminal vertex constitutes an articulated net and all other vertices are in null nets of unit order which are not articulated. The concept of maximal null net is evidently important in the formal representation of "redundancy" in graphs or networks.

A subgraph $H \subset G$ is said to be *complete in G* (or a *complete subgraph* of G) if, and only if, any vertex of G which is attainable from any vertex of H is attainable in one step. The *null graph* $G(\Lambda_n)$ is a complete articulated net; the *universal graph* $G(V_n)$ is the unique complete universal net of order n.

A graph G will be called a *permanent graph* if, and only if, any maximal net of G is either a complete articulated net or a complete universal net. A permanent graph G is then, evidently, complete.

THEOREM 2: *A Boolean relation matrix* R *is idempotent if, and only if,* G(R) *is a permanent graph.*

PROOF: *Sufficiency.* Assume $G(R)$ is a permanent graph. We show that R is transitive ($R^2 \leqslant R$) and compact ($R \leqslant R^2$). $G(R)$ is complete so that $R^2 \leqslant R$ follows directly. Thus, any pair of vertices α_i, α_j of $G(R)$ are either connected in one step or not connected at all. To establish $R \leqslant R^2$, we show that if any α_i is connected to an α_j, then it is connected to it in two and indeed any integral number of steps. Assume that α_i is connected to α_j. Then either there exists a cyclic traverse from α_i to α_j in $G(R)$ in which one or both vertices are in universal nets and, consequently (since $G(R)$ is complete), in cycles of unit order, or else both vertices are in articulated nets with an intervening vertex of a universal net between them. In both cases, two-step connection (in fact k-step connection, $k \geqslant 2$) obtains. $R^2 \leqslant R$ and R is idempotent.

Necessity. Assume R is idempotent. Since $R^2 \leqslant R$, $G(R)$ is complete. By Lemma 1, any maximal cyclic net of $G(R)$ is a complete universal net. Consider any maximal null net N which is connected to any vertex α of a maximal (null or cyclic) net L in $G(R)$. By Theorem 1, there exists a cyclic traverse from N to α of L so that N is a complete articulated net. $G(R)$ is, therefore, a permanent graph.

Theorem 2, in effect, states that in the graph $G(R)$ of an idempotent relation matrix R a vertex α is connected to a vertex β if, and only if, α is connected to β in one step and there exists a cyclic traverse from α to β which passes through a universal net.

COROLLARY 2a: *For any Boolean relation matrix* R *of order* n, *the group union* R^{**} *and the transitive closure* R_* *coincide if, and only if, every vertex of* G(R) *is in a cycle.*

PROOF: The idempotent matrices R^{**} and R_* both have permanent graphs. $G(R_*)$ contains all vertex accessibilities of $G(R)$ (and also slings or auto-accessibilities for all vertices), while $G(R^{**})$ exhibits all recurrent vertex accessibilities of $G(R)$. Thus, $R_* = R^{**}$ if, and only if, all vertex accessibilities of $G(R)$ are recurrent and, in fact coincide with 1-convergences of elements of R_* and R^{**}. But this is so if, and only if, every vertex of $G(R)$ is in a cycle.

COROLLARY 2b: *For any Boolean relation matrix* R *of order* n, *there exists a unique permanent graph in the sequence of graphs* $\{G(R^k); k = 1, 2, \ldots\}$.

PROOF: The sequence, in fact, contains at most $(n-1)^2 + 1$ distinct graphs (cf. [9]). The result follows from the Theorem and the unique idempotence of the identity element in any group and thus in the group of limit matrices of R.

To any subgraph H of G there corresponds a unique *complementary graph* $\overline{H}$ (called the *complement of* H) consisting of all edges in G which are not in H. The unique subgraph of G containing all the vertices of H and no other, and all edges of $\overline{H}$ with both endpoints in H will be called the *proper complementary graph* $\overline{H}'$ (or the *proper complement* of H). For any Boolean relation matrix R, the proper complement of $G(R)$ is identified with the graph $G(\overline{R})$ of the negation $\overline{R}$ of R.

A set of maximal nets in a graph G is said to be a *chain of maximal nets* if, and only if, any two distinct maximal nets in the set are strongly connected. A chain of maximal nets $H_j (j=1, \ldots, d)$ may then be written in the form $H_1 \succ H_2 \succ \ldots \succ H_d$, where the relation $\succ$ is asymmetric, transitive, and connected in the usual relation-theoretic sense.

We have the following result on complementary permanent graphs.

THEOREM 3: *A graph* G(R) *and its complement are both permanent graphs if, and only if,* G(R) *consists of a chain of complete maximal nets which are alternately universal and articulated.*

PROOF: *Sufficiency.* We show that the condition of the Theorem holds for $G(\overline{R})$ if it holds for $G(R)$. Let M be a maximal null (cyclic) net of $G(R)$. Then the proper complement $\overline{M}'$ of M is a maximal cyclic (null) net in $G(\overline{R})$. For if not, by augmenting $\overline{M}'$ to form a maximal net $\overline{M}''$, the proper complement of $\overline{M}''$ would be a null (cyclic) net in $G(R)$ properly containing M, which is a contradiction. Moreover, if a maximal net N is strongly connected to a maximal net M in $G(R)$ where $N \neq M$, then $\overline{M}'$ is strongly connected to $\overline{N}'$ in $G(\overline{R})$. The maximal nets of $G(\overline{R})$ are thus complete and $G(\overline{R})$ also satisfies the condition of the Theorem. $G(R)$ and $G(\overline{R})$ are, therefore, permanent graphs.

Necessity. Assume $G(R)$ and its complement to be permanent graphs. We show that any two distinct maximal nets in $G(R)$ ($G(\overline{R})$) which are connected without an intervening maximal net between them, cannot be both universal nets or both articulated nets. There clearly must exist an intervening (maximal) universal net between any two distinct and connected articulated nets in $G(R)$ ($G(\overline{R})$). By virtue of the evident correspondence between maximal nets of $G(R)$ and $G(\overline{R})$, it then follows that an intervening articulated net must exist between any two distinct and connected (maximal) universal nets in $G(R)$ ($G(\overline{R})$). Finally, we show that any two distinct maximal nets in $G(R)$ ($G(R)$) are strongly connected. Assume M and N to be maximal nets, $M \neq N$, neither of which is connected to the other in $G(R)$ ($G(\overline{R})$). Then the proper complements $\overline{M}'$, $\overline{N}'$ are symmetrically connected universal nets in $G(\overline{R})$ ($G(R)$), which is a contradiction. Moreover, if M is connected to N in $G(R)$ ($G(\overline{R})$), M must be strongly connected to N; for if not, $G(R)$ or $G(\overline{R})$ would not be complete, which is again a contradiction. Consequently, $G(R)$ and $G(\overline{R})$ are single chains of complete maximal nets which are alternately universal and articulated.

COROLLARY: *If* G(R) *and its complement are both permanent graphs,* G(R) *is symmetric if, and only if, it is the null graph or the universal graph.*

The following example provides an illustration of Theorem 3. Let ρ denote the relation "less than, or equal to and odd" defined over the set of positive integers $S_n = \{k;\ 1 \leqslant k \leqslant n\}$; thus for $x, y \epsilon S_n$, $x \rho y$ if, and only if, $x < y$, or $x = y$ and is odd. The negation $\overline{\rho}$ of ρ then denotes the relation "greater than, or equal to and even." The relations ρ, $\overline{\rho}$ are at once transitive and compact ($\rho \leqslant \rho^2$, $\overline{\rho} \leqslant \overline{\rho}^2$) so that the corresponding Boolean relation matrices $R, \overline{R}$ are both indempotent. $G(R)$ and $G(\overline{R})$ consequently consist of single chains of maximal nets of unit order which are alternately universal and articulated.

Not all idempotent relation matrices are accessible as limit matrices of nontrivial convergent sequences. An idempotent Boolean relation matrix S is said to have a *proper power primitive* if, and only if, there exists a convergent relation matrix R with limit matrix $R^* = S \neq R$ (cf. [6]).

THEOREM 4: *An idempotent Boolean relation matrix* S *of order* n (n $\geqslant$ 2) *has no proper power primitive if, and only if,* G(S) *contains at least* n $-$ 1 *maximal universal nets, and any path between distinct vertices in* G(S) *contains exactly one acyclic edge.*

PROOF: *Sufficiency.* Assume the condition of the Theorem holds. Consider any convergent relation matrix R with limit matrix $R^* = S$. We show $R = S$. The maximal nets of $G(R)$ and $G(S)$ coincide identically by Theorems 1 and 2. Any acyclic edge in $G(R)$ is in $G(S)$ for any such edge involves at least one vertex in a universal net which is by Theorem 1 in $G(S)$. Conversely, any acyclic edge in $G(S)$ is in $G(R)$. For if an edge $\overrightarrow{\alpha, \beta}$ of $G(S)$ is not in $G(R)$, there must exist an intervening path connecting α to β in $G(R)$. Such a path contains at most one vertex γ for all such vertices must be in null nets if only acyclic paths of unit length exist in $G(S)$. If α, β are both in univeral nets, then $\overrightarrow{\alpha, \gamma}$ and $\overrightarrow{\gamma, \beta}$ are in $G(S)$, which violates the hypothesis. If either α or β is in a null net, with γ there exist two articulated nets, which again violates the hypothesis. Consequently γ cannot exist and the edge $\overrightarrow{\alpha, \beta}$ is in $G(R)$. $G(R)$ and $G(S)$ coincide and $R = S$.

Necessity. Assume S is without proper power primitive. Consider the alternatives: (i) there exists a maximal net in $G(S)$ of order $m \geqslant 2$; (ii) $G(S)$ contains two or more articulated nets; (iii) there exists at least one path in $G(S)$ between distinct vertices involving two or more acyclic edges. If any of the three alternatives were to hold, it would be possible to construct a proper power primitive by adjoining (case (ii) and case (i) for null nets) or by removing (case (iii) and (i) for universal nets) a single edge in $G(S)$. The three alternatives are jointly excluded and lead to the condition of the Theorem. This completes the proof.

Theorem 4, in effect, states that an idempotent Boolean relation matrix S of order n ($n \geqslant 2$) has no proper power primitive if, and only if, S contains an identity matrix of order $n-1$ and the intersection $S \cap \overline{I}_n$ is nilpotent, in fact $(S \cap \overline{I}_n)^2 = \Lambda_n$.

COROLLARY: *Every Boolean relation matrix* R *of order* $n \geq 3$ *with* G(R) *and* G($\overline{R}$) *both permanent graphs has a proper power primitive.*

PROOF: Directly by Theorems 3 and 4 (for $n=2$, the proposition fails).

We now define the graph of the sequence of powers of an arbitrary relation matrix and consider it briefly. The *power graph* $\Phi(R)$ of a Boolean relation matrix R of order n ($n \geq 1$) is given by the finite directed graph with labeled vertices corresponding to the distinct powers of R and with edges existing only between successive powers. The *order* of $\Phi(R)$ is taken to be the cardinal number of vertices in $\Phi(R)$. $\Phi(R)$ clearly constitutes a many-one connected graph which is predominantly one-one with the exception of a distinguished pair of vertices which are both connected in one step to a unique vertex of $\Phi(R)$. The following properties of $\Phi(R)$ are readily verified: (i) $\Phi(R)$ is of order $m \leq (n-1)^2+1$; $m=(n-1)^2+1$ if, and only if, $G(R)$ is a universal net of order $n (n \geq 2)$ containing only two cycles, one of order n and the other of order $n-1$ [9]. (ii) $\Phi(R)$ contains a single cycle which is terminal and of an order given by the period of R (for R convergent, the period is unity). (iii) Let S denote the relation matrix corresponding to $\Phi(R)$; then the idempotent power of S has no proper power primitive if, and only if, the order of $\Phi(R)$ exceeds the period of R by at most unity (cf. Theorems 2 and 4). (iv) Consider the sequence of power graphs defined recursively by $\Phi^k(R) = \Phi(\Phi^{k-1}(R))$ for $k \geq 2$; then the order of $\Phi^w(R)$, where $w=(n-1)^2+1$, coincides with the period of R (cf. (ii) above).

In the succeeding sections of the paper we treat two applications of the present results on idempotence.

4. Application in a Process Formulation of Consistent Orderings

As a simple application of some of the previous results, we briefly consider a stationary Boolean process formulation of rank orderings. In this approach, the particular properties of consistent order are taken to be absent (or not fully present) at the outset of the process, and are sequentially generated so as to be exhibited in complete form only at equilibrium term of the process. An ordering process or computation of this type might, for example, represent some aspects of the "process of weighing or balancing alternatives" in a problem of preference rankings (for a formally related approach to orderings, cf. [2]).

For present purposes, we select two types of consistent ordering, weak ordering and quasi-ordering. In a weak ordering ρ, the binary relation ρ is required to be at once reflexive, transitive, and connected. Let R denote the weak ordering relation matrix of order n corresponding to ρ. R then satisfies the following conditions: (i) $I_n \leq R$, (ii) $R^2 \leq R$, (iii) $\overline{I}_n \leq R \cup \breve{R}$. R is clearly idempotent and the following identity obtains: (iv) $R \cup \breve{R} = V_n$. The identity states the complete "order-comparability" of all elements of the domain and range of ρ. In a quasi-ordering, comparability is not imposed as the binary relation satisfies only the two requirements of reflexivity and transitivity; the corresponding quasi-ordering matrix R is again idempotent. We exclude the case of a strong ordering ρ (ρ at once irreflexive, connected, and transitive) in the present formulation for the obvious reason that any strong ordering matrix R is nilpotent.

A finite sequence of successive powers of a Boolean relation matrix R of order n ($n \geq 2$) will be said to be a *weak ordering (quasi-ordering) process with generator or primitive* R if, and only if, all powers are distinct, except for the ultimate and penultimate powers which coincide and are weak ordering (quasi-ordering) relation matrices. A weak ordering process or a quasi-ordering process is called *nontrivial* if the sequence contains at least three powers, and is otherwise termed *trivial.*

It is clear that examples of nontrivial weak ordering and quasi-ordering processes may be readily constructed. For any relation matrix R such that $I_n \cup R \neq R_*$, the sequence $\{(I_n \cup R)^k;\ 1 \leq k \leq n\}$ contains a nontrivial quasi-ordering process in which the transitive closure R_* is the terminal power. If in $G(R)$ of the preceding example, one at least of any two distinct vertices is accessible from the other (cf. (iv) above), then the sequence contains a nontrivial weak ordering process.

We have the following result on ordering processes.

PROPOSITION: *For any weak ordering relation matrix* R *of order* $n \geq 3$, *there exists a nontrivial weak ordering process with* R *as terminal power.*

PROOF: The result follows from Theorem 4 by showing that R must have a proper power primitive (for $n=2$, the proposition fails). Assume the weak ordering matrix R has no proper power primitive. Then $G(R)$ contains only cycles of unit order. Consider three distinct vertices α, β, γ and suppose $G(R)$ to contain an edge $\overrightarrow{\alpha, \beta}$. Then one of the following triads must exist in $G(R)$: (i) $\overrightarrow{\alpha,\beta}$, $\overrightarrow{\beta,\gamma}$, $\overrightarrow{\alpha,\gamma}$; (ii) $\overrightarrow{\alpha,\beta}$, $\overrightarrow{\gamma,\beta}$, $\overrightarrow{\gamma,\alpha}$; (iii) $\overrightarrow{\alpha,\beta}$, $\overrightarrow{\gamma,\beta}$, $\overrightarrow{\alpha,\gamma}$. The graph then exhibits a path involving two acyclic edges, which is a contradiction. R therefore has a proper power primitive and is the limit matrix of some nontrivial weak ordering process.

The earlier results (and the observations on power graphs) consequently provide a basis for the design of processes of varying length for the evolution of consistent orderings where the latter, in effect, arise in the attainment of "maximal connectivities." Although we do not consider these further here, ordering processes of a more general character may be introduced as "subrelations" of terms in sequences which converge to limit forms of non-ordering type; as may, for example, be carried through with the aid of Theorem 3 (cf. Corollary, Theorem 4). Moreover, maximal null nets could in ranking problems be formally treated as representations of incommensurability.

5. Application in an Information System Model

We examine, lastly, the structure of a recursive model of a large-scale information system which was first proposed and studied in connection with a particular logical formulation of processes of communication or diffusion of information [8]. The model was originally stated in the formalism of the calculus of relations, but some of its basic properties were investigated, with greater convenience, by graph-theoretic means and associated relation matrix methods. In the earlier study, principal interest attached to the structure of transient information flow, and to algorithms for the computation of "minimum transmission times" in the propagation of information. At this point, we fix attention on issues of equilibrium or steady-state information flow. This leads naturally to an abstract connectivity approach to issues relating to the accessibility of information. In the following, we adhere for the most part to the order of development and terminology of the earlier study; some alterations of notation and language have, however, been made in the interests of clarity and conciseness [8, 10].

We consider a recursive system $\Omega(k)$ defined for all nonnegative integers k, as a composition of three fundamental binary relations, of which two are homogeneous and one nonhomogeneous. Two arbitrary and distinct sets Δ, Σ are assumed to be given, each finite and non-empty. A homogeneous binary relation ρ called the *communication relation* is defined on the elements of Δ. A homogeneous binary relation σ called the *derivation relation* (and also called the *primitive implication relation*) is defined on the elements of Σ; σ is not postulated to be either reflexive or transitive. Finally, a nonhomogeneous binary relation τ called the *assignment relation* is defined on the product set $\Delta \times \Sigma$, with domain and range respectively contained in Δ and Σ.

An element of the field (i.e., the domain or the range) of ρ is called a *communication element or entity*, and $x\rho y$ is read "x communicates with y." An element of the field of σ is called an *information element or entity*, where such entities may be of arbitrary logical or mathematical character; $\alpha\sigma\beta$ is read "β is derived from α" or "α primitively implies β." The relation τ is regarded as an "initial input relation" in the recursive system; for $x\epsilon\Delta, \alpha\epsilon\Sigma, x\tau\alpha$ is read "x is assigned and holds α."

The model structure is based on the perspicuous and well-known principle of computation in certain finite and closed information systems to the effect that *information is produced only by the operations of (initial) assignment, communication, and derivation.* In the present structure, communication and derivation are treated as equivalent or substitutable operations in respect to transmission of information and, in fact, proceed at the same unit rate. The recursive system of interest is completely defined by

$$\Omega(0) = \tau, \tag{1a}$$

$$\Omega(n) = \breve{\rho}\Omega(n-1) \cup \Omega(n-1)\sigma, \tag{1b}$$

$$(n = 1, 2, \ldots),$$

where $\breve{\rho}$ is the converse of ρ. $\Omega(n)$ is thus given as the union of the right and left relative products of $\Omega(n-1)$ by σ and $\breve{\rho}$ respectively, and with domain and range corresponding to that of τ. The relation $\Omega(k)$ is called the *thesaurus relation of order* k ($k \geqslant 0$); for any integer $k, x\epsilon\Delta, \alpha\epsilon\Sigma, x\Omega(k)\alpha$ is read "x holds α at stage k." Thus, (lb) may be regarded as providing a system analogue of *modus ponens*[2] in the form: If $x\Omega(k)\alpha$ and $\alpha\sigma\beta$, then $x\Omega(k+1)\beta$ $(x\epsilon\Delta;\alpha,\beta\epsilon\Sigma)$. In homologous fashion, the basic triadic relation "x communicates α to y" is rendered in the form: If $x\Omega(k)\alpha$ and $x\rho y$, then $y\Omega(k+1)\alpha$ $(x,y\epsilon\Delta;\alpha\epsilon\Sigma)$. The relations ρ and σ may then be regarded as expedient means of treating the fundamental operations of transfer and logical detachment. In a recent study, the relation τ of the present model has been construed as a state-description (in relation matrix form) of an information store.[3] The recursive system, in effect, affords a composite representation of two processes for the propagation of information which abstracts virtually every aspect except that of connectivity.

We turn to the problem of convergence in the system defined by (1a) and (1b). For this purpose, a sequence $\{\Omega(k); k \leqslant 0\}$ will be said to be *convergent* if, and only if, there exists a term $\Omega(p)$ in the sequence such that $\Omega(p) = \Omega(p+1)$. Let Ω_* denote the union $\bigcup_{k=0}^{\infty} \Omega(k)$ of all terms in a sequence (convergent or not), and let Ω^* denote the unique limit term of a convergent sequence. Ω_* and Ω^* are respectively called the *thesaurus relation closure* and the *thesaurus relation limit.*

From (1a) and (1b) we obtain in the powers of $\breve{\rho}$ and σ

$$\Omega(m) = \bigcup_{h=0}^{m} \breve{\rho}^h \tau \sigma^{m-h} \quad (m = 0, 1, 2, \ldots), \tag{2}$$

where $\breve{\rho}^0$, σ^0 are identity relations respectively defined on Δ and Σ. The system (1a), (1b) exhibits redundancy for $m \geqslant 2$ since

$$\Omega(m) = \breve{\rho}\Omega(m-1) \cup \tau\sigma^m$$
$$= \breve{\rho}^m\tau \cup \Omega(m-1)\sigma,$$

and $(\breve{\rho}\Omega(m-1)) \cap (\Omega(m-1)\sigma) \geqslant \bigcup_{h=1}^{m-1} \breve{\rho}^h\tau\sigma^{m-h}$.

If both σ and ρ were at once reflexive and transitive, the thesaurus relation sequence $\{\Omega(k); k \geqslant 0\}$ would converge rapidly. For then by (2), $\Omega^* = \Omega(k) = \breve{\rho}\tau \cup \tau\sigma \cup \breve{\rho}\tau\sigma$ for all $k \geqslant 2$. Moreover, $\Omega_* = \tau \cup \Omega^*$.

The general problem of convergence may be treated with advantage by the device of embedding the original system of recursions in a larger recursive system of

[2] Cf. Paul Hertz "Über Axiomensysteme für beliebige Satzsysteme," Math. Ann. 87 (1922), pp. 246–269.

[3] D. B. Hertz et al., "Research Study of Criteria and Procedures for Evaluating Scientific Information Retrieval Systems," Final Report (Contract NSF C–218), March 1962, prepared for Office of Scientific Information Service, National Science Foundation.

known structure. We proceed to reformulate the original recursive system as a stationary Boolean process (discrete parameter). Let R, S, T be Boolean relation matrices corresponding to ρ, σ, τ and let the sequence of rectangular Boolean matrices $\{W(k); k \geqslant 0\}$ (called the sequence of *information configurations*) correspond to the sequence $\{\Omega(k); k \geqslant 0\}$. Moreover, let W^*, W_* respectively correspond to Ω^* and Ω_*. The original system is then restated as

$$W(0)=T, \tag{1a$'$}$$

$$W(n)=\breve{R}W(n-1) \cup W(n-1)S, \qquad (n=1, 2, \ldots), \tag{1b$'$}$$

and the explicit form of $W(m)$ is given by

$$W(m)= \bigcup_{h=0}^{m} \breve{R}^h T S^{m-h} \qquad (m=0, 1, 2, \ldots). \tag{2$'$}$$

The last system of recursions can be concisely reformulated. Let Q denote the square Boolean relation matrix $\begin{bmatrix} \breve{R} & T \\ \varnothing & S \end{bmatrix}$, where $\varnothing$ denotes a rectangular null matrix. The new recursion constitutes a stationary Boolean process in the powers of Q and yields

$$Q^n=Q^{n-1}Q=\begin{bmatrix} \breve{R}^n & W(n-1) \\ \varnothing & S^n \end{bmatrix} \qquad (n=1, 2, \ldots), \tag{3}$$

where in nonredundant manner $W(n)=\breve{R}W(n-1) \cup TS^n=\breve{R}^nT \cup W(n-1)S$. The convergent behavior of the sequence $\{W(k); k \geqslant 0\}$ is then simply subsumed in the asymptotic behavior of the powers of Q. By Lemma 2c it is clear that a $W(k)$-sequence may converge even though R or S or both are oscillatory in their powers. For example, let the graph $G(Q)$ be specified as follows: (i) $G(\breve{R})$, $G(S)$ are respectively cycles of order r and s such that $(r, s)=1$; (ii) $G(T)$ contains only one edge. The $W(k)$-sequence then converges, and $W(rs-1)$ in Q^{rs} consists exclusively of nonnull entries.

More generally, Lemma 2c provides an essentially complete graph-theoretic characterization of the convergence conditions for $W(k)$-sequences and more, in the present embedding. In fact, for $x \epsilon \Delta$, $\eta \epsilon \Sigma$, $x\Omega^*\eta$ if, and only if, for some positive integer q_0 there exist one or more cyclic traverses from vertex α_x to vertex α_η in $G(Q)$ (involving communication relation or derivation relation cyclic traverses) which exhibit paths of all integral lengths $q \geqslant q_0$. It is worth noting that a single maximal null net of $G(\breve{R})$ may contribute to two or more maximal null nets in $G(Q)$ depending on the structure of the "coupling subgraph" $G(T)$. From (3) we readily conclude by Lemma 1 and Theorem 1 that a $W(k)$-sequence converges if R and S are both convergent. Consequently, if ρ, σ are both drawn from one of the following *classes of relations* the $W(k)$-sequence and the $\Omega(k)$-sequence exhibit convergent behavior (cf. [9]): reflexive, transitive, compact, equivalence, weak ordering, quasi-ordering, serial (strong ordering), hierarchical. From the preceding remark it also follows that the transitive closure Q_* of Q contains the union W_*, whether the $W(k)$-sequence is convergent or not.

We turn to a derivation of the explicit forms of W^* and W_* in the general case. Let $\breve{R}$, S be of dimensions d_r, d_s. Let $\Gamma(\breve{R})$, $\Gamma(S)$ denote the groups of limit matrices of $\breve{R}$ and S and of periods p_r, p_s. Let $\breve{R}^{**}$, S^{**} denote the unions of the elements of $\Gamma(\breve{R})$ and $\Gamma(S)$. Let $\Gamma(Q)$ denote the group of limit matrices of Q of period p_q, where $p_q \leqslant$ l.c.m. (p_r, p_s). An element Q'_α of $\Gamma(Q)$ will be written in the form

$$Q'_\alpha=\begin{bmatrix} \breve{R}'_\alpha W_\alpha \\ \varnothing S'_\alpha \end{bmatrix} \qquad (\alpha=1, \ldots, p_q),$$

where $\breve{R}'_\alpha \epsilon \Gamma(\breve{R})$, $S'_\alpha \epsilon \Gamma(S)$. Let Q^{**} denote the union $\bigcup_{\alpha=1}^{p_q} Q'_\alpha$ and let Q'_1 denote the identity of $\Gamma(Q)$; $\breve{R}'_1$, S'_1 are then the identities of $\Gamma(\breve{R})$, $\Gamma(S)$.

The group identity Q'_1 is idempotent so that

$$W_1=\breve{R}'_1W_1 \cup W_1S'_1. \tag{4}$$

Assume now that the $W(k)$-sequence converges. The idempotent matrix Q^{**} then leads to the relation

$$W^*=\breve{R}^{**}W^* \cup W^*S^{**}. \tag{5}$$

From the equations $QQ^{**}=Q^{**}Q=Q^{**}$ we obtain

$$W^*=\breve{R}W^* \cup TS^{**}, \tag{6a}$$

$$W^*=W^*S \cup \breve{R}^{**}T. \tag{6b}$$

The relations $\breve{R}^kTS^{**} \leqslant W^*$, $\breve{R}^{**}TS^k \leqslant W^*$ for all $k \geqslant 0$ follow from (6a) and (6b) respectively. The inclusion $\breve{R}^{**}TS \cup \breve{R}_*TS^{**} \leqslant W^*$ then follows readily (R_*, S_* are the transitive closures) and the converse inclusion follows directly from (2$'$) and (5) so that

$$W^*=\breve{R}^{**}TS_* \cup \breve{R}_*TS^{**}. \tag{7}$$

If both $\breve{R}$ and S are convergent, we write $\breve{R}^*$ for $\breve{R}^{**}=\breve{R}'_1$ and S^* for $S^{**}=S'_1$ so that (7) becomes

$$W^*=\breve{R}^*TS_* \cup \breve{R}_*TS^*. \tag{8}$$

The expression for W_* follows directly from (8) and is given by

$$W_*=\breve{R}_*TS_*. \tag{9}$$

From (4) it is clear that the $W(k)$-sequence converges if, and only if, $W_1=\breve{R}'_1W_1 \cup W_1S'_1=\breve{R}^{**}TS_* \cup \breve{R}_*TS^{**}$. This condition may be seen to imply the following two conditions for the group unions $\breve{R}^{**}$, S^{**}:

$$\breve{R}^{**}T \leqslant \breve{R}'_1T \cup \hat{\breve{R}}^{**}T\hat{S}, \tag{10a}$$

$$TS^{**} \leqslant TS'_1 \cup \breve{R}TS^{**}, \tag{10b}$$

the condition $W_\alpha = W^*$ can be shown to be equivalent to each of the following:

$$W_\alpha = \breve{R}'_\alpha TS_* \cup \breve{R}_* TS^{**}, \tag{11a}$$

$$W_\alpha = \breve{R}_* TS'_\alpha \cup \breve{R}^{**} TS_*, \qquad (\alpha = 1, \ldots, p_q). \tag{11b}$$

Finally, for convergent $W(k)$-sequences it may be verified that $[\max (d_s, d_r) - 1]^2 + \min (d_s, d_r)$ provides a sharp upper bound for the smallest integer q such that $W(q) = W^*$ (cf. [9]).

The expressions for W^* and W_* in (8) and (9) are of particular interest in the light of the present results on idempotence. The graphs $G(W^*)$ and $G(W_*)$ are by the embedding represented as subgraphs respectively of the permanent graphs $G(Q^{**})$ and $G(Q_*)$ of the composite information process. Theorems 1 and 2 (and Lemmas 1 and 2c as well) consequently provide the basis for a detailed description of the thesaurus relation limit Ω^* and the thesaurus relation closure Ω_*. The graph $G(Q^{**})$ could satisfy the requirements of Theorem 3. The formulas for W^* and W_* algebraically exhibit the components of information accessibility in the form of closure expressions of subsystem connectivity in $\breve{R}_*$ and S_*. By virtue of the presence of $\breve{R}^{**}$, S^{**} in the expression for W^* it is quite evident that the maximal nets of $G(\breve{R})$ and $G(S)$ constitute fundamental units of analysis in the study of the recursive system. This in turn leads to the clear possibility of constructing equivalence classes of communication elements relative to specified maximal nets of the graph of the derivation relation which may be called "information-accessibility equivalence classes." Such constructions could be effected for W_* as well as W^*.

The matrix W^* of a convergent sequence (for given R, T, and S) is viewed as the equilibrium or steady-state information configuration of the recursive system. For many significant information sequences, W^* is identically null since $\breve{R}$ and S are both null convergent, e.g., R, S both strict hierarchies with graphs $G(R)$, $G(S)$ given as trees. In such cases, the closure W_* regarded as the summary computational history of the sequence provides a useful description of the information process. In systems of greater complexity W^* is not identically null (or the limit does not indeed exist) so that the persistence of nonnull information configurations is always mediated through "reverberating circuits" in the form of maximal cyclic nets of $G(R)$ or of $G(S)$.

The organization of information in the present recursive system may be outlined with the aid of (8) and (9), and a supplementary classification of maximal nets. Theorem 4 and the earlier noted "information-accessibility equivalence classes" for W^* and W_* are also relevant here. We consider a "hierarchical" classification of the maximal nets of $G(R)$ and $G(S)$. A maximal net (null or cyclic) of $G(R)$ ($G(S)$) is said to be of the *first level* if no vertex of the net is attainable from any other maximal net of $G(R)$ ($G(S)$). A maximal net of $G(R)$ ($G(S)$) is recursively said to be of *level* k if, and only if, it is attainable in one step from a distinct maximal net of level $k-1$ and is not attainable in one step from any maximal net of lower level. In the general case, $G(R)$ and $G(S)$ are each possibly composed of disjoint and frequently similar (i.e., one-one corresponding) subgraphs. The transmission and distribution of information in the present recursive system for specified R and S are then governed by the mode of assignment of the maximal net "hierarchies" of $G(R)$ to the maximal net "hierarchies" of $G(S)$. It would, consequently, be of interest to consider alternative general system designs for the achievement of preassigned W^* or W_*.

This work was supported in part by the U.S. Patent Office.

6. References

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(Paper 67B4–107)

Publications of the National Bureau of Standards*

Selected Abstracts

The maximum number of zeros in the powers of an undecomposable matrix, M. Marcus and F. May, *Duke Math. J.* **20,** *No. 4, 581–588 (Dec. 1962).*
Let A be an N-square matrix. A is called decomposable if there is a permutation matrix P such that PAP^T is a subdirect sum. The question here is: given an n-square indecomposable matrix with complex entries, how many fixed positions (i, j), $1 \leqslant i, j \leqslant n$ can be zero in every positive integral power of A. This is answered for several classes of matrices, including normal matrices with distinct characteristic roots.

Packing inequalities for circles, P. J. Davis, *Michigan Math. J.* **10,** *25–31 (1963).*
The theory of complex variables is employed to obtain inequalities for non overlapping circles.

Lower bounds to eigenvalues using operator decompositions of the form B*B, N. W. Bazley and D. W. Fox, *Arch. Rat. Mech. Anal.* **10,** *352 (1962).*
A procedure is given for obtaining lower bounds to the eigenvalues of a self-adjoint operator; the difference between the quadratic form of the operator and that of a second operator having a smaller quadratic form is assumed to be of the form $(Bu, Bu)'$ in which B is an operator from one Hilbert space $\mathcal{H}$ to another $\mathcal{H}'$. The eigenvalues and eigenvectors of the smaller quadratic form are also assumed known. The lower bounds are obtained by instructing operators that have quadratic forms intermediate between the smaller and that of the given operator. The construction depends on the use of the operator B^* adjoint to B.

Statistical interpretations, W. J. Youden, *Book, Standard Methods of Chemical Analysis* **IIA,** *Chapt. 16, 319–325 (D. Van Nostrand Co., Princeton, New Jersey, N.J. 1963).*
This paper has been prepared for the chapter on Statistics in the new (6th) edition of Scott's Standard Methods of Chemical Analysis published by Van Nostrand. The first part of the paper goes into detail regarding the steps required to obtain meaningful estimates of the errors in analytical determinations. The remainder takes up a few recurring problems in analytical chemistry along with the appropriate statistical techniques.

Precision of simultaneous measurement procedures, W. A. Thompson, Jr., *J. Am. Stat. Assoc.* **58,** *No. 302, 474–479 (June 1963).*
We consider the problem of measurement under the following conditions: The process of gathering the data is such that on any given item only one opportunity for measurement occurs, but it can be observed simultaneously by several instruments. The items to be measured are variable so that one cannot obtain replicate observations with the same instrument which would show directly the variance of the instrument readings. Procedures are discussed for estimating the precisions of the instruments and the variability of the items being measured.

Determining fastest routes using fixed schedules, B. M. Levin and S. Hedetniemi, *Proc. Conf. American Federation for Information Processing Soc.* **23,** *1–9 (1963).*
In determining routings between locations having no direct connections, most published timetables leave to the patron the job of selecting the most desirable routing. A program has been written which determines routings using speed and cost as criteria. The solution stems from and is related to the Shortest Route Problem. This paper discusses the program and its relationship to the Shortest Route Problem.
The present program is designed for use in routing airmail. The techniques can be used to route passengers, or articles on any type of scheduled transportation trip. It can also be useful in evaluating proposed transportation networks.

Dielectric relaxation in a high temperature dipole lattice, R. Zwanzig, *J. Chem. Phys.* **38,** *No. 11, 2766 (1963).*
The effect of rotational Brownian motion on the dielectric susceptibility of a rigid cubic lattice of permanent point dipoles is calculated. All dipolar interactions are taken into account by means of a high temperature perturbation expansion. The most significant result is that dipolar interactions give rise to additional new relaxation times, increasing dielectric loss at high frequencies.

Some free products of cyclic groups, M. Newman, *Michigan Math. J.* **9,** *369–373 (1962).*
Let $\lambda_n = 2\cos(\pi/n)$. The principal result of this paper is that the group generated by the linear fractional transformations

$$\tau' = -1/(\tau - \lambda_p),\ \tau' = -1/(\tau + \lambda_q)$$

is the free product of a cyclic group of order p and a cyclic group of order q for $p \geqslant 2$, $q \geqslant 3$.

Multipliers of difference sets, M. Newman, *Can. J. Math. 121–124 (Nov. 1961).*
Let D be a difference set with parameters $v, k, \lambda, v > k > \lambda > 0$. Let q be a prime divisor of $n = k - \lambda$. The Hall-Ryser theorem states that if $(q, v) = 1$ and $q > \lambda$ then q is a multiplier of D. In this paper a proof of theorem is given by incidence matrices alone. The restriction $q > \lambda$ is also removed in certain cases, e.g. $n = q$, or $n = 2q$ and $(v, 7) = 1$.

The segemental variation of Blaschke products, G. T. Cargo, *Duke Math. J.* **30,** *No. 1, 143–150 (Mar. 1963).*
We say that a function which is regular in the open unit disk D has finite segmental variation at a point $e^{i\theta}$ provided every line segment connecting $e^{i\theta}$ to a point of D is mapped onto a rectifiable curve by the function.
Seidel and Walsh proved that a univalent function has finite segmental variation almost everywhere; and Tsuji proved that, if a function has a finite Dirichlet integral, it has finite segmental variation off a set whose (logarithmic) capacity is zero.
In this paper, analogous local and global theorems are established for a well-known class of functions whose members are not univalent and do not have finite Dirichlet integrals, namely, (infinite) Blaschke products.

The structure of some subgroups of the modular group, M. Newman, *Illinois J. Math.* **6,** *No. 3, 480–487 (Sept. 1962).*
The groups Γ^m generated by the mth powers of all elements of the 2×2 modular group Γ are studied and their structure determined. The connection with the Burnside problem is discussed.

Conditions for second-order waves in hypo-elasticity, B. Bernstein, *Trans. Soc. Rheology* **VI,** *No. 1, 263–273 (1962).*
The study in hypo-elasticity of second order waves, regarded as propagating singular surfaces of order two in displacement, yields results similar to those in general elasticity theory. The amplitude of a wave must be an eigenvector of a certain matrix, the acoustical tensor, which depends on the stress and direction of propagation, and the product square of the wave speed with the mass density must be the corresponding eigenvalue. Conditions are stated that at least one wave is possible in each direction and that the existence of a longitudinal amplitude imply the possibility of a longitudinal wave. The condition that the acoustical tensor be always symmetric is the same as that a Cauchy-elastic hypo-elastic material possess a strain energy.

The invariance of symmetric functions of singular values, M. Marcus and H. Minc, *Pacific J. Math.* **12,** *No. 1, 327–332 (1962).*

Let M denote the vector space of all $m \times n$ matrices over the complex numbers and let T be a linear map of M into itself. Let $f(A)$ denote the r^{th} elementary symmetric function of the squares of the singular values of A, $A \epsilon M$. We determine completely the structure of the group M_f of all linear T satisfying the invariance condition: $f(T(A)) = f(A)$ for all $A \epsilon M$.

Bounds for cofactors and arithmetic minima of quadratic forms, M. Newman, *J. London Math. Soc.* **38,** *p. 215–217 (Apr. 1963).*
Let A be a real, symmetric $n \times n$ positive definite matrix of determinant d. It is shown that after a suitable integral congruence transformation bounds for the determinants of the leading principal minor matrices of A and for the diagonal elements of A may be derived which depend only on n and d. It is also shown that the Hermite constant γ_n satisfies

$$\gamma_{m+n}^{m+n} \geqslant \gamma_m^m \gamma_n^n .$$

Convergence to normality of powers of a normal random variable, N. Severo and L. J. Montzingo, Jr., *Bull. Intern. Stat. Inst.* **XXXIX,** *Pt. 2, 491–500 (1962).*
Let Y be a normally distributed random variable, and let p be a real number such that y^p is also real, $-\infty \leqslant y \leqslant +\infty$. Let μ_y and σ_y be the mean and standard deviation of Y, respectively. Let $X = Y^p$. It is shown that if p is a positive integer, then X is asymptotically normally distributed with mean μ_x and standard deviation $\sigma_{x'}$ as $\mu_y/\sigma_y \rightarrow \infty$. Otherwise, X is asymptotically normally distributed with mean $\mu_0 = \mu_y^p[1 + 0(\eta^{-2})]$ and standard deviation

$$\sigma_0 = (|p| \mu_y^{p/\eta}) [1 + 0(\eta^{-2})]^{1/2}$$

as $\mu_y/\sigma_y \rightarrow \infty$. When μ_x and σ_x exist, it is shown that $\mu_y/\sigma_y \rightarrow \infty$ is necessary and sufficient to insure that $\mu_x/\sigma_x \rightarrow \infty$.

Non-linear effects in spectra of the iron group, R. E. Trees, *Phys. Rev.* **129,** *No. 3, 1220–1224 (Feb. 1, 1963).*
As known for the last ten years, calculations made with the linear theory agree poorly with experiment in the $3d^6$ configuration of Fe III, compared to the agreement obtained for the $3d^54s$ configuration of that spectrum. It is shown that effects of $3s3d^64s$ on $3s^23d^54s$ satisfy the necessary conditions for linear behavior, consistent with the close agreement obtained in $3d^54s$ of Fe III. Effects of $3s3d^7$ on $3s^23d^6$ are not linear; it is shown that if they are accounted for in the calculations, then close agreement is obtained in the $3d^6$ configuration also (the mean error is ± 65 cm^{-1}). The magnitude of the parameter γ, introduced to define this non-linear interaction, is found to be 40% smaller than expected from Hartree-Fock calculations, but this is a minor inconsistency in view of the known inadequacies of s.c.f. orbitals. It is pointed out that the parameters of the linear theory are changed considerably when the non-linear effects are accounted for.

Normal functions, the Montel property, and interpolation in H$^\infty$, G. T. Cargo, *Michigan Math. J.* **10,** *141–146 (1963).*
This paper exhibits an interrelationship among three apparently unrelated subjects, namely, normal functions, Montel's property, and interpolation with bounded holomorphic functions in the open unit disk. As a by-product, new and somewhat shorter proofs of several known results are obtained.

Copositive and completely positive quadratic forms, *M. Hall,* Jr., and M. Newman, *Proc. Camb. Phil. Soc.* **59,** *329–339 (1963).*
The relation between quadratic forms which are sums of squares of non-negative linear forms, and quadratic forms which are non-negative for non-negative values of the variables is investigated and a duality theorem is proved.

Covers and packings in a family of sets, J. Edmonds, *Bull. Am. Math. Soc.* **LXVIII,** *No. 5, 494–499 (Sept. 1962).*
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J. C. Tanner has discussed the problem of the delay to pedestrians trying to cross a road, and being blocked by traffic. In his treatment it was assumed that the pedestrians arrived at the intersection completely at random and crossed either upon arrival or in a group, after the departure of a car. Tanner also assumed that the criterion for crossing depended on whether the gap seen by the pedestrian exceeded a given one, T_c, and that the gaps were described by a Poisson distribution. Mayne generalized this problem by allowing arbitrary gap and arrival distributions with the gap acceptance criterion given by a critical gap T_c, as in Tanner's treatment. We shall consider the same problem assuming a Poisson arrival distribution for pedestrians, but allowing for a probabilistic gap acceptance criterion, i.e., we let $\alpha(t)$ be the probability that a pedestrian will cross the road if he sees a gap of t in the oncoming traffic. In this paper the imbedded Markoff process generated by this situation will be studied, an expression for the generating function for the number of pedestrians at the intersection will be given, and various other statistics discussed.

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Note on a subgroup of the modular group, M. Newman and J. R. Smart, *Proc. Am. Math. Soc.* **14,** *No. 1, 102–104 (Feb. 1963).*
Let Γ be the 2×2 modular group, Γ^2 the subgroup of Γ generated by the squares of the elements of Γ, Γ^3 the subgroup of Γ generated by the cubes of the elements of Γ. Let G' denote the commutator subgroup of G for any subgroup G of Γ. Then it is shown that

$$(\Gamma^2)' (\Gamma^3)' = \Gamma',$$
$$(\Gamma^2)' \cap (\Gamma^3)' = \Gamma(6)$$

where $\Gamma(6)$ is the principal congruence subgroup of Γ of level 6.

Linear velocity-gradient term in time-dependent pair distribution function, R. E. Nettleton, *J. Chem. Phys.* **38,** *No. 8, 179–1802 (Apr. 1963).*
A Chapman-Enskog type perturbation solution, in gradients of energy, velocity, and density, is proposed for the hierarchy of integro-differential equations obeyed by the time-dependent Ursell-Mayer functions. The hierarchy is then terminated by an *ansatz* relating the three-particle Ursell functions to those of lower order, which yields a closed system for the perturbations to the singlet and pair distribution functions (it is assumed the equilibrium functions are known). Attention is focused on the equation for the two-particle functions, and on the term therein proportional to the divergence of fluid velocity, from which the bulk viscosity may be calculated. An expansion for this term is assumed in powers of scalar products of particle separations and velocities, in which the coefficients in turn are expanded in Sonine polynomials in the velocities. The functions which multiply these polynomials satisfy a system of integral equations, for which a first approximation is written down and the solution discussed. Finally, it is shown that substitution of the solution for the pair distribution function into the hierarchy equation for the singlet distribution yields an infinite number of conditions which may be used to evaluate a corresponding number of parameters in the *ansatz* employed to close the hierarchy.

Other NBS Publications

J. Res. NBS 67A (Phys. and Chem.), No. 5 (Sept.–Oct. 1963) 70 cents.

Reduction of space groups to subgroups by homogeneous strain. H. S. Peiser, J. B. Wachtman, Jr., and R. W. Dickson.

High-temperature thermodynamic functions for zirconium and unsaturated zirconium hydrides. T. B. Douglas.

Heat of oxidation of aqueous sulfur dioxide with gaseous chlorine. W. H. Johnson and J. R. Ambrose.

Thickness of adsorbed polystyrene layers by ellipsometry. R. R. Stromberg, E. Passaglia, and D. J. Tutas.

Melting temperature and change of lamellar thickness with time for bulk polyethylene. J. J. Weeks.

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Resolution of limits of analyzers and oscillatory systems. E. L. R. Corliss.

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Reactions of polyfluorobenzenes with nucleophilic reagents. L. A. Wall, W. J. Pummer, J. E. Fearn, and J. M. Antonucci.

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Limitations on electron beam density of unipotential electron guns at low voltages. J. A. Simpson and C. E. Kuyatt.

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Development of filters for a thermoelectric colorimeter. S. H. Emara and R. P. Teele.

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Radio pulse propagation by a reflection process at the lower ionosphere. J. R. Johler.

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Curves of ground proximity loss for dipole antennas (a digest). L. E. Vogler and J. L. Noble.

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