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THE SEARCH FOR THE GOLDEN SEQUENCE

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This is an expository draft !!!

Preamble

Let us begin by asserting that all structures and changes in nature are fundamentally discrete or discontinuous. Here we use **structure** in biology as a general term to mean an arrangement of parts and hence a structure may be an organism or even a dismembered organism.

By **discrete** we mean that the structure has distinct parts. These parts may of course be interconnected and thereby arranged in some sort of a pattern . What is more these patterns usually characterise the structure perhaps even more so than the nature of the parts. An elephant tusk and a Rhinoceros horn are typed by us as closely allied forms despite the fact that one is of ivory and the other of hair.

The structure may be changed by either rearranging the interconnections between its parts , for example by stretching, or it may change by having new parts(or new connections) added , as with fission when a single cell splits into two . However in both cases the total amount of material in the structure immediately before and after the change is the same. Consequently both of these changes are basically changes in the arrangement of material.

What are the characteristics of a structure ? We have described two ; the arrangement of the constituent parts and the nature of the material in the constituent parts. Each of these parts may now be considered to be another structure and so on and we are faced with the hypothesis that our material is nothing but a hierarchy of patterns. So does a structure have any more substance than pattern? Why should it ? After all our knowledge of structure comes to us only through our sensory organs which are essentially pattern processors .

It follows that all structures change discretely in time.Perhaps the only quantity that might be considered to change continuously in nature is time . However it makes sense where there are an enormous number of parts , such as the molecules in a component part , to use a continuous model as an approximation.

The aphorism of Heracleitus that "Everything flows " might be more aptly replaced by "Time flows everything else **jumps** ". (Actually what he really said is better translated as " All is flux , nothing is stationary ".)

In what follows an attempt is made, using a minimum amount of mathematical jargon , to describe some discrete patterns which have only recently been shown to occur widely in nature . A more detailed treatment may be found in the references.

It is believed that these patterns have great intrinsic beauty and consequently are worth studying for their own sake. Furthermore following D'Arcy Thompson (' On Growth and Form '-abridged edition edited by J.T.Bonner -Cambridge University Press) it is believed that it is primarily because of this beauty that such patterns occur in nature ; particularly associated with the emergence of form in plants (morphogenesis) .

Because these patterns are coded in binary it also comes as no surprise to find that they are used in computers ; namely for the storage of data in memory (hashing codes).

Self Replication

If we look down in plan view upon a sea shell such as the Nautilus we often see a curve which can be approximated by a logarithmic spiral. As the material in the shell is made up of an enormous number of component structures it is plausible to consider it as continuously distributed so this continuous curve is an appropriate model. One of the important characteristics of this spiral is that it is **self similar.** By this we mean the following . If we take any part of the curve , rotate it about the origin and then magnify it we will find that it now corresponds to another part of the curve (in fact this property can be used to uniquely define the logarithmic spiral) . Observing that the straight line and the circle are particular cases of the logarithmic spiral it is seen that there are no curves in the plane , other than the logarithmic spiral , with this self similar property .

This spiral is a very fundamental blueprint for the self replication of continuous structures in nature as all that has to be encoded in the genes is the information necessary to reproduce one basic shape . We observe that the genes are in fact discrete structures themselves and what is more they are arranged on a helix . Unfortunately we do not have the time to untangle the convoluted delights of this structure here.

We now ask the question " when we are concerned with biological discrete structures **what is the discrete blueprint that corresponds to self replication** ? ".

Well firstly we must guess that it must be some sort of a sequence rather than a continuous curve and that parts of it must look like other parts . Furthermore we would hope that the basic code might be simple but simple in an unexpected way . So off we go to inspect simple discrete models in Nature. We can hardly expect to see a discrete structure replicating itself exactly otherwise the number of parts would remain the same. So what we might hope to find is an evolving structure where the number of parts is increasing as we go from one generation to the next but there is some sort of similarity in the arrangement of the parts .

One of the simplest such structures is that concerning the arrangement of the leaves around the stem of a plant (**Phyllotaxis** from the Greek : Phyllon = leaf, taxis = arrangement) . Perhaps the most elementary model of such a process is as follows :

We look down on the stem and regard it as a circle . At each unit of time we allow a leaf to emerge at a constant angle μ from its predecessor and represent this as a point lying on the circumference. What we seek is the 'best ' angle μ which would result in the leaves being 'spread out uniformly ' no matter how many leaves we place .For example if we knew that we were to place only 5 leaves then the best value for μ would be 1/5 revolution. But then if we were to place yet another leaf it would be over the first leaf . We will show that to minimise the ratio of largest gap between leaves compared to the smallest gap we make μ equal to the golden section . But first we must start with-

THE 2-3 GAP THEOREM

Suppose that at time zero a point ,which we will call 0 (our origin) , is placed at some arbitrary position on the circumference of a circle .

At time one unit we place another point, which we call 1, at a clockwise angle of μ from the origin (or an anti clockwise angle of $1-\mu$) . Then at time two units we place a third point, which we call 2, at a clockwise angle of μ from point 1 and so on. Hence at time n , point n+1 is placed at a clockwise angle of μ from point n . In this manner at time n we have placed the n+1 points 0,1,2, . . . ,n .

From now on we adopt the convention that the circle has **unit circumference** (and thus the radius is $1/2\pi$) in which case our angle is most conveniently measured in terms of revolutions .

If μ is rational that is we have $\mu=p/q$ where p and q are integers then for n greater than q-1 we must have coincidence of points . For example if $\mu=1/10$ the point 10 would coincide with point 0 . Now the plant has to arrange an **unknown number of leaves** in such a way that there is as little overlap as possible and thus rational angles are to be avoided . Hence from now on we consider **only irrational values of** μ (that is μ cannot be represented as the ratio of two integers) and this means that it is impossible for two leaf centres to coincide no matter how many points are placed .

GOLDEN HOPS

In **particular** we examine the case where
$\mu = \tau = 1/\phi = \phi - 1 = (\sqrt{5} - 1)/2 =$ **.618 .., where ø is the golden section** .

That is we place points around a circle of unit circumference at each time unit so that two successive points in time are at a distance of .618 . revolutions .

The golden section is referred to as ø from Phideas the name of the famous Greek sculptor who used it in the design of sculptures as well as buildings eg the Parthenon . Alternatively the Greek letter τ for cut is often used .

We will restrict our use of τ to the positive root of $x^2 + x - 1 = 0$. It is readily shown that $\phi = 1.618\ldots = (1 + \sqrt{5})/2$, which is the positive root of $x^2 - x - 1 = 0$. ø can be expressed in terms of the **continued fraction**

$$\phi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}$$

If we chop off the last terms of this fraction we obtain the chops (or convergents)

1/1 , 1+1/1 = **2/1** , 1+1/(1+1/1) = **3/2**, 1+1/(1+1/{1+1/1}) = **5/3** . . .

It is seen that each chop is a ratio of successive members of the **Fibonacci series 1, 1, 2, 3, 5, 8, 13, . . .** which is easily generated by writing the first two terms 1,1 and then each subsequent term is simply the sum of the preceding two terms. It is readily demonstrated that the ratio of these successive pairs approaches ø as the number of terms increases . Furthermore if the ratio of one pair is less than ø (say 3/2) then the ratio of the next pair (5/3) is greater than ø . We also note that 5/3 is closer to ø than 3/2 .

If we consider the difference between ø and any nearest fraction p/q then it can be shown that , in a certain way , as q increases this difference becomes greater with ø than any other angle . Hence not only is ø irrational but , it is more difficult to approximate by rationals than any other number and in this sense the Golden section may be considered to be the **most irrational of all numbers .**

An Example

We will now examine the pattern of gaps formed by two successive points in distance as we go about the circle

Fig 1 shows the arrangement of points when n = 12 and thus the **total number of points is the Fibonacci number 13** and the angle is $\mu = 1/\phi = .618 \ldots$ (we note that this gives exactly the same pattern as when $\mu = \phi$ because whole revolutions are cast away) .

++++++++++ **FIG 1 HERE** ++++++++++

It is no coincidence that the two points on either side of 0 that is point 5 on the clockwise side and point 8 on the other side are each Fibonacci numbers whose sum is 13 , the total number of points .

This is quite general whenever the number of points is a Fibonacci number. Also there is a simple rule to determine the successor of a point as we go around the circle in a clockwise direction. We look at the successor to the origin 0 this is point 5 . Then we add this to our point and thus obtain its successor if the sum does not exceed 13. For example the successor to 2 is 2 + 5 = 7 . However 9+5 = 14 which exceeds 13 . In this case we subtract 13 obtaining 1 . Hence 1 is the successor to 9. Using this simple rule we proceed to generate successors as we go clockwise around the circle thus obtaining the cyclic sequence 0, 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8, 0.

Now consider the size of the gaps between these points . By a gap we will mean the arc between consecutive points on the circle . By enumeration it can be confirmed that **the gaps occur only in two sizes** 0.055721 . . and 0.0901699 revolutions.

If we were to then construct the case where the total number of points was 14 we would find three different gap sizes . It might be thought that these are only particular cases and that if for example we had 500 points then the gap sizes would range over many values . But this is not so !

It can be shown that whenever the angle is the **golden section the two-gap case always occurs when the total number of points (and hence**

gaps) is a Fibonacci number. Otherwise there are exactly three different gap sizes.

Another interesting pattern with the golden section is that the gap sizes are always a power of $1/ø$. In our example, where the number of points is 13, it is easily checked that the larger gap is $\mathbf{1/ø^5}$ = 0.0901699 and the smaller gap is $\mathbf{1/ø^6}$ = 0.055721 . Now F_7 = 13 and it is always found that the exponent of the smaller gap size is always one less than the subscript 7 and the larger gap size is always two less than this subscript .

Furthermore there are always more large gaps that small . In the case of 13 points there are in fact 5 small and 8 large . What is more **5 and 8 are the two predecessors to 13 in the Fibonacci sequence** (as well as being the adjacent points to the origin on our circle) ! It follows that as the number of points becomes very large the ratio of the number of large to small gaps approaches ø as does the ratio of the size of the larger gap to the smaller gap .

In the general case **where μ is other than the Golden section** the same general result holds -the two gap situation occurs when the total number of points is equal to the denominator of a convergent (the above chop obtained from expressing our angle as a continued fraction) and the three gap situation occurs when the total number of points is any other value.

One of the most interesting properties of this golden angle, **1/ø** , relates to the way the points are spread out uniformly . In terms of the original problem of leaf arrangements the best situation occurs when the largest gap is made as small as possible or the smallest gap is made as large as possible ; in each case the overlap of the leaves will be lessened . It can be shown that the golden angle optimises both of these measures simultaneously .

Thus in the general case , where an **unknown number** of radiation collectors in the form of discs are are to be centred around the circumference of a circle, the golden angle will result in the last placed collector being able to receive the most radiation whilst the other collectors have the least amount of overlap.

The two gap result when the angle is the golden section is undoubtedly one of the great treasures of the mathematics of nature as we will now demonstrate.

THE TWO GAP SEQUENCE

From now on we consider only the two-gap situation in which the total number of gaps is a Fibonacci number. In this case the gaps are either large or small and we now ask "**what is the pattern of these gap types** as we go around the circle ? " .

To simplify we label our gaps **l for large and s for small** . Let us now list the sequence of gap types for the case where the **number of points is 13** . As we are on a circle the number of points is of course equal to the number of gaps.

Clockwise from the origin, point 0 , we start with gap (0,5) which is large ; next is gap(5,10) which is also large; then gap(10,2) which is small, . . . and then ; gap(8,0) , our last gap, which is small . Hence our sequence of gap types is

llsllslsllsls

We will call this **the 2-Gap sequence** . Immediately we see that the small gaps always occur singly separating large gaps which sometimes occur singly and sometimes in pairs .

Now let us list the sequence when we have **8 points**

slsllsll

As we can see it is an entirely different sequence . Or is it ?

Well the first and last gaps are certainly different types. If we were to watch these two gaps, then as the total number of points changes from one Fibonacci number to its successor , this pair switches in type. Thus for 8 terms the first is type s and the last is type l and this pair switches when the number of points becomes 13.

But what about the other terms ? Look at them and you will see that they do not change !

Let us write them out again starting with the second term (that is beginning the sequence with the gap (5,10) to the left of point 5) and finishing with the second last gap (3,8) . Then add the first term, gap (0,5) followed by the last gap (8,0) . Alternatively we could simply rearrange our sequence by shifting the first term gap (0,5) so that it is placed between the last two terms, gap (3,8) and gap (8,0) .

In this way we obtain as our **new** sequences

lsllslsl , for 8 points and ,
lsllslsllslls , for 13 points .

As can be seen the subsequence formed from the first 8 terms of the second sequence is identical to the first sequence ! Furthermore if we write the sequence for 5 terms, lslls , it is immediately seen that the sequence for 13 points is obtained by joining the sequence for 5 points to that for 8 points .

This demonstrates that our new sequence has an invariant pattern where our angle is the golden section and the number of points is a Fibonacci number . In terms of our original 2-gap sequence , this means that if the total number of points is Fibonacci then the second gap is always large the third is always small and so on .

THE GOLDEN SEQUENCE

We now list the first 55 terms of our new sequence and uncover further interesting properties

lsl ls (<u>5</u>) lsl (<u>8</u>) lsl ls (<u>13</u>) lsl ls lsl (<u>21</u>)

lsl ls lsl lsl ls (<u>34</u>) lsl ls lsl lsl ls lsl ls lsl (<u>55</u>). . . G

It is noted that we have interspersed Fibonacci numbers in brackets to indicate running totals.

We will refer to **G** , the **infinite** sequence of characters l and s , as **The Golden Sequence** . The transformation from the 2-Gap sequence to the Golden Sequence is done simply as follows . Remove the first term from the 2-Gap sequence and insert it between the last two terms of that sequence . Alternatively we can look at the transformation as follows. Cycle the terms of the 2-Gap sequence so that the last term runs into the first term (in other words look at them the way they are arranged on the circle) . Now start the new sequence with the second term and then switch the last two terms of this new sequence. In both cases it is seen that we have generated the first part of G with the same number of terms as with the 2-Gap sequence .

Bernoulli or Beta sequence

If we now change to numeric values by making the substitutions **l**=1 and s=0, this sequence becomes

1011010110110 . . . . (ß)

We will refer to this sequence , consisting of **units and zeros** , as the **Bernoulli (or Beta** for short) **sequence** . It can be shown that the kth term which we will call **ß(k)** has the value

$$ß(k)= \mathrm{INT}\{ (k+1)ø \} - \mathrm{INT}(k ø) - 1 , \quad k = 1, 2, . . \qquad (*)$$

where **INT (x)** is the integer part of x , that is INT (16.8) = 16 and INT(π) = 3 .

This immediately allows us to calculate the nth term of our Golden Sequence G : if ß(k) is 1 this term is large , if zero this term is small .

We also define { x } as the fractional part of x . That is { 16.8 } =.8, and of course

INT (x) + { x } = x.

Hence defining x_n as the position in revolutions measured clockwise from the origin our problem can be formulated as placing the (n + 1) th point at the distance x_n where

$$x_n = \{ x_{n-1} + \mu \} = \{ n \mu \} .$$

Historical Note

Sequences of this type where ø is replaced by a **general value** μ were first studied by **Johann Bernoulli III** (and hence the name Beta) who was the astronomer grandson of the famous mathematician Johann Bernoulli I. As the calculation of integer parts is very cumbersome arithmetic especially without a computer he developed these techniques in effect to use the symbolic properties of G so as to calculate the arithmetic values ß(k) and hence INT (k ø)

Johann III studied sequences generated by (*) where ø may become any value whatsoever . It is rather tedious to calculate ß(k) directly so he searched for some sort of a pattern that might help him . He observed in 1772 [6a] , but did not prove that, in such sequences , having calculated the first few terms by the above integer parts formula (*) , these terms could then be used to generate a larger number of terms and then this new subsequence could be used to generate an even larger subsequence and so on . Each time the increase in the number of terms would itself be increasing which allows us to generate the sequence much more rapidly rather than with the usual recurrence relations where only the next term in the sequence is added (for example with the Fibonacci sequence we form the next term as the sum of the last pair of terms) . The proof for this result had to wait over a century until 1882 when A . Markov established the result using continued fractions .

The Generating Rule

This technique applied to the Golden Sequence reduces to the following . We start with the first two terms of the Bernoulli sequence which we calculate as follows

$\beta(1) = INT(2\phi) - INT(\phi) - 1$
$= INT(3.236\ldots) - INT(1.61803\ldots) - 1 = 1,$
$\beta(2) = INT(3\phi) - INT(2\phi) - 1$
$= INT(4.8541\ldots) - INT(3.236\ldots) - 1 = 0.$

These as we might hope match with the first two terms of (ß) . Now we convert them into the first two terms of G obtaining ls .

The third term is got by repeating the first term l , then from these three terms lsl the succeeding two terms are got by repeating the first two (ls) . From these five terms , lslls , the next three are obtained by repeating the first three obtaining eight terms . Then the next five terms are obtained from these eight terms by repeating the first five thus obtaining thirteen terms , and so on .

In this way we have chanced upon the **generating rule for the Golden Sequence** : From now on we will use " **thread**" to mean an initial subsequence of G equal in length to a Fibonaccci number .

Starting with any thread of length equal to say F_j , we can generate a thread of length F_{j+1} by simply duplicating the thread made up from our first F_{j-1} terms and adding this to the right of our original thread . This new thread we will call the **stretched thread**. This can be checked against G where for example we can obtain a stretched thread of 55 terms from the thread consisting of the first 34 terms by repeating the thread made up of the first 21 terms.

Another way of looking at this is to generate a sequence of different threads as follows. Suppose that the second last thread is of length F_{j-1} and the last is of length F_j , then the next thread of F_{j+1} terms is got by simply adding (concatenating) the second last thread to the right of the last thread . That is the Fibonacci arithmetic property that the length of the new thread is the sum of the lengths of the last two threads is carried over to string properties : the new thread is simply got by placing the last two threads alongside of each other .

As F_j is proportional to ϕ^j for large values of j , it can be seen that the sequence is very soon generated at an exponential rate .

That is the increase in the number of terms is about ϕ , as the number of terms jumps from one Fibonacci number to its successor. This is in accordance with the well known result that F_{j+1}/F_j approaches ϕ as j increases .

Other Patterns

Many patterns , derived from G , also replicate the above sequence ! This certainly applies if we consider two parts , made up of threads whose lengths are successive Fibonacci numbers . For example if we look at the pattern of large groups lsl , which we replace by a single **L** , and small groups ls which we replace by a single **S** , then the new sequence is

LSLLSLSLLSLLS

And as can be seen the resultant pattern is identical to G . Similarly for the groups lslls and lsl we again regenerate our G .

Generalising this it is seen that if we have any thread of length F_j (which we will call the **original** thread) then as we have already discovered this may be considered to consist of two parts :the L- part which is a thread of length F_{j-1} followed by; a second or S-part which is a thread of length F_{j-2}. Then we can replace every term l in our original thread by an L-part and every term s in this original thread by an S-part, thus obtaining a very much stretched thread. This is the basis for a **very fast technique** for generating G , which we describe in detail in the section on the Fibonacci tree.

We should warn against the false implication from this that we can have at most only two consecutive identical copies of threads in G. For example we see that there are three consecutive occurrences of the thread lsl starting at the 6th term and finishing at the 14th. This follows because if we represent this thread by L and ls by S, then we know that the string LLSL, which always occurs whenever we find a double LL, can be broken up into LLLsl. As sl cannot be the beginning of a thread we have also shown that four consecutive copies of any thread are impossible.

The Length of Runs.

We see that each **s** occurs singly between runs of **l** . Also if we count the number of l's in each run we will note that there are two different run lengths , one of length 1 and the other of length 2 . Thus we obtain another sequence of digits - the **lengths of runs of l's sequence**. Ignoring the first term -

2, 1, 2, 2, ...

By representing the smaller digit by s and the larger by l we get another new sequence . The remarkable result is that this new sequence is also identical to G .

The Length between Occurrences of a Double l

As can be checked by inspection the lengths between the first l in a double to the first l in the next double, form the following sequence

5, 3, 5, 5, 3, 5, 3, 5, . .

This is yet another sequence which matches with the Golden sequence if we replace the 5 by an l and the 3 by an s.

We show this by examining the following two subsequences l ls ls ll and l lsl ls . Now these are the only two patterns in which successive double l's can occur. In the first of these the length is 5 and in the second the length is 3. Furthermore the first is associated with a double occurrence of ls and the second with a single occurrence of ls. But we know that if we replace ls by L we see that through the ' length of runs ' result that there is a correspondence between double and single occurrences of ls as between l and s in G . Hence the result follows.

Symmetry.

Another interesting pattern that we find in G is one of **symmetry**. We have already noted that as the thread length increases from one Fibonacci number to its successor the last two terms switch . Now if we remove these last two terms and form a string (note that it is no longer a thread) of length F_j-2 this reduced string is symmetric . Another way of describing this is to say that this symmetric string is a **palindrome** that is it reads the same backwards as forwards. We list some of these symmetric strings, note their lengths and enclose the central character or characters in bars

l-**s**-l(3=5-2) ls-**ll**-sl (6=8-2)lslls-**l**-sllsl (11=13-2) lsllslsll -s -llslsllsl (19=21-2) lsl lsl sl lsl lsl s-**ll**-s lsl lsl ls lsl lsl (32=34-2)

lsl ls lsl lsl ls lsl ls lsl lsl ls l-s-l l sl lsl sl lsl sl lsl ls lsl lsl (53)

This gives us yet another way of generating G . Knowing where the middle term is located we simply reflect our string about this term taking care to distinguish the case where there are two middle terms rather than one . We then add the pair ls onto the end remembering that the pair switches as we go from one thread to its successor .

Relation to Beatty's Theorem

We consider the pair of sequences B_1 and B_2 where

B_1 = INT (r) , INT (2r), INT (3r) , . .

B_2 = INT (t) , INT (2t), INT (3t) , . .

where r and t are irrational numbers each greater than 1 and

$1 / r + 1 / t = 1$.

Beatty's theorem (see [1] and [6c]) shows that each integer will show up in either B_1 or B_2 but not in both (that is the sequences form a partition of the positive integers) . Each sequence in the pair is called the complement of the other .

Now $r = ø$, and $t = ø^2$, satisfy these conditions as

$1 / ø + 1 / ø^2 = 1$,

and hence it follows that B_1 and B_2 are complementary where

$B_1 = \{ 1, 3, 4, 6, 8, 9, \ldots \} = \{ x_1, x_2, x_3, \ldots)$,
where $x_k = \{ INT(k ø) \}$

$B_2 = \{2, 5, 7, 10, 13, \ldots \} = \{ y_1, y_2, y_3, \ldots)$,
where $y_k = \{ INT(k ø^2) \}$.

As each integer is in either B_1 or B_2 , this gives us an alternative way of deciding whether the jth term in ß is either a unit or zero.

That is

ß(j) = 1 , if j is in B_1 ,
= 0 , if j is in is in B_2 .

Thus for example as 8 is in the first set B_1 , the 8th character of G is an " 1 "and as 13 is in B_2 , the 13th character of G is an "s" .

We also note another interesting pattern in B_1 and B_2 - to reveal this consider the list of integers included in the sets up to and including 10 the fourth term in $\mathbf{B_2}$. It is noted that 8 and 9 are missing from this list and that 8 , the smaller of these , is the next (the fifth) term in B_1 . This pattern always holds .

We see that the difference between the jth terms in B_1 and B_2 is always j. Thus we are able to generate the sets progressively . Start with the terms up to j = 2. As 4 is missing this becomes the next (the third) term in B_1 . To this we add 3 to obtain 5 the third term in B_2 . 6 is now the missing number, and so must be the next term in B_1. To proceed we must keep a list of the missing terms at each step adding to this list a number between the newly formed pair of terms in the sets, and removing the smallest number in the list to become the next term in B_1 .

A Truly Beautiful Result

Suppose we convert the Bernoulli sequence (ß) into a **binary fraction** by putting a point at the beginning (do not call it a decimal point as it not in decimal - the correct name is binary point or in general radix point)

$$\mathbf{x = .1011010110110 \ldots} = \sum_{k=1}^{\infty} \frac{1}{2^{x_k}} \quad \text{where } x_k = INT(k ø)$$

.

From which x is about .709834 .

Davison [4a] has shown that the continued fraction for x is

$$\cfrac{1}{1+\cfrac{1}{2^{F_1}+\cfrac{1}{2^{F_2}+\cfrac{1}{2^{F_3}+\dots}}}}$$

That is, the terms of this continued fraction are

$a_0 = 0,\ a_1 = 1,\ a_2 = 2^{F_1}\ , \dots ,\ a_k = 2^{F_{k-1}}$.

Be warned that the proof although elegant is not easy . Now why do we claim that this relationship is beautiful ? The answer is that it allows us to look upon an infinite sum, involving integer parts as exponents, being transformed into a continued fraction with Fibonaccci numbers as exponents . What is more in both cases the base is two.

On The Self Matching of G

The following is based on the article [6e]. Suppose that we form another sequence from G by simply chopping off the first F_i terms at the beginning . This new **displaced** sequence we will call G_i . When $i = 4$, this sequence is G_4 = lsl <u>sl</u> lsllsl <u>sl</u> lsl <u>sl</u> lsl <u>sl</u> lsllsl <u>sl</u> lsl <u>sl</u> ls . .

If we compare this displaced sequence with the original sequence G, we will see that the first three terms match, then we have a double mismatch (as underlined) that is the original pair ls is switched into sl in the displaced sequence. Then the next 6 terms match followed by another double mismatch and so on .

Now we list the lengths of the matching runs, before each double mismatch , to obtain the following sequence 3, 6, 3, 3, 6, 3, 6, 3, . .

Yes this is a Golden sequence with l replaced by 3 and s replaced by 6 . Furthermore the term 3 is simply $F_5 - 2$, and the term 6 is simply $F_6 - 2$. And this holds in general .

That is a displacement of $\mathbf{F_i}$ terms results in runs of lengths , $F_{i+1} - 2$, and $F_{i+2} - 2$. With l corresponding to the smaller of this pair we now have a Golden sequence of run lengths . This tells us that subsequences of G remote from the origin are similar to the beginning of G , over run lengths which increase with remoteness .

Suppose now that the mean length of a run is M then if we evaluate the mean number of matches in a string of length M + 2 we obtain a measure of the self similarity of G . This measure for a displacement of $\mathbf{F_i}$ terms , can be shown to have the value $1 - 2/\phi^{i}$.

This means that if we compare a long subsequence of G, starting at the term $F_{11} = 89$, with its beginnings , then it will match for over 99% of its terms .

Hofstadter's Extraction Hypothesis

Suppose that we again form our displaced string from G which we will call the chop

G_4 = lsl s l l s l l s l sl lsl sl lsl sl lsllsl sl lsl sl ls . .

G* = lsl * l * s l * s l lsllsl sl lsl sl lsl sl lsllsl sl lsl sl ls . .

G = lsl l s l s l lsllsl sl lsl sl lsl sl lsllsl sl lsl sl ls . .

That is in this case we have chopped off the first $F_4 = 3$ terms

Again starting with the first term in G we try to find a match in G_4 . If we are successful we go on to our next term in G and compare this with the next term in G_4 .However if we find a mismatch then we insert an * in G , jump over that term in G_4 and compare with the next term in G_4 . In this way we construct the string G*. It follows that G* is the same as G_4 with an * covering each term that had a mismatch with G . Now construct the string made up of the elements in G_4 matching with an * in G.This we will call the extracted sequence

for our example this is seen to be slls. .

Now this is found to be G_2 ; that part of G beginning two back from where G was chopped to form G_4 .

What is more this pattern always holds. No matter where we chop the extracted string and not just for Fibonacci numbers the extracted string always corresponds to that part of G beginning two before the chop.

Even more remarkable - this pattern holds for for all Beta sequences and not just the Golden string. Any chop of the Beta sequence results in the extracted sequence matching with another chop starting two back from the original chop.

An Interesting Fibonacci Function

In carrying out our self matching analysis we stumbled upon

$\mathbf{u_{i,k} = kF_{i+1} + [\ km\ ]F_i}$.

A most interesting function indeed. It can be shown that $u_{i,k}$ is the distance to the kth mismatch pair of terms. Furthermore if we define

$w_{i,k} = u_{i,k+1} - u_{i,k} = (1 - b_k)F_{i+1} + b_k F_{i+2}$.

That is whenever $b_k = 1$, $w_{i,k} = F_{i+2}$, and when $b_k = 0$, $w_{i,k} = F_{i+1}$.

Then $w_{i,1}$, $w_{i,2}$, ... , $w_{i,k}$, . . form another Golden sequence. Furthermore it is readily shown that with k fixed and i varying each of $w_{i,k}$ and $u_{i,k}$ satisfy the Fibonacci relationship that each term is the sum of the preceding two.

Penrose Tiles

Now we explain how G can turn up in Penrose tiles . See reference [4] and in more detail [5].

Fig 2 here ***

These tiles are obtained from the rhomb (that is a parallelogram with equal sides) in fig 2, which has each side of length ø and an included angle of 72 degrees . Now place a point on the main diagonal dividing it into two parts; one of length unity, the other of length ø . Join this point to the other two vertices forming two quadrilaterals ; the one associated with the part of length unity has an obtuse angle - call this the **dart** ; the other , associated with the part of length ø we call the **kite**.

In this way we have constructed the two master Penrose tiles which will tile the plane **non periodically** - that is any other tile, no matter what the pattern, can be overlaid by one of these master tiles by a translation and sometimes a rotation . In the usual regular tiling patterns (hexagons , triangles or squares) that we see in most bathrooms a rotation is never necessary . In this case the tile pattern is called periodic .

These Penrose tiles generate some of the most remarkable tile patterns imaginable - full of interest and unexpected delights. As one stares at a large patch of such tiling, more and more relationships between the patterns reveal themselves. Penrose used the expression " unexpected simplicity " to capture this experience and we have found this expression to be most useful in assessing art objects in general.

The essence of these relationships is that they are all based on the pentagon . This observation led to the discovery of the first crystals with fivefold symmetry and also to the discovery of pentagonal patterns in streamlines just before a fluid becomes turbulent.

Now if each tile is marked in a particular way various interesting curves are displayed when the tiles are assembled . One of the most delightful of these is due to the insights of Robert Amman who discovered a way of marking each tile with straight lines such that the assembled pattern displays systems of parallel straight lines .

Fig 3 here ***

Fig 3 shows, how the individual tiles are marked and also, how these become aligned into such a system. Now if one measures the distances(gaps) between these lines they are found to be only of two sizes , large or small - and here is the punchline - the sequence of these gap sizes as one runs across such a system of parallel lines matches G .
It should be emphasised that Peter Pleasants, now at Macquarie University has been one of the founding figures in this area of 5-fold symmetry; in fact he invented the term 'quasicrystal'. His co-authored work [????] shows how the Golden Sequence is a consequence of projections from icosahedral group symmetries in higher dimensions.

THE FIBONACCI TREE, HOFSTADTER and the GOLDEN THREAD

The following is based on reference 6d.

" *If a tree puts forth a* ***new branch after one year*** *, and always* ***rests for a year,*** *producing* ***another new branch only in the year following*** *, and if the same law applies to each branch , - then, in the first year we should have only the principal shoot , in the second - two branches , in the third - three , then 5,8,13* " .

So observed the brilliant Polish mathematician, Hugo Steinhaus in that mathematical treasurehouse " Mathematical Snapshots " , (see reference [8]) from which we have produced Fig 4 .

Figure 4 ** here

This appears to be the first recorded description of a tree with Fibonaccci properties . Steinhaus observed that the numbers of branches as we go up the levels in the tree are the same as the populations in the successive generations of the Fibonacci rabbits .

Fig 5 is a reflection of the tree in Fig 4 showing vertex numbering as first proposed by Hofstadter [2, pp134-137] . We will refer to this as **the Fibonacci Tree** . In contrast what is referred to as a Fibonacci Tree by some other computer scientist is a quite different structure.

Figure 5 ** here

Additionally we have indicated the **type** of each vertex as " **l**" (for large) for those vertices of degree 3 and " **s** " (for small) for those vertices of degree 2.

There are some real world trees that are roughly approximated by the Fibonacci tree (see Stevens [9]) at some stages of their growth. However the main use of the Fibonacci tree is that it is sometimes a useful way of displaying the Fibonacci relationship when it occurs in other biological and computer structures.

Firstly we explore the properties of the Hofstadter numbering and then demonstrate that the sequences of vertex types at every level are golden threads. We observe that :

1. A tree has a unique chain between any pair of vertices hence we define the **level d** of a vertex as the number of edges in the path between that vertex and the origin vertex which we have numbered 1 . Hence vertices 6,7 and 8 are at the level 4 and vertex 1 which we have called our origin is at level 0 .

It is seen that for $d > 0$, the number of vertices at the level d , is the Fibonacci number F_d . The total number of vertices in the tree below level d is F_{d+1} .

These results easily follow from the Fibonacci rabbit problem , when we relate type l vertices to large rabbits and type s vertices to small (immature) rabbits.

2. Hofstadter- Numbering

As with the level-numbering the label on each vertex is increased by unity as we go from left to right across a level. Additionally we increase the number by unity, as we go up a level from the rightmost vertex on one level to the left most vertex on the next level. We start the numbering with the number one for the origin .

Thus we number the extreme left most vertex at level d with $F_{d+1} + 1$ and the right most vertex with F_{d+2}.

In this way for example we number the three vertices at level 4 with 6, 7, 8.

3. Property F - Vertex Generation Rules

A type **s** vertex at level d is always joined to a type **l** vertex at level d +1.

A type **l** vertex at level d is always joined to a pair of vertices at level d +1, in such a way that the left vertex in the pair is always of type **l** , and the right vertex in the pair is always of type s .

These are referred to as generation rules to emphasise that they tell us how vertices are related **between** levels .

We can now view each level as a generation and thus we can say that a type l vertex in generation d is the **father** of the pair of sons, ls , in generation d +1 . Similarly any type s vertex in generation d is the father of a type l vertex in generation d + 1.

It is seen that these rules correspond to those for the rabbit problem except that here the order is important.

It follows from Property F that, **across a level**, a type s vertex must occur singly and a type l vertex may occur singly or as a double but never in longer runs.

We can see that each vertex at a given level contributes exactly one type l vertex to the level above . That is there are **as many type l vertices at level d as there are vertices of both types at level d -1** .

At level $d > 0$, of the total of F_d vertices, F_{d-1} are type l and F_{d-2} are type s, from which it is seen that the type l vertices are more numerous than the type s. That is a fraction of F_{d-1} / F_d are type l vertices and F_{d-2} / F_d are type s vertices. As the level d increases these proportions approach $\phi - 1$ and $2 - \phi$, respectively. That is the ratio of type l to type s vertices at levels high up in the tree approaches ϕ .

We must be careful not to confuse the successor of a vertex with its son . In contrast to the father son relation between levels, the **successor** relation applies to the sequence, usually **within** a level, except when the vertex is rightmost. Thus as we proceed **across a level** we see that the successor of a type s vertex is always a type l vertex. In contrast it is not clear what type follows a type l vertex. All we know is that a double occurrence of a type l is always followed by a type s.

G - The Hofstadter Recursive Tree

We now investigate the properties of the type sequence through the following recursive representation of a tree which Hofstadter defined in [2]

Fig 6 ** here

Henceforth we will refer to this structure as R for recursive. It follows that the Fibonacci tree is simply , root -- s -- R .

The tree R is seen to be made up of a left tree, which is simply a copy of itself shifted up one level and a right tree which is another copy shifted up two levels.

This means that we can view the Fibonacci tree as follows . From level 3 on we can see a left tree, which is simply a copy of the Fibonacci tree above level 1 shifted up one level and a right tree, which is another copy of the Fibonacci tree above level 0 shifted up two levels.

It follows that when we observe the sequence of vertex types, at any level after level 2, in the Fibonacci tree we see a left part which is identical to the complete sequence of vertex types one level below, together with a right part which is identical to the complete sequence of vertex types two levels below. Proceeding in this way we eventually prove that

The type sequence at each level is a Golden thread, that is at level d we see the thread of length F_d .

It follows that the type sequence at each level simply uncovers more of the infinite Golden sequence.

Hofstadter observed that the father of the Hofstadter numbered vertex j is simply

INT((j + 1)/ø) , j > 1. (This is proved in [6d]).

Thus as we go from level d - 1 to level d , the thread is increased in length by the fraction F_d / F_{d-1} which approaches d high up the tree .

It is also seen that if we are at level d, for d > 3, and we trace ancestors down the tree to level d =3, which is simply the string "ls", then, all of the vertices in the leftmost thread of length F_{d-1} have the common ancestor "l" and all of the vertices in the rightmost thread of length F_{d-2} have the common ancestor "s".

This is the basis for the proof of our **fast method** for generating threads - supposing we have generated the F_d vertices at level d , then we simply replace every type l vertex by the thread of length F_{d-2} , and every type s vertex by the thread of length F_{d-2} .

Knowing that $(F_{d-1})^2 + (F_{d-2})^2 = F_{2d-3}$, it is seen that the fractional increase in the length of our Golden thread $S(F_d)$ with this fast technique is F_{2d-3} / F_d, which at the top of the tree approaches $ø^d / ø^3 = .236..(1.618..)^d$

••••••••••••••••••

Thus it is seen that any partition of a thread, into two threads equal in length to successive Fibonacci numbers, generates a stretched Golden thread, if we replace the longer thread by "l" and the shorter by "s".

For example using this fast method with an original thread of length $F_5 = 5$, we see that our stretched thread is of length $2^2 + 3^2 = 13 = F_7$. Stretching again we see that $8^2+5^2 = 89 = F_{11}$. And again $34^2 + 55^2 = 4181 = F_{19}$.

We note that the last of these stretch ratios, namely $4181/89 = 46.9$, is very close to $.236(1.618)^{11}$ as claimed.

Consider now a fragment of a **Golden string of length j characters** then within this string (we cannot call it a thread as the number of characters is not Fibonacci)

a) the number of type l vertices is $INT((j + 1)/ø)$.

b) the number of type s vertices is $INT\{([j / d] + 1)/ø\}$.

For j Fibonacci these reduce to Fibonacci numbers.

Hofstadter [2] also observed that if H(n) is the father of the vertex numbered n then this satisfies the following recurrence relationship

$$\mathbf{H(n) = 0, \quad n=0,}$$
$$\mathbf{= n - H(H(n-1)), \quad n \geq 1.}$$

That is as $n = H(H(n-1)) + H(n)$, the label on a vertex is equal to the sum of the label of its father together with that of the grandfather of its predecessor.

The Zeckendorf Representation

This is simply the representation of a number as the sum of distinct Fibonacci numbers. As $F_1 = F_2 = 1$, for convenience we never use F_1. Thus 16 can be represented $F_7 + F_4 = 13 + 3$, $F_6 + F_5 + F_4 = 8 + 5 + 3$, $F_7 + F_3 + F_2 = 13 + 2 + 1$, $F_6 + F_5 + F_3 + F_2 = 8 + 5 + 2 + 1$.

The first involves a minimal number of terms, the last involves a maximal number of terms. There are no other representations for the minimal or maximal and this holds for all numbers. In other cases there will be several different representations. Thus the second and third are two different representations where we use three terms .

It will be noted that the **maximal** always ends with an F_3 or an F_2 . Now write down the maximal Zeckendorf representations for the **integers** 1, 2, 3, 4, . . . These are F_2, F_3, F_3+F_2, F_4+F_2, . . . From this it is seen that the **sequence of last terms** of each of these representations is $F_2F_3F_2F_2F_3$ . . Thus we have the pretty result that this is a **Golden sequence** in which the l is replaced by F_2 and the s is replaced by F_3 .

On the Ancestory of a Male Bee

The Fibonacci tree is also relevant to tracing the antecedents of a honeybee as we will now show.

There is only a single Queen bee to each hive and she alone is capable of laying eggs . Before laying these eggs the Queen mates with a single male(who blissfully expires on completion of the conjugal act). However the Queen does not fertilise all eggs used in breeding. There is a very strict gender rule: those she fertilised always become female and the remaining unfertilised eggs always hatch into males. That is a male has only one parent, the Queen, whereas a female has both a mother and the happily deceased father . Thus after some time all progeny in the hive have the same parents .

With "l" for lady representing the female (queen) and "s" representing the male it is seen that our Fibonacci tree represents the ancestors of a single bee, placed at vertex 2 if it is a male, or vertex 3 if it is a female. In either case all the vertices above this bee apply to queens and their consorts. Thus the vertex generation rules (Property F) represents the bee to parent relationship . But note that the parent to son relationship for rabbits in this tree has to be reversed for our bees; a type l vertex at level d is now the daughter of the parents l-s at level d +1, a type s vertex at level d is the parthogenetic(begot without a father) son of the single mother at level d +1.

We also note the initial bee and its parent or parents all come from the same hive. All other mothers are queens from different hives. Each male of course comes from the same hive as its queen.

Thus **starting with a single male** bee at level 1 we see that level 2 represents the single parent, the queen from the bee's hive . The parents of this female in level 3 are of course a male and female (a royal pair from another hive) and hence are the grandparents. In this way we see that there are three ancestors (grandparents) in the fourth level, five ancestors in the fifth level and so on. Of course in the case of the bees the order of the ancestors within a particular level has no biological significance.

Chaos and Fractals

It is of some topical interest to note that the mapping
$x_n = \{ x_{n-1} + \mu \} = \{ n \mu \}$,
is both chaotic and fractal .

We will offer the following simplistic definition of **chaos** in terms of what it is not. A non chaotic process may be characterised as follows. Suppose we start the process with two points close together and observe their fate as time increases. As time increases it will be seen that they steadily separate. What is more if we were to start with the points even closer together then this separation is found to be decreased .

For example consider a current of in which the velocity is small. If we place two tiny corks initially close together we observed that separate after a certain time interval . We now repeat the experiment this time halving the initial distance between the corks and we would find that when we observe the corks after the same time interval they are even closer together . On the other hand if the velocity is increased until turbulence results those original points will be found to separate from each other in a very haphazard way. Halving their initial separation in this case could result in an even more irregular separation - we are observing chaos.

Back to our problem and let us look at what happens if we consider two processes simultaneously . That is we take two initial points $x_0 = \mu$ and $x_0' = \mu'$ and then iterate the two processes according to the above map ; one with a jump of μ and the other with a jump of μ'. It is found that after a large number of steps the corresponding points x_n and x_n' will appear to be distributed haphazardly around the circle (don't forget that the x is measured as a fraction of a revolution).

A simplistic definition of a **fractal** is that it is a shape or pattern made up of components each of which is but a shrunken copy of the main shape. Consider the situation where we have placed three points 0, 1, 2 which corresponds to the two gap sequence sll. Now look at the large gap between points 2 and 1 and observe what happens as we place extra points around the circle such that another two fall in this gap.That is if we place another 5 points(8 in total) the original large gap has the points 7 and 4 added forming the sequence of points 2,7,4,1 . The old large gap is now made up of three new gaps. What is more this new gap sequence is sll.

That is if we were to stretch this old gap around the circle with the points 2 and 1 coinciding with the origin, then we would see that the other two points(7 and 4) would overlap the points 1 and 2 that is the new sequence in this old gap is but a shrunken copy of the original three point gap arrangement!

We have demonstrated that the pattern of new gaps within a large gap, as we place new points, is an exact copy of the gaps sequence within the full circle at an earlier stage. It can be shown that this result holds in general with two gap partitioning (see Tony van Ravenstein reference [6i] .). So in this way we see that our Golden hops produce fractal patterns.

Conclusion

As is the logarithmic spiral a blueprint for the replication of continuous structures , such as occur in sea shells , the Golden threads appears to be a fundamental and ubiquitous blueprint in nature for the replication of discrete structures such as occur for example with the placement of leaves around a stem.

Golden threads hide themselves in all sorts of natural structures as well as in artifacts such as the Penrose tiles and computer storage schemes.

The threads will probably be found associated with any structure where collectors , whose number is unknown, are to be arranged around a circle, in such a way as to maximise radiation collected by them . This follows because it can be shown that for large numbers of points, the minimum gap between them is largest and the maximum gap is smallest when the successive angle of placement is equal to the golden section .

Because of this association with radiation collection it is not inconceivable that we might find extra terrestrial evidence of such structures. In that case , on the basis of its simplicity , it might be appropriate to transmit golden threads as a dot dash radio signal in the hope that a message of beauty is received from us instead of our weapons of war.

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" **Godel , Escher, Bach** - An Eternal Golden Braid ". (Vintage Books , 1980)

This is basically a metaphysical fugue on minds and machines in the spirit of Lewis Carrol. Bach is in the book because he has composed music (fugues) that refer back to themselves, Escher the Dutch artist because he has produced etchings with subjects that refer back to themselves such as his pair of hands drawing each other, and Godel is there because he has produced mathematics that refers back to itself (his famous theorem in effect constructs a sentence within arithmetic which claims " this sentence can't be proved " . This will show once and for all that no system of rules entirely within arithmetic can capture all the truths of arithmetic).

This concept of self reference is a thread that runs throughout the book and Hofstadter makes us dizzy with his delightfully convoluted examples. It is brilliant , exhilarating, ingenious , entertaining , amusing and educational; one of the most significant intellectual achievements of the 20th century . Although the reader is being led towards a theory of the mind the lasting value of this book will be of the sort now attributed to D'Arcy Thompson's " On Growth and Form ". No research group was ever directly associated with that book but the ideas and insights that Thompson communicated as a classicist-scientist profoundly influenced the way in which all biologists thought about their models from then on .

He is also the successor to Martin Gardener famous for his column ' Mathematical Games ' in the Scientific American. Hofstadter changed the name of the column to ' Metamagical Themas'. Not only is this a clever anagram of 'Mathematical Games ' but the material in Hofstadter's themas has much more structure than just mathematical games. It has now been collected into a book of the same name .

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7. Roger Penrose, Rouse Ball Professor of Mathematics at Oxford .

Background - Born in 1932 , elected FRS in 1972, father Lionel S. also FRS , as is his brother Oliver also a Professor of Mathematics. Another brother is a professor of psychology. His geneticist father shared his passion for recreational mathematics - in particular they discovered the impossible triangle featured by Escher in his etching " The Waterfall " .

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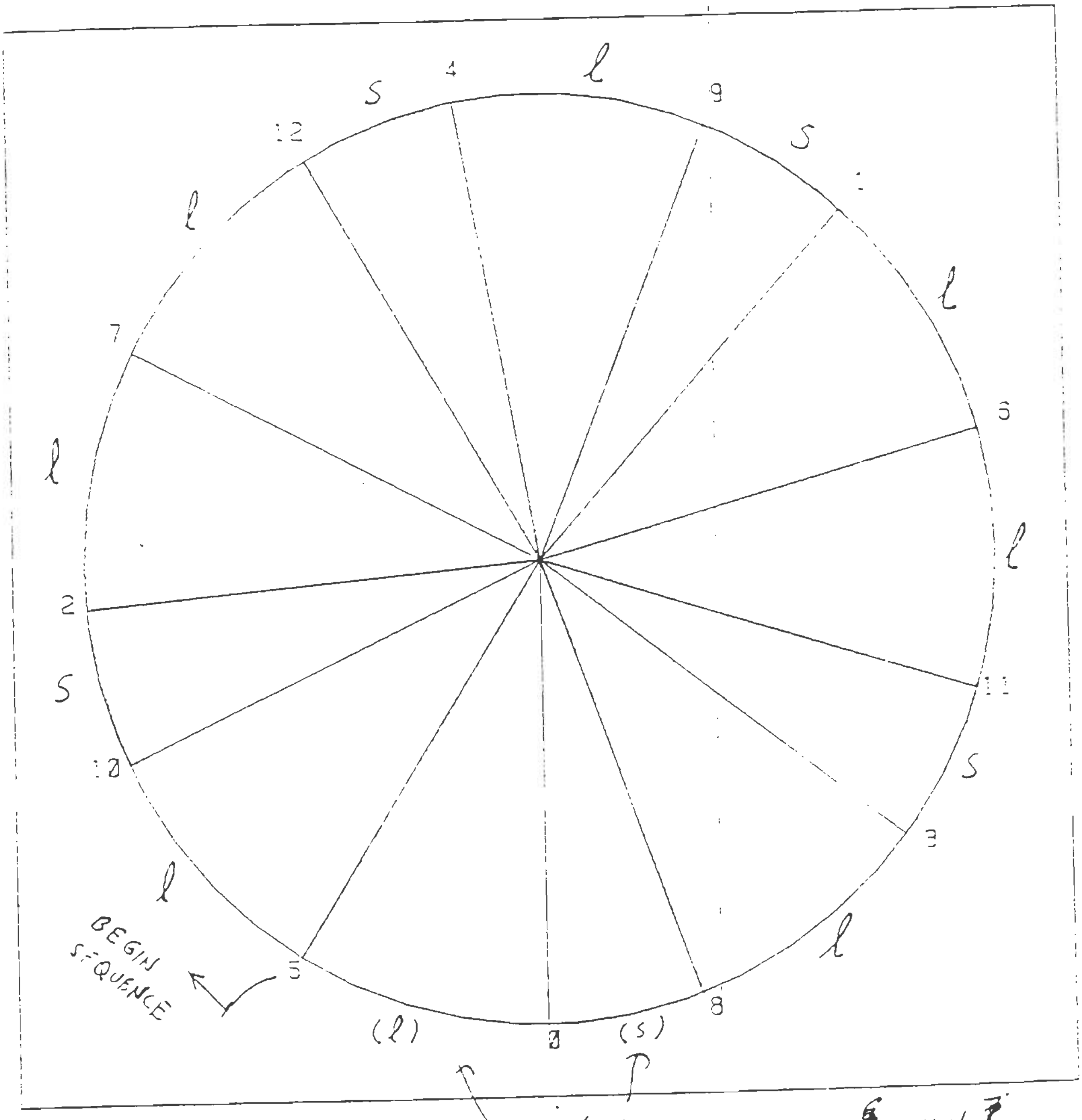


switch

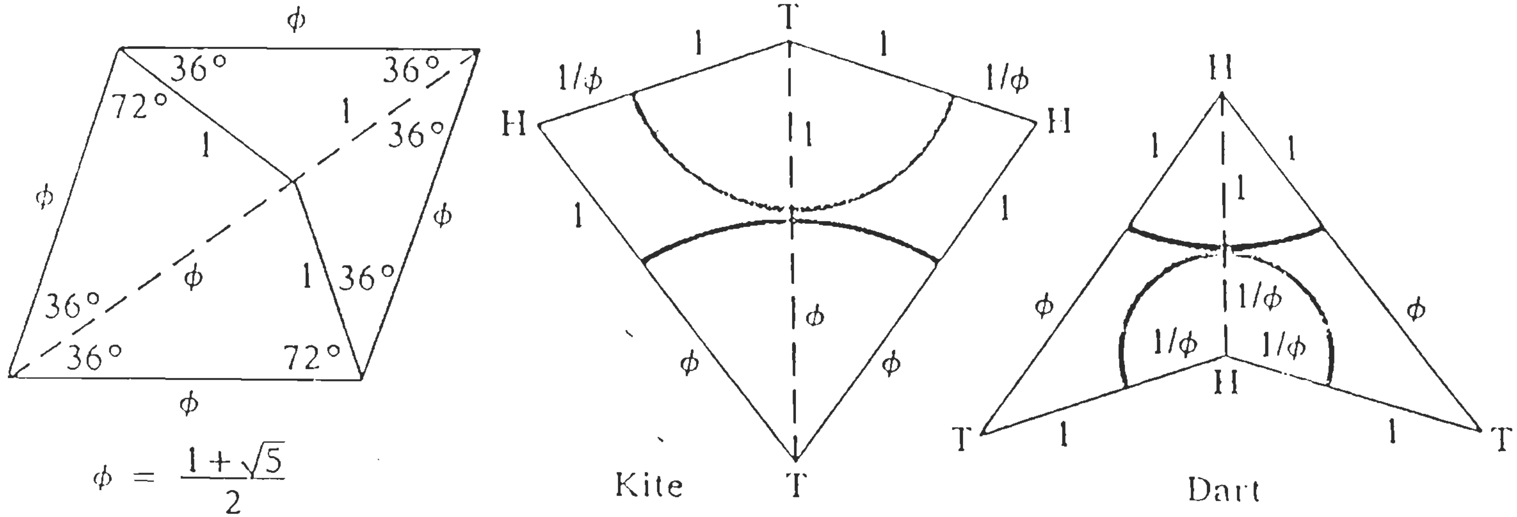
large $(\frac{1}{\tau})^{7}$ small $(\frac{1}{\tau})^{8}$

Figure 1

$N = 13$, $\frac{1}{\tau^5}$ large, $\frac{1}{\tau^6}$ small,
up 5, down 8

$F_8 = N = 21$, $F_7 = 13$, $F_6 = 8$
up 13, down 8

5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8
l s l l s l s l l s l



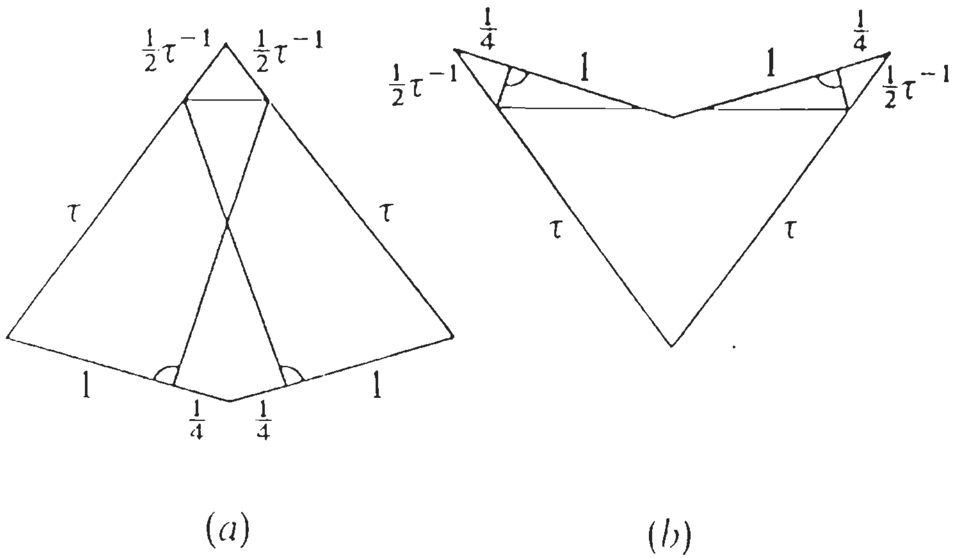
$$\phi = \frac{1+\sqrt{5}}{2}$$

FIG 2

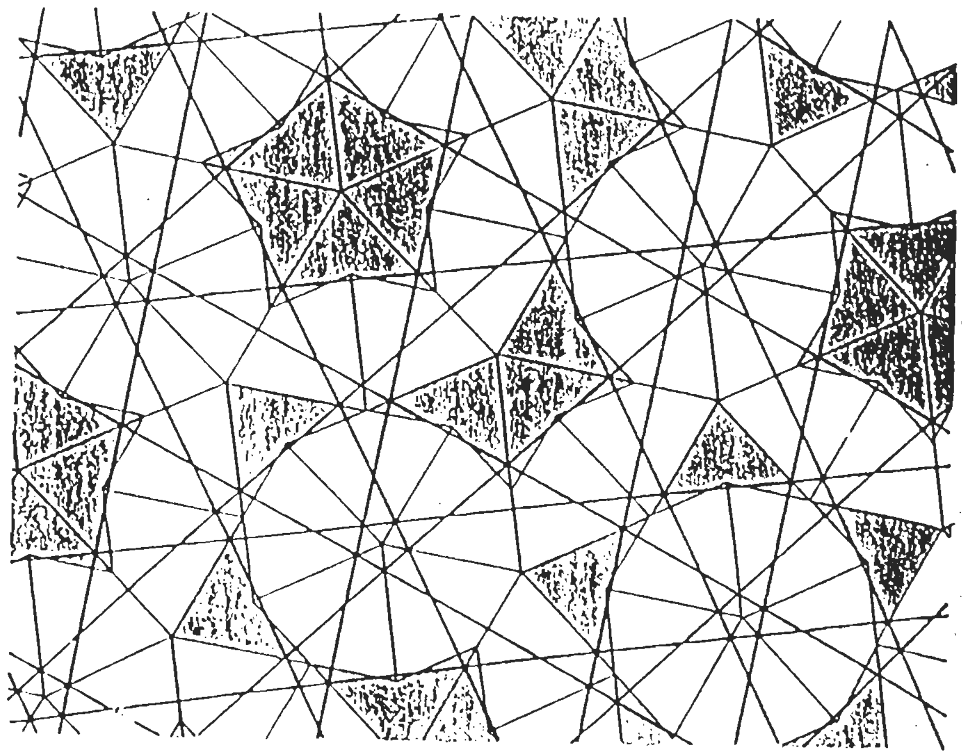
(from Gardner's Book)

from Greenbaum

FIG 3



Ammann bars on the Penrose kite and dart.



A tiling by Penrose kites and darts with the tiles marked by Ammann bars.

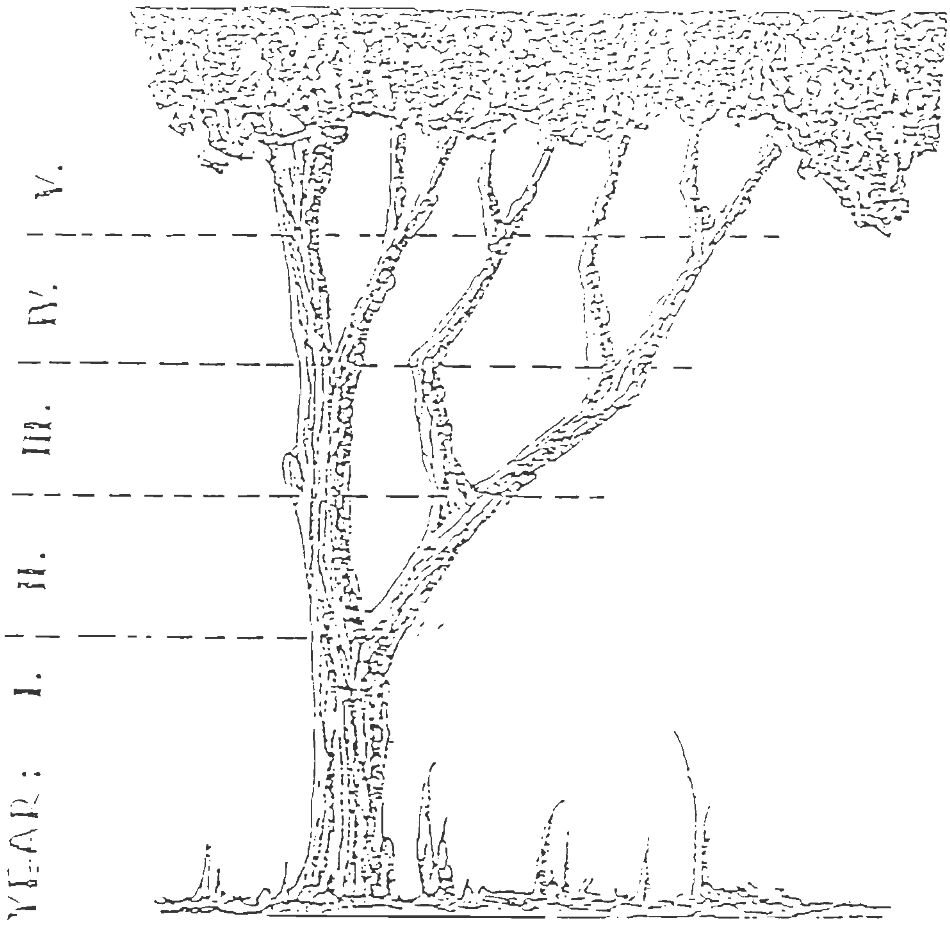


Fig 4. The Steinhaus tree

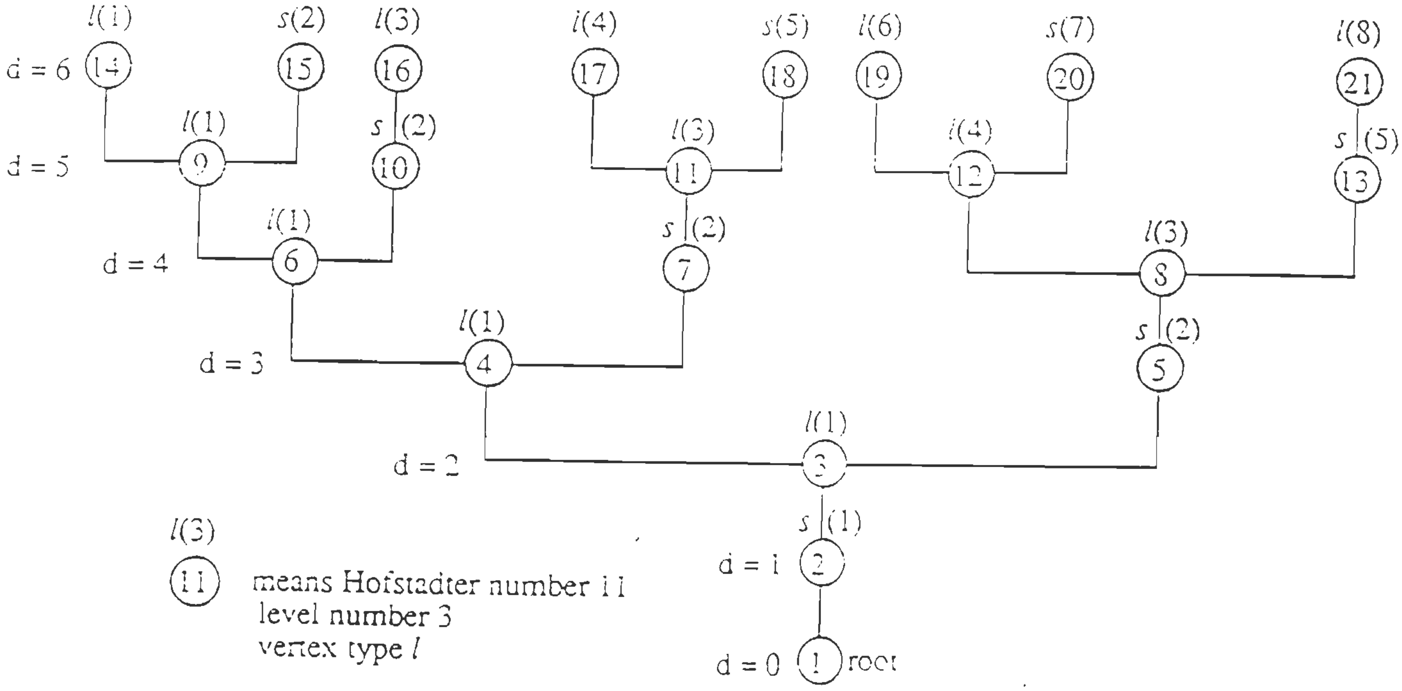


Figure 5. The Fibonacci Tree after Hofstadter

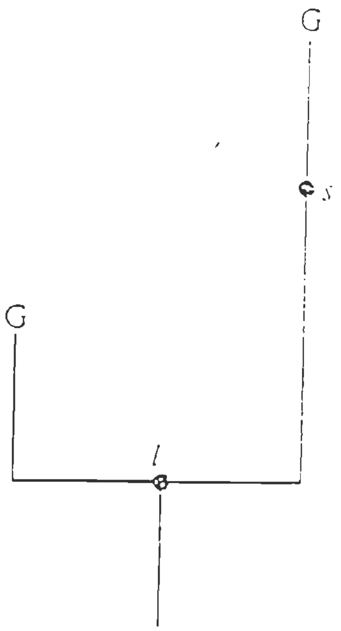


Figure 6. The Hofstadter Recursive Tree G.