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ENUMERATION OF NON-SEPARABLE PLANAR MAPS

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1. Introduction. In **(2)**, Tutte has shown that the number, B_n, of ro... *non-separable planar maps* having n edges is $[2(3n-3)!]/[n!(2n-1$... Rooting was accomplished by designating one edge as the *root*, orienti... and distinguishing between its sides as *left* and *right*. We shall here comp... the number, $B_{n,m}$, of rooted non-separable planar maps having n edges a... such that the face to the left of the root is incident with exactly m ed... which maps will be said to be *of type* $[n, m]$. Following this we shall comp... the number, ${}_rB_{n,m}$, of rooted non-separable planar maps of type $[n, m]$ wh... are invariant under automorphisms induced by a rotation of period r of th... boundary of the face to the left of the root. It will then be possible to calcul... *the number, up to orientation-preserving isomorphisms, of non-separable pla... maps having n edges, where rooting is accomplished by designating a face* (du... *a vertex*) *as the root, and assigning to it an orientation.* This is a first step tow... enumerating the isomorphism classes of planar non-separable maps.

2. We adopt the terminology of Tutte in **(2)**; all *maps* referred to will b... *planar*. The following lemma will be required in our construction.

(2.1) LEMMA. *v is a cut-vertex of a map M if and only if there is a face of M which is incident with v more than once.*

Proof. This is (6.1) of **(2)**, q.v.

(2.2) COROLLARY. *Let M be a map in which an edge E is incident with two distinct faces, F_1 and F_2. Merging E with F_1 and F_2 produces a simply conn... region F; thus the resulting dissection of the 2-sphere is a map, which we den... by M'. The set of cut-vertices of M' is the union of the set of cut-vertices of M and the set of vertices of singularity of the boundary of F in M'.*

In our enumeration we shall not include the loop-map or link-map amon... the non-separable maps. Because of the latter exclusion the faces to the le... and right of the root will be distinct, and will be designated as the *extern... root-face* (or simply the *external face*) and the *internal root-face*, respectiv... of the map; edges in the boundary of the external face will be called *extern... edges*. Exclusion of the loop-map implies that every face is incident with mor... than one edge.

We define, as *formal power series*

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$$B_{n.}(y) = \sum_{m=2}^{\infty} B_{n,m} y^m, \tag{2.3}$$

$$B_{.m}(x) = \sum_{n=2}^{\infty} B_{n,m} x^n, \tag{2.4}$$

$$\begin{aligned} B(x, y) &= \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} B_{n,m} x^n y^m \\ &= \sum_{n=2}^{\infty} x^n B_{n.}(y) = \sum_{m=2}^{\infty} B_{.m}(x)\, y^m. \end{aligned} \tag{2.5}$$

It is clear that $B_{n,m} = 0$ for $n < m$. Hence $\sum_{m=2}^{\infty} B_{n,m}$ is a finite sum, and

$$B(x, 1) = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} B_{n,m} x^n$$

is a well-defined power series.

3. An equation for $B(x, y)$. Let M be a non-separable map of type $[n, m]$, with root-edge A directed from vertex p to vertex q. Let Γ_1 and Γ_2 denote the boundaries of the external and internal root-faces, respectively. The orientation of A induces an orientation in the edges of Γ_1 such that the number of edges directed towards and away from each vertex is one. Let M' be the map obtained from M by erasing A and merging the external and internal root-faces. By (2.2), the cut-vertices of M' are those vertices (excluding p and q) common to Γ_1 and Γ_2. Proceeding from $q = a_0$ along Γ_1 in the direction of the orientation, we label the cut-vertices (if any) of M', $a_1, a_2, \ldots, a_k$, and set $p = a_{k+1}$ ($p = a_1$ if M' is non-separable). Then the arcs not containing A which a_i and a_{i+1} intercept in Γ_1 and Γ_2 either together constitute a simple closed curve (which, by the Jordan curve theorem, separates the 2-sphere into two residual simply connected domains), or coincide as a single link ($i = 0, 1, \ldots, k$). In the former case, by erasing all edges of M in the residual region containing A to obtain the external face, and taking the oriented edge of Γ_1 which emanates from a_i as root, we obtain a rooted non-separable map M_i. Schematically, M has the form shown in Figure 1, where any of the submaps M_i may be degenerate as a link-map (cf. **2**, § 8). Thus M determines uniquely a sequence of $k + 1$ rooted non-separable maps, M_i, of types $[n_i, m_i]$ say (allowing links as type $[1, 2]$) with each of which is associated an index h_i, the number of edges which M_i contributes to the external face of M ($i = 0, 1, \ldots, k$). The following conditions are necessary:

$$\left\{\begin{array}{l} \left.\begin{array}{l} 0 < h_i < m_i \\ 1 - \delta_{m_i,2} < n_i \end{array}\right\} \qquad (i = 0, 1, \ldots, k), \\ \displaystyle\sum_{i=0}^{k} h_i = m - 1, \\ \displaystyle\sum_{i=0}^{k} n_i = n - 1. \end{array}\right. \tag{3.1}$$

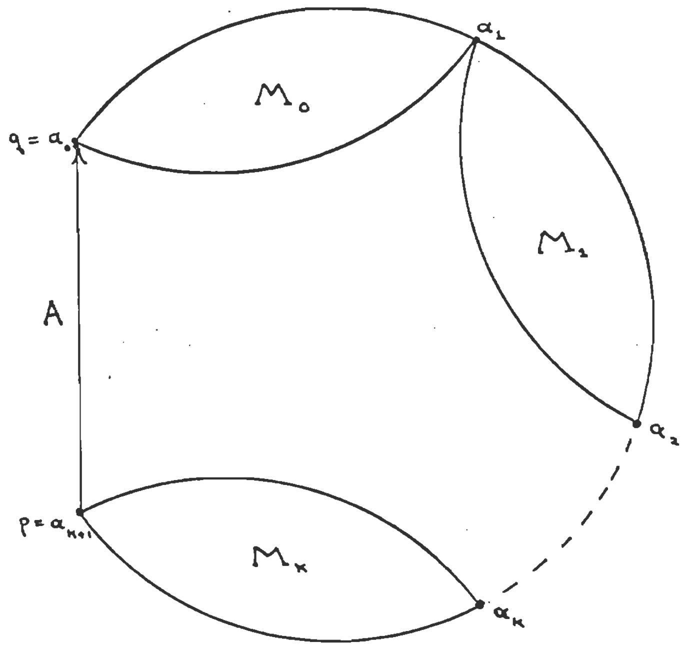

FIGURE 1.

Conversely, any ordered selection of $k+1$ rooted non-separable maps of types $[n_i, m_i]$ (allowing links as type $[1, 2]$) with associated indices h_i $(i = 0, 1, \ldots, k)$ satisfying (3.1) determines a rooted non-separable map. Thus

$$B_{n,m} = \sum_{k=0}^{\infty} \sum \prod_{i=0}^{k} (B_{n_i,m_i} + \delta_{n_i,1}\,\delta_{m_i,2}), \tag{3.2}$$

where the second sum is taken over all ordered sets of positive integers $(n_0, n_1, \ldots, n_k; m_0, m_1, \ldots, m_k; h_0, h_1, \ldots, h_k)$ satisfying conditions (3.1) for each k. (3.2) is equivalent to

$$B(x, y) = xy \sum_{k=0}^{\infty} \left\{ \sum_{m=2}^{\infty} B_{.m}(x)[y + y^2 + \ldots + y^{m-1}] + xy \right\}^{k+1}. \tag{3.3}$$

Multiplying both sides of (3.3) by

$$(1-y)\left\{1 - \sum_{m=2}^{\infty} B_{.m}(x)[y + y^2 + \ldots + y^{m-1}] - xy\right\}$$

we obtain

$$(1-y)\left\{1 - \sum_{m=2}^{\infty} B_{.m}(x)[y + y^2 + \ldots + y^{m-1}] - xy\right\} B(x, y) \tag{3.4}$$
$$= xy(1-y)\left\{\sum_{m=2}^{\infty} B_{.m}(x)[y + y^2 + \ldots + y^{m-1}] + xy\right\}.$$

i.e.

$$\left\{(1-y)(1-xy)-\sum_{m=2}^{\infty} B_{.m}(x)[y-y^m]\right\} B(x,y) \\ = xy\left\{\sum_{m=2}^{\infty} B_{.m}(x)[y-y^m]+xy(1-y)\right\}, \tag{3.5}$$

which we can rewrite as

$$\{(1-y)(1-xy)-yB(x,1)+B(x,y)\}B(x,y) \\ = xy\{yB(x,1)-B(x,y)+xy(1-y)\},$$

or, after rearrangement,

$$[B(x,y)]^2+[1-y+xy^2-yB(x,1)]B(x,y) \\ -xy^2[B(x,1)+x(1-y)]=0. \tag{3.6}$$

This is analogous to equation (3.8) in **(1)**.

Thus far we have defined $B(x, y)$ only as a formal power series, having made no hypothesis as to its representing an analytic function in some neighbourhood. This accounts for the method used in (3.4) and (3.5) for "summing" geometric series.

By comparing coefficients of powers of x in (3.6) we obtain

$$\begin{cases} (1-y)B_{2.}(y) = y^2(1-y), \\ (1-y)B_{3.}(y) = -y^2B_{2.}(y)+y^2B_{2.}(1), \\ (1-y)B_{n.}(y) = -\sum_{s=2}^{n-2} B_{s.}(y)\,B_{n-s.}(y)+y\sum_{s=2}^{n-2} B_{s.}(1)\,B_{n-s.}(y) \\ \qquad -y^2B_{n-1.}(y)+y^2B_{n-1.}(1) \qquad (n>3). \end{cases} \tag{3.7}$$

Multiplying both sides of equations (3.7) by the infinite formal power series $1 + y + y^2 + \ldots$ we see that the series $B_{n.}(y)$, and hence $B(x, y)$, are completely determined by (3.6). Thus the only solution-pair $(\sigma(x, y), \tau(x))$ of the functional equation

$$[\sigma(x,y)]^2+[1-y+xy^2-y\tau(x)]\sigma(x,y)-xy^2[\tau(x)+x(1-y)]=0 \tag{3.8}$$

in non-negative powers respectively of x and y, and x alone, satisfying the conditions

(3.9) $\quad$ *$\sigma(x, 1)$ is well defined and equal to $\tau(x)$*, and

(3.10) $\quad$ *the smallest power of x appearing is x^2*,

is $(B(x, y), B(x, 1))$. For *any* function $\tau(x)$ which is *analytic* at $x = 0$ we can solve (3.8) as a quadratic equation in $\sigma(x, y)$ and obtain two solutions in terms of $\tau(x)$. If one of these solutions satisfies (3.9) and

$$\sigma(x,0)=\sigma_y(x,0)=0 \qquad \text{for all } x, \tag{3.11}$$

then the Taylor series expansion of $\sigma(x, y)$ about $(x, y) = (0, 0)$ will be $B(x, y)$.

4. Solution of equation (3.6). Solving equations (3.7) for $n < 8$ we obtain

$$\begin{aligned}
B_{2.}(y) &= y^2,\\
B_{3.}(y) &= y^2 + y^3,\\
B_{4.}(y) &= 2y^2 + 3y^3 + y^4,\\
B_{5.}(y) &= 6y^2 + 9y^3 + 6y^4 + y^5,\\
B_{6.}(y) &= 22y^2 + 32y^3 + 26y^4 + 10y^5 + y^6,\\
B_{7.}(y) &= 91y^2 + 129y^3 + 112y^4 + 60y^5 + 15y^6 + y^7,
\end{aligned}$$

from which we *conjecture* that

$$B(x, 1) = 2 \sum_{n=2}^{\infty} \frac{(3n-3)!\, x^n}{n!\,(2n-1)!} \quad (4.1)$$

(This statement is known, of course, to be true from **(2)**; however, our argument does not require this fact.) As we shall see in (4.16), (4.1) can be expressed parametrically by

$$x = u(1-u)^2, \quad (4.2)$$

$$B(x, 1) = u^2(1-2u), \quad (4.3)$$

for u sufficiently small, in particular for u satisfying

$$|u| \leqslant \tfrac{1}{8}. \quad (4.4)$$

For convenience we set

$$v = 1 - u, \quad (4.5)$$

$$z = vy. \quad (4.6)$$

Substituting

$$\tau(x) = u^2(1-2u) \quad (4.7)$$

in (3.8) we obtain

$$\sigma^2 + [1 - (1 + u + 2u^2)z + uz^2]\sigma - u^2z^2[(1 - u - u^2) - vz] = 0, \quad (4.8)$$

where $\sigma = \sigma(x[u], y[u, z])$. The discriminant of (4.8), considered as a quadratic equation in σ, is

$$\begin{aligned}
&[1 - (1 + u + 2u^2)z + uz^2]^2 + 4u^2z^2[(1 - u - u^2) - vz]\\
&\qquad = (1 - 2z + z^2)[1 - 2u(1 + 2u)z + u^2z^2].
\end{aligned}$$

By the binomial theorem,

$$\begin{aligned}
[1 - 2u(1+2u)z + u^2z^2]^{1/2} &= [(1 - uz)^2 - 4u^2z]^{1/2} \quad (4.9)\\
&= 1 - uz - 2\sum_{t=1}^{\infty} \frac{(2t-2)!}{t!\,(t-1)!} u^{2t}(1-uz)^{1-2t}z^t.
\end{aligned}$$

This expansion is absolutely convergent for all u, z satisfying

$$(1 - uz)^2 - 4u^2 |z| > 0,$$

in particular, for u, z satisfying (4.4) and

(4.10) $$|z| \leqslant 2,$$

for then

$$(1 - uz)^2 - 4u^2 |z| \geqslant 1 - 2uz + u^2z^2 - \tfrac{1}{8} > \tfrac{7}{8} - 2uz;$$

but

$$|uz| \leqslant \tfrac{1}{4}.$$

The solutions of (4.8) are

(4.11) $$\sigma(x, y) = \tfrac{1}{2}\{-1 + (1 + u + 2u^2)z - uz^2 \pm (1 - z)[1 - 2u(1 + 2u)z + u^2z^2]^{1/2}\}.$$

Setting $y = 1$, i.e. $z = v$, in (4.11) and selecting the positive sign, we obtain

$$\sigma(x, 1) = u^2(1 - 2u).$$

This is true in particular for u satisfying (4.4), in which case it follows that (4.10) is also satisfied. In general, selecting the positive sign in (4.11), we obtain

(4.12) $$\begin{aligned}\sigma(x, y) &= \frac{1}{2}\Bigg\{[-1 + (1 + u + 2u^2)z - uz^2 + (1 - z)(1 - uz)] \\ &\qquad - (1 - z) \sum_{t=1}^{\infty} \frac{(2t - 2)!}{t!\,(t - 1)!} u^{2t}(1 - uz)^{1-2t}z^t\Bigg\} \\ &= u^2z - (1 - z) \sum_{t=1}^{\infty} \sum_{s=0}^{\infty} \frac{(2t - 2)!\,(2t - 2 + s)!}{t!\,(t - 1)!\,s!\,(2t - 2)!} u^{2t+s}z^{t+s} \\ &= \sum_{m=2}^{\infty} \sum_{j=m}^{2m-2} \frac{(j - 2)!}{(j - m + 1)!\,(j - m)!\,(2m - j - 2)!} u^jz^m \\ &\qquad - \sum_{m=2}^{\infty} \sum_{j=m+1}^{2m} \frac{(j - 2)!}{(j - m)!\,(j - m - 1)!\,(2m - j)!} u^jz^m \\ &= \sum_{m=2}^{\infty} \sum_{j=m}^{2m} \frac{(j - 2)!\,m(3m - 2j - 1)}{(j - m + 1)!\,(j - m)!\,(2m - j)!} u^jv^my^m\end{aligned}$$

for u, y sufficiently small, in particular for u satisfying (4.4) and y satisfying

(4.13) $$|y| \leqslant 16/7.$$

We note that conditions (3.9) and (3.11) are fulfilled; hence the power series expansion of $\sigma(x, y)$ about the origin will indeed be $B(x, y)$.

Applying Lagrange's theorem (1, § 5; 4, p. 132) to

$$u = xv^{-2}$$

we obtain

$$(4.14)\qquad v^{-t} = (1-u)^{-t} = 1 + t\sum_{i=1}^{\infty}\frac{x^{i}}{i!}\frac{d^{i-1}}{du^{i-1}}\{(1-u)^{-(2i+t+1)}\}_{u=0}$$

$$= t\sum_{i=0}^{\infty}\frac{(3i+t-1)!}{i!\,(2i+t)!}x^{i}\qquad \text{for } t>0.$$

Hence

$B(x, y)$

$$= \sum_{m=2}^{\infty}\sum_{j=m}^{2m}\sum_{i=0}^{\infty}\frac{m(3m-2j-1)\,(2j-m)\,(j-2)!\,(3i+2j-m-1)!}{(j-m+1)!\,(j-m)!\,(2m-j)!\,i!\,(2i+2j-m)!}x^{j+i}y^{m}$$

$$= \sum_{m=2}^{\infty}\sum_{n=m}^{\infty}\sum_{j=m}^{\min(n,2m)}\frac{m(3m-2j-1)\,(2j-m)\,(j-2)!\,(3n-j-m-1)!}{(n-j)!\,(2n-m)!\,(j-m)!\,(j-m+1)!\,(2m-j)!}x^{n}y^{m}.$$

i.e.

$$(4.15)\qquad B_{n,m} = \frac{m}{(2n-m)!}\times$$

$$\sum_{j=m}^{\min(n,2m)}\frac{(3m-2j-1)\,(2j-m)\,(j-2)!\,(3n-j-m-1)!}{(n-j)!\,(j-m)!\,(j-m+1)!\,(2m-j)!}\qquad (n \geqslant m \geqslant 2).$$

Also

$$(4.16)\qquad B(x,1) = u^{2}(1-2u) = x^{2}v^{-4} - 2x^{3}v^{-6}$$

$$= 4x^{2}\sum_{i=0}^{\infty}\frac{(3i+3)!}{i!\,(2i+4)!}x^{i} - 12x^{3}\sum_{i=0}^{\infty}\frac{(3i+5)!}{i!\,(2i+6)!}x^{i}$$

$$= 2\sum_{n=2}^{\infty}\frac{(3n-3)!}{n!\,(2n-1)!}x^{n}$$

as conjectured in (4.1).

5. Isomorphisms. Let M_i^0, M_i^1, M_i^2, respectively, denote the classes of vertices, edges, and faces of maps M_i $(i = 1, 2)$. An *isomorphism* $f : M_1 \to M_2$ is defined to be a triple of one-to-one mappings, $f^j : M_1^j \to M_2^j$ $(j = 0, 1, 2)$ which preserve incidence relations.

(5.1) LEMMA. *Let $f : M_1 \to M_2$ be an isomorphism between non-separable maps. Let p, q be the end-points of an edge A incident with a face F of M_1, and suppose that the action of f on p, q, A, F is given. Then f is completely determined.*

Proof. By (2.1), as M_i is non-separable, the boundary of every face is a simple polygon $(i = 1, 2)$. Thus f maps edges and vertices of the boundary of F onto edges and vertices of the boundary of $f^2(F)$. This mapping of the boundary of F is clearly determined by the action of f on A and its end-points. Let G be any face adjacent to F along an edge B. Its image is determined as the face $\neq f'(F)$ incident with $f'(B)$ in M_2. Further, the images of the edge B and its end-points in the boundary of G are known. By the connectedness of the graph of M_1, f is determined.

Let M_1, M_2 be *rooted* non-separable maps. An isomorphism $f : M_1 \to M_2$ will be called a *boundary isomorphism* if it carries the external face of M_1 onto the external face of M_2. A boundary isomorphism will be said to be *orientation-preserving* or *-reversing* according to its action on the orientation induced in the boundary of the external face by the root.

An isomorphism of a map onto itself will be called an *automorphism*. It follows immediately that

(5.2) LEMMA. *An orientation-preserving boundary automorphism induces a rotation of the boundary-graph of the external face.*

(5.3) COROLLARY. *An orientation-preserving boundary automorphism is completely determined by a rotation of the boundary-polygon of the external face.*

(5.4) COROLLARY. *The orientation-preserving boundary automorphisms of a non-separable map M of type $[n, m]$ form a group, $\mathfrak{R}[M]$, isomorphic to a subgroup of the rotation group of the m-gon; thus $\mathfrak{R}[M]$ is a cyclic group whose order divides m.*

For any set-theoretic mapping $\phi : A \to A$, $\tilde{\phi}_r$ will represent the mapping which associates with each element a of A the ordered set

$$\tilde{\phi}_r a = \{a, \phi a, \phi^2 a, \ldots, \phi^{r-1} a\} \qquad (r = 1, 2, \ldots). \tag{5.5}$$

6. Rooted non-separable maps of type $[n, m; r]$.

A rooted non-separable map M of type $[n, m]$ is said to be *of type* $[n, m; r]$ if r divides the order of $\mathfrak{R}[M]$; thus $r|m$. We define ${}_rB_{n,m}$ to be the number of rooted non-separable maps of type $[n, m; r]$. The corresponding generating functions are

$${}_rB_{n.}(y) = \sum_{m=2}^{\infty} {}_rB_{n,m} y^m, \tag{6.1}$$

$${}_rB_{.m}(x) = \sum_{n=2}^{\infty} {}_rB_{n,m} x^n, \tag{6.2}$$

$$\begin{aligned} {}_rB(x, y) &= \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} {}_rB_{n,m} x^n y^m \\ &= \sum_{n=2}^{\infty} x^n \, {}_rB_{n.}(y) = \sum_{m=2}^{\infty} {}_rB_{.m}(x) y^m. \end{aligned} \tag{6.3}$$

We have seen in § 4 that $B(x, y)$ defines an analytic function for (x, y) in the rectangle defined by (4.13) and

$$-7^2 / 8^3 < x < 9^2 / 8^3. \tag{6.4}$$

Hence ${}_rB(x, y)$, being majorized by $B(x, y)$, defines a function analytic in the same rectangle.

A rooted non-separable map M of type $[n, m; r]$ will be said to be *of type* $[n, m; r]^*$ if the order of $\mathfrak{R}[M]$ is *exactly* r. Let ${}_rB^*_{n,m}$ be the number of rooted maps of type $[n, m; r]^*$. Then

$$(6.5) \qquad {}_rB_{n,m} = \sum_{k=1}^{\infty} {}_{rk}B^*_{n,m}$$

and hence, by the Möbius Inversion Theorem (**3**, p. 36),

$$(6.6) \qquad {}_rB^*_{n,m} = \sum_{k=1}^{\infty} \mu(k)\, {}_{kr}B_{n,m},$$

where μ is the Möbius function. Clearly the number, up to orientation-preserving isomorphisms, of rooted non-separable maps of type $[n, m; r]^*$ is $(r/m)\, {}_rB^*_{n,m}$. Hence, the number, up to orientation-preserving isomorphisms, of rooted non-separable maps of type $[n, m]$ is

$$(6.7) \qquad \begin{aligned} L_{n,m} &= \sum_{r=1}^{\infty} (r/m)\, {}_rB^*_{n,m} = (1/m) \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \mu(k) r\, {}_{kr}B_{n,m} \\ &= (1/m) \sum_{s|m} \sum_{k|s} \mu(k) \,.\, (s/k)\, {}_sB_{n,m} \\ &= (1/m) \sum_{s|m} \phi(s)\, {}_sB_{n,m}, \end{aligned}$$

where ϕ is the Euler function (**3**, p. 27). Thus, if we define a *face-rooted* (*vertex-rooted*) non-separable map of type $[n, m]$ to be a non-separable map containing n edges, in which one face (vertex) of valency m is designated as the *root* and assigned an orientation, then the number of such maps up to isomorphisms which preserve the root and its orientation is $L_{n,m}$.

7. An equation for ${}_rB(x, y)$. Clearly ${}_1B(x, y) = B(x, y)$.

Assume now that $r > 1$, and let M be a rooted non-separable map of type $[n, m; r]$ with root-edge A directed from vertex p to vertex q. Let F_1, F_2 respectively denote the external and internal root-faces, and Γ_1, Γ_2 their respective boundary-polygons. We shall represent the sets of vertices and edges in Γ_i respectively by $\Gamma_i{}^0$, $\Gamma_i{}^1$ $(i = 1, 2)$. Let f be the generator of $\mathfrak{R}[M]$ which induces a rotation through $w = m/r$ edges of Γ_1 in the direction of the orientation of A. (We shall suppress the superscripts j of the mappings f^j $(j = 0, 1, 2)$.)

We shall show in what follows that M corresponds to a combination of rooted maps chosen from six different classes; and, conversely, that any combination of rooted maps from these classes chosen subject to restrictions to be determined corresponds uniquely to a map of type $[n, m; r]$. The restrictions will be of two types:

(a) specifying the number of maps to be chosen from each class;

(b) requiring that the total number of edges in the maps chosen be equal to n, and that the total number of edges in these maps which correspond to external edges of M be equal to m.

Thus, subject to each set of specifications of type (a), any partition of n and m consistent with restrictions (b) will yield a unique map of type $[n, m; r]$. Each set of specifications of type (a) will therefore correspond to a term in a sum of generating functions which will be set equal to ${}_rB(x, y)$; each term in this sum will be the product of generating functions corresponding to each of the classes from which maps may be chosen, according to the particular set of specifications (a); the variable y will enter into these generating functions according as the corresponding maps contribute to Γ_1.

Representative elements of the six classes are:

Class I. A single edge in Γ_1, corresponding to the generating function xy.

Class II. An element obtained by erasing the root from a non-separable rooted map, and including its remaining external edges as external edges of M. The corresponding generating function is $B(x, y)/xy$.

Class III. An element similar to those of Class II, but not contributing to Γ_1, corresponding to the generating function $B(x, 1)/x$.

Class IV. An element consisting of a rooted non-separable map of type $[n, m; r]$ (allowing the link-map as type $[1, 2; 2]$) but contributing no edges to Γ_1. It corresponds to the generating function ${}_rB(x, 1) + x\delta_{r,2}$.

Class V. An element consisting of a non-separable rooted map of type $[n, m; r]$ (allowing the link-map as type $[1, 2; 2]$) and contributing hr external edges to Γ_1, where $m = rw$, $0 \leqslant h \leqslant w$. The generating function for this class is

$$\sum_{h=1}^{\infty} {}_rB_{.hr}(x)[1 + y^r + y^{2r} + \ldots + y^{(h-1)r}] + x\delta_{r,2}$$
$$= (1 - y^r)^{-1} \sum_{h=1}^{\infty} {}_rB_{.hr}(x)(1 - y^{hr}) + x\delta_{r,2}$$
$$= (1 - y^r)^{-1}[{}_rB(x, 1) - {}_rB(x, y)] + x\delta_{r,2}$$

for y sufficiently small.

Class VI. An element consisting of a rooted non-separable map of type $[n, m]$ which contributes to Γ_1 h of its external edges, beginning at the originating vertex of the root and proceeding along the boundary of its external face in the direction of the orientation induced by the root, and in which a vertex is chosen from among those $m - h - 1$ which remain in the boundary but which are not incident with any edge selected for inclusion in Γ_1 $(0 < h < m - 1)$. The corresponding generating function is

$$\sum_{m=2}^{\infty} B_{.m}(x)[(m - 2)y + (m - 3)y^2 + \ldots + (1)y^{m-2}]$$
$$= \sum_{m=2}^{\infty} B_{.m}(x)[(m - 2)y(1 - y)^{-1} + (y^m - y^2)(1 - y)^{-2}]$$
$$= y(1 - y)^{-1}[B_y(x, 1) - 2B(x, 1)] + (1 - y)^{-2}[B(x, y) - y^2B(x, 1)]$$

for y sufficiently small.

Elements of Classes I, II, III, and VI will never appear alone in our decomposition, but always as a set of r identical copies; thus each edge in the representative element will correspond to r edges in M, and the relevant generating functions will be obtained by replacing x, y in the generating functions for their classes respectively by x^r, y^r.

Consider now the "dissection" M' of the 2-sphere obtained from M by merging the edges of $\tilde{f}_r A$ with F_1 and the faces of $\tilde{f}_r F_2$ to form a single region F; cf. (5.5). M' must fall into one of the following categories:

(1) F is not simply connected, i.e. M' is not a map;

(2) M' is a map, but may have cut-vertices.

Case (1). Suppose F is not simply connected. If the elements of $\tilde{f}_r F_2$ were all distinct, M' would be a map, by (2.2); thus two, and hence all, of the elements of $\tilde{f}_r F_2$ coincide. Proceeding from $q = a_0$ along Γ_1 in the direction of the orientation, let us label the vertices common to Γ_1, Γ_2 as $a_1, a_2, \ldots, a_{k+1} = fp$. We see that M is a combination of r identical copies of one element from each of Classes I and II (allowing the element from Class II to be degenerate, as a vertex-map); thus the generating function corresponding to this case is $x^r y^r J(x^r, y^r)$, where

$$J(x, y) = \frac{B(x, y)}{xy} + 1. \tag{7.1}$$

Case (2). Suppose now that M' is a map. By (2.2) the cut-vertices of M' are the elements of the two sets

$\Theta =$ vertices, excluding those of $\tilde{f}_r p, \tilde{f}_r q$, common in M to Γ_1 and the boundary-graph of some element of $\tilde{f}_r F_2$;

$\Lambda =$ vertices not in Θ common to the boundaries of two or more faces of $\tilde{f}_r F_2$.

In M, beginning at $q = a_0$ and proceeding along Γ_1 in the direction of the orientation, we denote successive cut-vertices (if any) of M' between faces F_1 and F_2 by $a_1, a_2, \ldots, a_j$, where a_j is the last such vertex before fp; similarly, proceeding from $fp = b_0$ along Γ_1 in the opposite direction, we denote successive cut-vertices between faces F_1 and fF_2 by $b_1, b_2, \ldots, b_k$, where b_k is the last such vertex before q. These two sequences may coincide at most in their last vertices a_j and b_k ($j \geqslant 0$, $k \geqslant 0$). Each sequence determines an r-tuple of identical elements of Class II (which may be degenerate); thus the factor $x^r y^r [J(x^r, y^r)]^2$ will appear in the final expression for the generating function in this case.

Case (2*a*). Suppose $\Lambda = \emptyset$. Then the remainder of the map, after exclusion of the portions enumerated above, is of Class V, and the generating function for this sub-case is

$$x^r y^r [J(x^r, y^r)]^2 \{(1 - y^r)^{-1} [{}_rB(x, 1) - {}_rB(x, y)] + x\delta_{r,2}\}.$$

Case (2*b*). Suppose $\Lambda \neq \emptyset$. Proceeding along Γ_2 in M from a_j in the direction of the orientation of A, we denote successive vertices of $\Gamma_2{}^0 \cap f\Gamma_2{}^0$ by

$c_1, c_2, \ldots, c_k$. If $c_1 \neq a_j$, then the region bounded by the arcs (not contai
4) a_jc_1, b_kc_1, and a_jb_k respectively in the boundaries of F_2, fF_2, and F_1 c
sponds to an element of Class VI. The vertices $c_1, c_2, \ldots, c_k$ determin
element of Class III which may be degenerate. If $c_1 = a_j$, the eleme
Class VI is degenerate, while the element of Class III may not be degene
The remainder of M, all cut-vertices having been accounted for, corresp
to an element of Class IV, which may be degenerate. Thus the gener
function for this sub-case is

$$x^r y^r [J(x^r, y^r)]^2 [{}_rB(x, 1) + x\delta_{r,2} + 1][C(x^r, y^r)J(x^r, 1) - 1],$$

where

$$C(x, y) = y(1 - y)^{-1}[B_y(x, 1) - 2B(x, 1)] + (1 - y)^{-2}[B(x, y) - y^2B(x, 1) \tag{7.2}$$

It follows that

$$\begin{aligned}{}_rB(x, y) = {} & x^r y^r [J(x^r, y^r)]^2 \{ [{}_rB(x, 1) + x\delta_{r,2} + 1] C(x^r, y^r) J(x^r, 1 \\ & + (1 - y^r)^{-1} [y^r \, {}_rB(x, 1) - {}_rB(x, y)] - 1\} + x^r y^r J(x\end{aligned} \tag{7.3}$$

i.e.

$$\begin{aligned}& \{1 + x^r y^r (1 - y^r)^{-1} [J(x^r, y^r)]^2\} \, {}_rB(x, y) \\ & = x^r y^r [J(x^r, y^r)]^2 \{ [{}_rB(x, 1) + x\delta_{r,2} + 1] C(x^r, y^r) J(x^r, 1) \\ & \qquad + (1 - y^r)^{-1} y^r \, {}_rB(x, 1) - 1\} + x^r y^r J(x^r,\end{aligned} \tag{7.4}$$

for y sufficiently small.

Cases (1), (2a), and (2b) are represented for $r = 3$ in Figures 2, 3,
respectively.

8. Solution of equation (7.4). We begin by reducing (7.4) to a
convenient form. For ease of computation, we set

$$K(x, y) = 1 - (1 + u + 2u^2)z + uz^2, \tag{8.1}$$

$$L(x, y) = -u^2z^2[(1 - u - u^2) - vz], \tag{8.2}$$

$$D(x, y) = (1 - z)[1 - 2u(1 + 2u)z + u^2z^2]^{1/2}. \tag{8.3}$$

We temporarily drop the arguments from $B(x, y)$, $C(x, y)$, $D(x, y)$, $J($
$K(x, y)$, $L(x, y)$, and set

$$B_1 = B(x, 1), \tag{8.4}$$

$$B_{y1} = B_y(x, 1). \tag{8.5}$$

Then (3.6) and (4.11) (with positive sign) yield

$$B^2 = -KB - L, \tag{8.6}$$

$$B = \tfrac{1}{2}(-K + D). \tag{8.7}$$

538

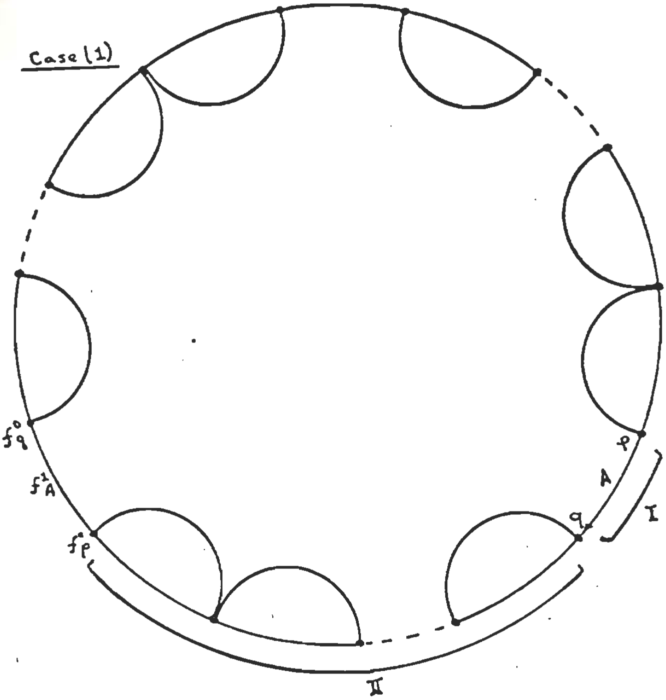

FIGURE 2.

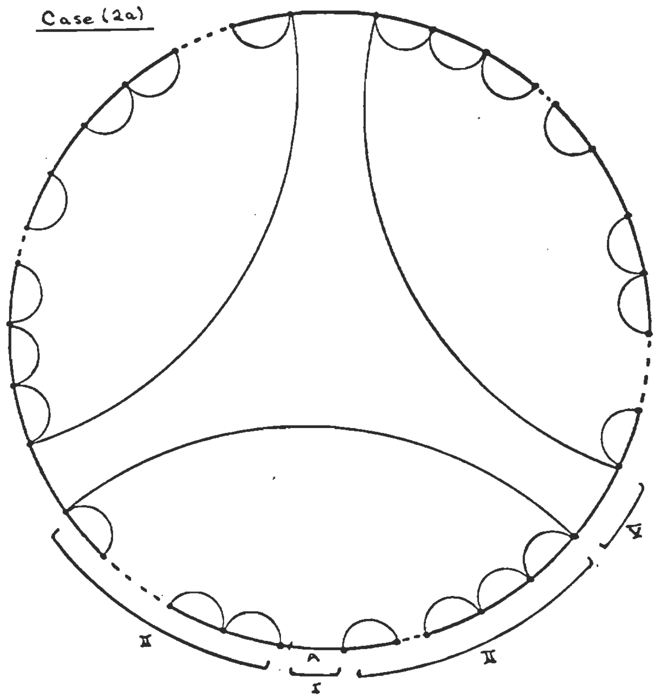

FIGURE 3.

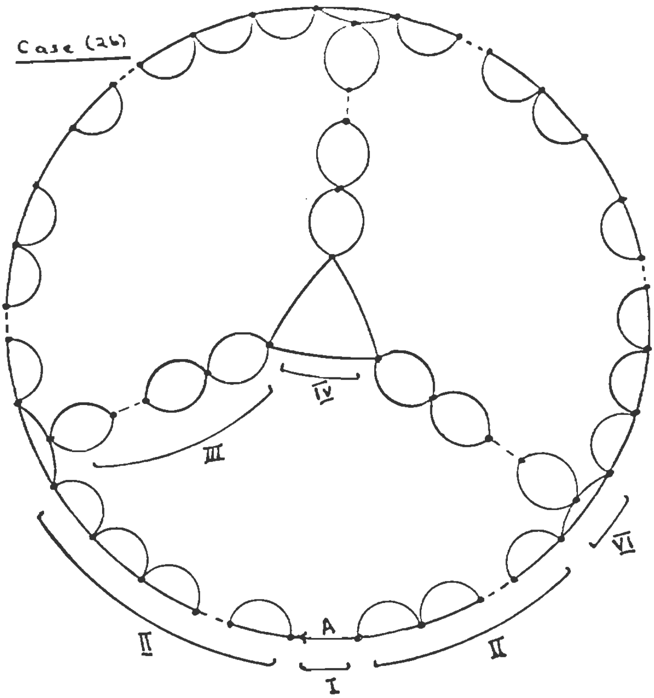

FIGURE 4.

Hence

$$(8.8)\quad \begin{aligned} J^2xy(1-y)^{-1}+1 &= [xy(1-y)]^{-1}[B^2+2uvBz+(u^2v^2z+uvz-uz^2)] \\ &= [xy(1-y)]^{-1}[(2uvz-K)B+(u^2v^2z^2+uvz-uz^2-L)] \\ &= [2xy(1-y)]^{-1}[-(2uvz-K)K+(2uvz-K)D \\ &\qquad +2(u^2v^2z^2+uvz-uz^2-L)] \quad \text{by (8.6) and (8.7)} \\ &= [2xy(1-y)]^{-1}[D^2+(2uvz-K)D] \\ &= D(1-y)^{-1}J. \end{aligned}$$

Now

$$(8.9)\qquad -y^2B_1-K+xy = -(1-y)[1-u(1-u-u^2)y],$$

$$(8.10)\qquad -y^2B_1-L/xy = uy(1-y)(1-u-u^2).$$

Hence

$$\begin{aligned} JC &= [xy(1-y)^2]^{-1}\{B^2+[y(1-y)(B_{y1}-2B_1)-y^2B_1+(1-y)^2+xy]B \\ &\qquad +xy[y(1-y)(B_{y1}-2B_1)-y^2B_1+(1-y)^2]\} \\ &= [xy(1-y)^2]^{-1}\{[y(1-y)(B_{y1}-2B_1)+(1-y)^2-y^2B_1-K+xy]B \\ &\qquad +xy[y(1-y)(B_{y1}-2B_1)+(1-y)^2-y^2B_1-L/xy]\} \quad \text{by (8.6)} \\ &= [x(1-y)]^{-1}[(B+xy)(B_{y1}-2B_1-1+u-u^2-u^3)+x] \\ &\qquad \text{by (8.9) and (8.10).} \end{aligned}$$

540

From (4.11) we find that

$$B_{y1} = u^2v(2 - 3u - 3u^2)(1 - u - u^2)^{-1}. \tag{8.11}$$

Hence

$$JC = -v^2y[(1 - y)(1 - u - u^2)]^{-1}J + (1 - y)^{-1}. \tag{8.12}$$

Substituting in (7.4) where u, z are now redefined by

$$x^r = uv^2, \tag{8.13}$$

$$z = vy^r \tag{8.14}$$

(v remains equal to $1 - u$), we obtain

$$\begin{aligned}(1 - y^r)^{-1}&D(x^r, y^r)J(x^r, y^r)\ {}_rB(x, y)\\ &= J(x^r, y^r)[(1 - y^r)^{-1}\{[-x^ry^rB(x^r, y^r) + u(1 - u - u^2)]\\ &\times [{}_rB(x, 1) + x\delta_{r,2} + 1]y^r + [-1 + y^r + y^r\ {}_rB(x, 1)][B(x^r, y^r) + x^ry^r]\} + x^ry^r],\end{aligned}$$

which reduces to

$$\begin{aligned}D(x^r, y^r)\ {}_rB(x, y) &= u(1 - u - u^2)y^r\ {}_rB(x, 1)\\ &\quad + [-B(x^r, y^r) + uy^r(1 - u - u^2) - x^ry^{2r}]\\ &\quad + xy^2\delta_{r,2}[-B(x^2, y^2) + u(1 - u - u^2) - x^2y^2]\end{aligned} \tag{8.15}$$

for u, y sufficiently small.

Unlike (3.6), this equation can be solved directly, without the need for a preliminary conjecture. Setting $y^r = 1/v$ in (8.15) we find that the left side vanishes, and we obtain after reduction

$${}_rB(x, 1) = u(1 - u - u^2)^{-1} + x\delta_{r,2}u(1 + u)(1 - u - u^2)^{-1}, \tag{8.16}$$

which, substituted in (8.15), yields

$$\begin{aligned}D(x^r, y^r)\ {}_rB(x, y) &= [-B(x^r, y^r) + uy^r(1 - u^2) - uv^2y^{2r}]\\ &\quad + xy^2\delta_{r,2}[-B(x^2, y^2) + u - uv^2y^2]\\ &= \tfrac{1}{2}\{[-D(x^r, y^r) + (1 - z)(1 + uz)]\\ &\quad + xy^2\delta_{r,2}[-D(x^2, y^2) + (1 - z)(1 + 2u - uz)]\}.\end{aligned}$$

Hence, for u, y sufficiently small,

$$\begin{aligned}{}_rB(x, y) &= \tfrac{1}{2}[\{(1 + uz)[1 - 2u(1 + 2u)z + u^2z^2]^{-1/2} - 1\}\\ &\quad + xy^2\delta_{r,2}\{(1 + 2u - uz)[1 - 2u(1 + 2u)z + u^2z^2]^{-1/2} - 1\}].\end{aligned} \tag{8.17}$$

By the binomial theorem,

$$\begin{aligned}[1 - 2u(1 + 2u)z + u^2z^2]^{-1/2} &= \sum_{s=0}^{\infty} \frac{(2s)!}{s!\,s!}(u^2z)^s(1 - uz)^{-2s-1}\\ &= \sum_{s=0}^{\infty}\sum_{j=s}^{\infty} \frac{(s + j)!}{s!\,s!\,(j - s)!}u^{s+j}z^j.\end{aligned} \tag{8.18}$$

Hence

(8.19)
$$(1 + uz)[1 - 2u(1 + 2u)z + u^2z^2]^{-1/2} - 1$$
$$= 2 \sum_{s=0}^{\infty} \sum_{j=s}^{\infty} \frac{j(s + j - 1)!}{s!\, s!\, (j - s)!} u^{s+j} z^j - 1$$
$$= 2 \sum_{j=1}^{\infty} \sum_{s=0}^{j} \sum_{i=0}^{\infty} \frac{j(2s + j)\, (s + j - 1)!\, (3i + 2s + j - 1)!}{s!\, s!\, (j - s)!\, i!\, (2i + 2s + j)!} \times x^{r(s+j+i)} y^{rj} \qquad \text{by (4.14)}$$
$$= 2 \sum_{j=1}^{\infty} \sum_{w=j}^{\infty} \sum_{k=j}^{\min(w,2j)} \frac{j(2k - j)\, (k - 1)!\, (3w - k - j - 1)!\, x^{rw} y^{rj}}{(k - j)!\, (k - j)!\, (2j - k)!\, (w - k)!\, (2w - j)!}.$$

Also

(8.20)
$$(1 + 2u - uz)[1 - 2u(1 + 2u)z + u^2z^2]^{-1/2} - 1$$
$$= 2 \sum_{j=0}^{\infty} \sum_{s=1}^{j+1} \frac{(j + 1)\, (s + j - 1)!}{s!\, (s - 1)!\, (j - s + 1)!} u^{s+j} z^j$$
$$= 2 \sum_{j=0}^{\infty} \sum_{s=1}^{j+1} \sum_{i=0}^{\infty} \frac{(j + 1)\, (2s + j)\, (s + j - 1)!\, (3i + 2s + j - 1)!}{s!\, (s - 1)!\, (j - s + 1)!\, i!\, (2i + 2s + j)!} \times x^{r(s+j+i)} y^{rj} \qquad \text{by (4.14)}$$
$$= 2 \sum_{j=0}^{\infty} \sum_{w=j+1}^{\infty} \sum_{k=j+1}^{\min(w,2j+1)} \frac{(j + 1)\, (2k - j)\, (k - 1)!\, (3w - k - j - 1)!}{(k - j)!\, (k - j - 1)!\, (2j - k + 1)!\, (w - k)!\, (2w - j)!} x^{wr} y^{jr}.$$

Hence ${}_rB_{n,m} = 0$ everywhere, except in the following cases:

(8.21)
$$\begin{cases} {}_rB_{rw,rj} = \dfrac{j}{(2w - j)!} \displaystyle\sum_{k=j}^{\min(w,2j)} \frac{(2k - j)\, (k - 1)!\, (3w - k - j - 1)!}{(k - j)!\, (k - j)!\, (2j - k)!\, (w - k)!} \\ {}_2B_{2w+1,2j} = \dfrac{j}{(2w - j + 1)!} \\ \qquad \times \displaystyle\sum_{k=j}^{\min(w,2j-1)} \frac{(2k - j + 1)\, (k - 1)!\, (3w - k - j)!}{(k - j + 1)!\, (k - j)!\, (2j - k - 1)!\, (w - k)!} \end{cases}$$
$$(w \geqslant j;\ w, j = 1, 2, \ldots;\ r = 2, 3, \ldots).$$

We note that the value of ${}_rB_{rw,rj}$ is independent of r $(r > 1)$. For small w, j, the values of ${}_rB_{rw,rj}$ and ${}_2B_{2w+1,2j}$ can be found respectively in Tables I and II. By means of formula (6.7), the numbers $L_{n,m}$ have been computed for $n < 8$, $m < 8$, and are listed in Table III. The non-separable face-rooted maps of type $[n, m]$ (for $n < 6$) are shown in Figure 5.

US 542 63
542

TABLE I

$_rB_{rw,rj}$

$w \backslash j$	1	2	3	4	5	6	$\sum_j {}_rB_{rw,rj}$
1	1						1
2	2	1					3
3	6	6	1				13
4	24	26	12	1			63
5	110	120	75	20	1		326
6	546	594	416	174	30	1	1761

141 out 394 out (259)

TABLE II

$_2B_{2w+1,2j}$

$w \backslash j$	1	2	3	4	5	6	$\sum_j {}_2B_{2w+1,2j}$
1	1						1
2	2	2					4
3	7	8	3				18
4	30	34	21	4			89
5	143	160	114	44	5		466
6	728	806	609	308	80	6	2537

~~889~~ 152 out 164 305

TABLE III

$L_{n,m}$

$n \backslash m$	2	3	4	5	6	7	$\sum_m L_{n,m}$
2	1						1
3	1	1					2
4	2	1	1				4
5	4	3	2	1			10
6	14	12	8	2	1		37
7	49	43	30	12	3	1	138

89 out (87)

We also compute directly from (8.16) the coefficients of $_rB(x, 1)$ $(r > 1)$.

(8.22) $$_rB(x,1) = uv^{-1}(1-u^2v^{-1})^{-1} + x\delta_{r,2}[-1+v^{-1}(1-u^2v^{-1})^{-1}]$$

$$= \sum_{p=0}^{\infty} u^{2p+1}v^{-(p+1)} + x\delta_{r,2}\left[-1+\sum_{p=0}^{\infty} u^{2p}v^{-(p+1)}\right]$$

$$= \sum_{p=0}^{\infty}\sum_{s=2p+1}^{\infty} \frac{(5p+3)\,(3s-p-1)!}{(s-1-2p)!\,(2s+p+1)!}\,x^{rs}$$

$$+\,\delta_{r,2}\left[-x+\sum_{p=0}^{\infty}\sum_{s=2p}^{\infty} \frac{(5p+1)\,(3s-p)!}{(s-2p)!\,(2s+p+1)!}\,x^{2s+1}\right].$$

87

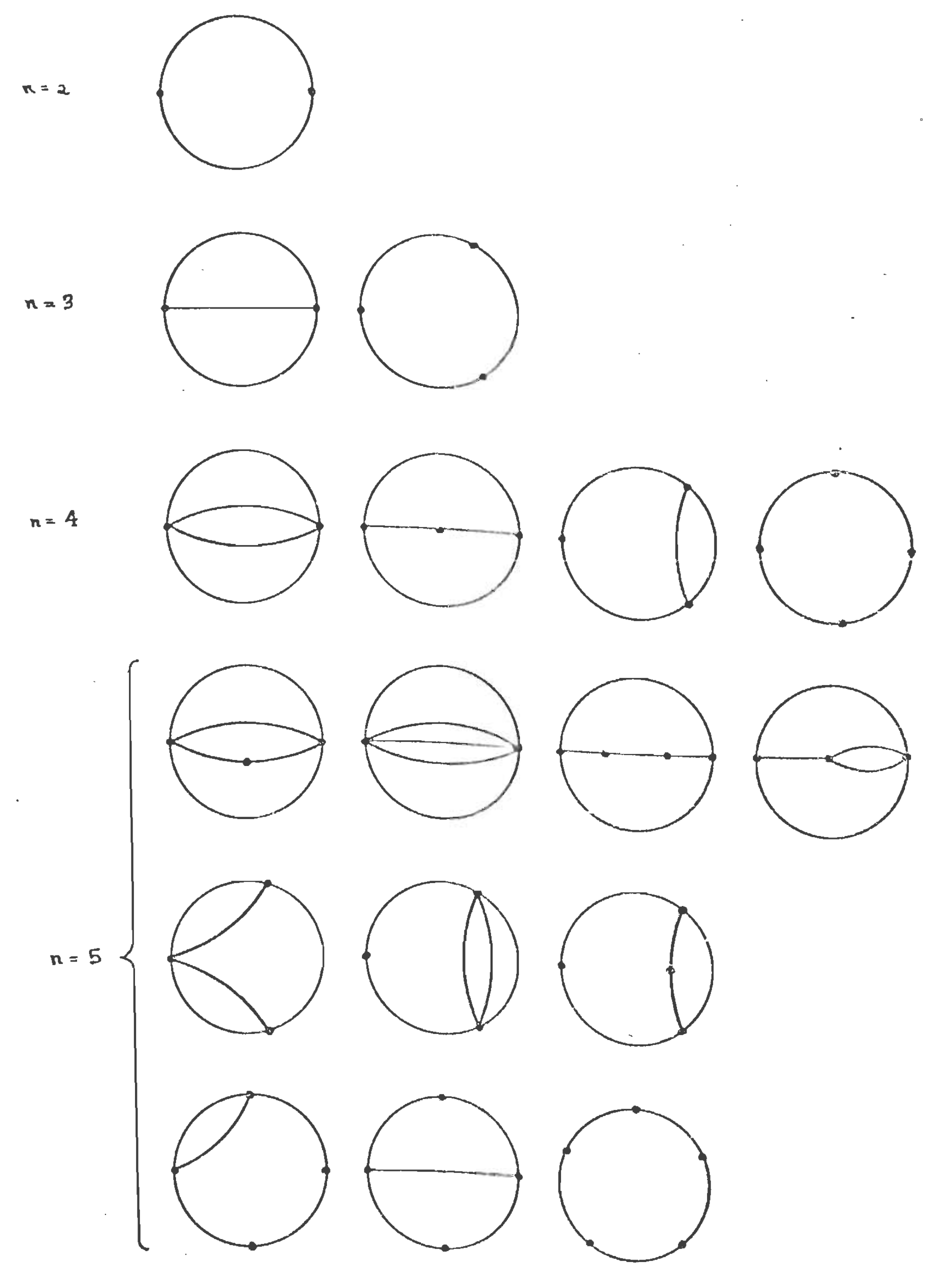

FIGURE 5.

by (4.14). Hence

$$
\begin{aligned}
{}_rB(x, 1) &= \sum_{s=1}^{\infty} \sum_{p=0}^{[(s-1)/2]} \frac{(5p+3)\,(3s-p-1)!}{(s-1-2p)!\,(2s+p+1)!} x^{rs} \\
&\qquad + \delta_{r,2} \sum_{s=1}^{\infty} \sum_{p=0}^{[s/2]} \frac{(5p+1)\,(3s-p)!}{(s-2p)!\,(2s+p+1)!} x^{2s+1} \\
&= (x^r + 3x^{2r} + 13x^{3r} + 63x^{4r} + 326x^{5r} + 1761x^{6r} + \dots) \\
&\qquad + \delta_{r,2}(x^3 + 4x^5 + 18x^7 + 89x^9 + 466x^{11} + 2537x^{13} + \dots).
\end{aligned}
$$

9. Asymptotic behaviour of $\sum_m L_{n,m}$. By well-known properties of the binomial coefficients,

$$
\begin{aligned}
(9.1) \quad \sum_{p=0}^{[(s-1)/2]} \frac{(5p+3)\,(3s-p-1)!}{(s-1-2p)!\,(2s+p+1)!} &= \sum_{p=0}^{[(s-1)/2]} \frac{5p+3}{3s-p} \binom{3s-p}{2s+p+1} \\
&\leqslant \sum_{p=0}^{[(s-1)/2]} \frac{5(s-1)+6}{6s-(s-1)} \binom{3s}{2s+p+1} = \sum_{p=0}^{[(s-1)/2]} \binom{3s}{2s+p+1} \\
&\leqslant \sum_{p=0}^{[(s-1)/2]} \binom{3s}{2s} \leqslant \frac{s+1}{2} \binom{3s}{2s}.
\end{aligned}
$$

Hence, by Stirling's formula,

$$
\sum_{j=1}^{\infty} {}_rB_{rs,rj} \leqslant \frac{s+1}{2}\binom{3s}{2s} \sim \frac{s}{2} \frac{e^{-3s}(3s)^{3s+(1/2)}(2\pi)^{1/2}}{e^{-s-2s}s^{s+(1/2)}(2s)^{2s+(1/2)}2\pi} = \frac{1}{4}\left(\frac{27}{4}\right)^s \sqrt{\frac{3s}{\pi}},
$$

i.e.

$$
(9.2) \qquad \sum_{j=1}^{\infty} {}_rB_{n,rj} \lesssim \frac{1}{4}\left(\frac{27}{4}\right)^{n/r} \sqrt{\frac{3n}{\pi r}} \qquad \text{if } n \equiv 0 \pmod{r}.
$$

Similarly it can be shown that

$$
(9.3) \qquad \sum_{j=1}^{\infty} {}_2B_{n,2j} \lesssim \frac{3}{2}\left(\frac{27}{4}\right)^{n/2} \sqrt{\frac{3}{\pi}} \left(\frac{n}{2}\right)^{3/2} \qquad \text{if } n \equiv 1 \pmod{2}.
$$

Hence

$$
\begin{aligned}
\sum_{r=2}^{\infty} \sum_{m=2}^{\infty} {}_rB_{n,m} &\lesssim \sum_{r=2}^{n} \frac{1}{4}\left(\frac{27}{4}\right)^{n/2} \sqrt{\frac{3}{\pi}} \left(\frac{n}{2}\right)^{1/2} + \frac{3}{2}\left(\frac{27}{4}\right)^{n/2} \sqrt{\frac{3}{\pi}} \left(\frac{n}{2}\right)^{3/2} \\
&< \left(\frac{1}{2}+\frac{3}{2}\right)\left(\frac{27}{4}\right)^{n/2} \sqrt{\frac{3}{\pi}} \left(\frac{n}{2}\right)^{3/2}.
\end{aligned}
$$

But, also by Stirling's formula,

$$(9.4)\qquad \sum_{m=1}^{\infty} B_{n,m} = \frac{2(3n-3)!}{n!\,(2n-1)!}$$
$$\sim \frac{2e^{-(3n-3)}(3n-3)^{3n-(5/2)}(2\pi)^{1/2}}{e^{-n-(2n-1)}n^{n+(1/2)}(2n-1)^{2n-(1/2)}2\pi}$$
$$= \frac{2}{9}\left(\frac{27}{4}\right)^n \frac{n^{-5/2}}{(3\pi)^{1/2}} \frac{e^2[1-(1/n)]^{3n-(5/2)}}{[1-(1/2n)]^{2n-(1/2)}}$$
$$\sim \frac{2}{9}\left(\frac{27}{4}\right)^n n^{-5/2}(3\pi)^{-1/2}.$$

Since $(27/4)^{1/2} < (27/4)$,

$$\lim_{n\to\infty}\left[\left(\sum_{r=2}^{\infty}\sum_{m=2}^{\infty} {}_rB_{n,m}\right)\Big/\left(\sum_{m=2}^{\infty} B_{n,m}\right)\right] = 0.$$

In this sense, almost all non-separable rooted maps M possess no orientation-preserving automorphisms other than the identity, i.e. $\mathfrak{N}[M] = I$. It follows that

$$(9.5)\qquad \sum_{m=2}^{n} L_{n,m} \sim \sum_{m=2}^{n} (1/m)\, B_{n,m} \qquad \text{as } n \to \infty.$$

References

1. W. T. Tutte, *A census of planar triangulations*, Can. J. Math., *14* (1962), 21–38.
2. —— *A census of planar maps*, Can. J. Math. *15* (1963), 249–271.
3. I. M. Vinogradov, *Elements of number theory* (New York, 1954).
4. E. T. Whittaker and G. N. Watson, *A course of modern analysis* (Cambridge, 1927).

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