~~[illegible]~~ 255!
153?
MAG 52(1968)
23052
166

ι, and $\theta = \alpha$,

$)$, and $\theta = 0$.

$:/\pi$.

e fraction of OD as the arc EA is

$$\Big) - \tan^{-1}\frac{b}{a}\sqrt{\left(\frac{a^2-r^2}{r^2-b^2}\right)}.$$

$$\overline{{}^2)} = \int_{AB}\frac{dr\,.\sec\phi}{\sqrt{(a^2-r^2)}}$$

$$\frac{\iota^2-b^2)\,dr}{-r^2)(r^2-b^2)]}$$

$^2)\,d$ with $r^2 = a^2\sin^2 t + b^2\cos^2 t$,

$^2)/a$.

$\sqrt{[(a^2-b^2)/ag]}$, compared with the $\iota/g)$.

H. Martyn Cundy

s that I have seen, **a** × **b** is defined × **b** is made to seem a *convention*; uence of (1) **a** × **a** = 0 and (2) the plication. For:

+ **a** × **b** + **b** × **a** + **b** × **b**

iately.

A. R. Pargeter

MATHEMATICAL NOTES

3207. A sequence connected with the sub-factorial sequence

We define a sequence

$$1, 1, 3, 11, 53, 309, \ldots \text{ ——— } 255$$

which satisfies the conditions

$$u_0 = 1, \quad u_1 = 1, \quad u_r = ru_{r-1} + (r-1)u_{r-2} \qquad (r \geqslant 2).$$

This sequence has the interesting property that *if* $(1+u)^n$ *denotes symbolically the expression*

$$u_0 + nu_1 + \tfrac{1}{2}n(n-1)u_2 + \ldots + nu_{n-1} + u_n,$$

with binomial coefficients, then

$$(1+u)^n = (n+1)!.$$

Proof:

In the symbolic notation, we have

$$\begin{aligned}(1+u)u^r &= u^r + u^{r+1}\\ &= u^r + (r+1)u^r + ru^{r-1}\\ &= ru^{r-1}(1+u) + 2u^r \quad \text{if} \quad r \geqslant 1 \quad (u^0 = u_0 = 1).\end{aligned}$$

Now

$$\begin{aligned}(1+u)^{s+1} &= (1+u)(1+u)^s\\ &= (1+u)\sum_{r=0}^{s}\binom{s}{r}u^r\\ &= \sum_{r=1}^{s}\binom{s}{r}[ru^{r-1}(1+u) + 2u^r] + 1 + u\\ &= \sum_{r=1}^{s}s\binom{s-1}{r-1}u^{r-1}(1+u) + \sum_{r=0}^{s}2\binom{s}{r}u^r \quad (\text{since } u_1 = 1)\\ &= s(1+u)^s + 2(1+u)^s\\ &= (s+2)(1+u)^s.\end{aligned}$$

Since $1 + u_1 = 2$, it follows easily by induction that

$$(1+u)^n = (n+1)!.$$

The given sequence is closely connected with the sub-factorial sequence v_r, where v_r is the number of permutations of r objects in which no object is left unchanged.* This sequence satisfies the conditions†

* This is the old problem of the number of ways in which r letters can be put into r addressed envelopes so that each letter is wrongly addressed.

† The choice of 1 for v_0 is purely conventional, but very convenient later.

p. 381

$$v_0 = 1, \quad v_1 = 0, \quad v_r = (r-1)(v_{r-1}+v_{r-2}) \quad (r \geqslant 2),$$

and the first few terms are

166

$$1, 0, 1, 2, 9, 44, 265, \ldots.$$

Now it is easily proved by induction that

$$u_r = v_r + v_{r+1}.$$

For the result is clearly true for $r = 0$ and 1, and if it is true for $r = k-2$ and $k-1$, so that

$$u_{k-2} = v_{k-2} + v_{k-1}$$

and

$$u_{k-1} = v_{k-1} + v_k,$$

then

$$\begin{aligned} u_k &= ku_{k-1} + (k-1)u_{k-2} \\ &= k(v_{k-1}+v_k) + (k-1)(v_{k-2}+v_{k-1}) \\ &= (k-1)(v_{k-1}+v_{k-2}) + k(v_k+v_{k-1}) \\ &= v_k + v_{k+1}. \end{aligned}$$

We now prove a result for the v sequence similar to that for the u sequence, namely

$$(1+v)^n = n!$$

Proof:

Since $u^r = (1+v)v^r$,

$$(1+v)^n = (1+v)\sum_{r=0}^{n-1}\binom{n-1}{r}v^r = \sum_{r=0}^{n-1}\binom{n-1}{r}u^r = (1+u)^{n-1} = n!$$

Finally, we obtain an explicit expression for u_r. It is well known that

$$v_r = r!\left(\frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r\frac{1}{r!}\right).$$

Hence

$$\begin{aligned} u_r = v_r + v_{r+1} &= \frac{v_{r+2}}{r+1} \\ &= (r+2)r!\left(\frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{r+2}\frac{1}{(r+2)!}\right). \end{aligned}$$

141B *Upper Clapton Road, London, E.5.* MAX RUMNEY
University of Leicester E. J. F. PRIMROSE

3208. Powerful numbers

Mr. ApSimon's implicit challenge in Mathematical Note 3172 (A very odd number) spurred me to find a larger such number, viz.

$$4679307774 = 4^{10}+6^{10}+7^{10}+9^{10}+3^{10}+0^{10}+7^{10}+7^{10}+7^{10}+4^{10}$$

I suggest the name *powerful* numbers be given to numbers having this property and give a list of those I have found in addition to the one above:

23052

$$\begin{aligned}
153 &= 1^3+5^3+3^3\\
370 &= 3^3+7^3+0^3\\
371 &= 3^3+7^3+1^3\\
407 &= 4^3+0^3+7^3\\
1634 &= 1^4+6^4+3^4+4^4\\
8208 &= 8^4+2^4+0^4+8^4\\
9474 &= 9^4+4^4+7^4+4^4\\
4150 &= 4^5+1^5+5^5+0^5\\
4151 &= 4^5+1^5+5^5+1^5\\
54748 &= 5^5+4^5+7^5+4^5+8^5\\
92727 &= 9^5+2^5+7^5+2^5+7^5\\
93084 &= 9^5+3^5+0^5+8^5+4^5\\
194979 &= 1^5+9^5+4^5+9^5+7^5+9^5\\
548834 &= 5^6+4^6+8^6+8^6+3^6+4^6\\
1741725 &= 1^7+7^7+4^7+1^7+7^7+2^7+5^7\\
4210818 &= 4^7+2^7+1^7+0^7+8^7+1^7+8^7\\
9800817 &= 9^7+8^7+0^7+0^7+8^7+1^7+7^7\\
9926315 &= 9^7+9^7+2^7+6^7+3^7+1^7+5^7\\
14459929 &= 1^7+4^7+4^7+5^7+9^7+9^7+2^7+9^7\\
24678050 &= 2^8+4^8+6^8+7^8+8^8+0^8+5^8+0^8\\
24678051 &= 2^8+4^8+6^8+7^8+8^8+0^8+5^8+1^8\\
88593477 &= 8^8+8^8+5^8+9^8+3^8+4^8+7^8+7^8\\
146511208 &= 1^9+4^9+6^9+5^9+1^9+1^9+2^9+0^9+8^9\\
472335975 &= 4^9+7^9+2^9+3^9+3^9+5^9+9^9+7^9+5^9\\
534494836 &= 5^9+3^9+4^9+4^9+9^9+4^9+8^9+3^9+6^9\\
912985153 &= 9^9+1^9+2^9+9^9+8^9+5^9+1^9+5^9+3^9
\end{aligned}$$

15 *Farcliffe Terrace, Bradford, 8, Yorkshire.* JEREMY RANDLE

3209. An inequality

Most of our pleasant discoveries turn out to have been anticipated by a century or more and the following is no exception, being associated with the name of Jensen. But although it can be found in standard works of algebra if one happens to look in the right place, it may be as new to others as it was to me.

E