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F. R. Bernhart

Fundamental
Chrom. Numbers
(unpublished)

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FUNDAMENTAL CHROMATIC NUMBERS

Three integer sequences denoted $\mu(n)$, $\nu(n)$, $\rho(n)$ are found by Bifkhoff and Lewis [3, pp. 432-434] to be of particular importance in the analysis of rings, as part of the study of chromatic polynomials of plane maps. They are able to give a formula for the last one only, and also say

> "For the complete theory of the n-ring it would be even more important to determine the function $\nu(n)$ than the function $\mu(n)$. It is hard to say which of these problems is more difficult."

It is good to report that due to recent study all of these functions (or sequences) can be determined.

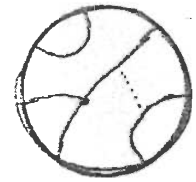
There is a simple algorithm which they give for $\rho(n)$. You simply express $(x-1)^n + (-1)^n(x-1)$ as a linear combination of falling factorials $x(x-1)\ldots(x-t)$ for $t = 1,2,\ldots, n-1$, and then sum the coefficients. A direct formula was also given, but it is somewhat more involved. The meaning of $\rho(n)$ is the number of essentially distinct patterns for coloring a ring of regions, with no limit on the number of colors. (Today we would dualize, and talk of coloring the vertices of a simple circuit graph.) Values are as follows

n	2	3	4	5	6	7	8	9	10	
$\rho(n)$	1	1	4	11	41	162	715	...		✓296
We add $\rho'(n)$	1	1	4	10	31	91	274	820	2461	... 6342

See notes attached

where $\rho'(n)$ is obtained by limiting the number of colors to four; these numbers show up prominently in the Appel/Haken proof of the Four Color Theorem, and other work on Four Color Reducibility [1, 2]. Without difficulty $\rho'(n) = \frac{[3^{n-1}+(-1)^n+2]}{2}$.

Function $\mu(n)$ counts the 'elementary maps' of order n. This is also the same as the number of ways of joining by lines inside a circle a set of n distinguished points on the circle, as follows. (a) Each point in turn is joined to another point on the circle, or to an interior point of a line already drawn. (b) The lines stay in the circle, and do not cross. (c) More than three lines cannot meet at a point. The inside of the circle is a forest with all valencies (or degrees) less than four. A typical example is shown at the right. When there are exactly



n regions in the interior, then the figure is no more than the geometric dual of a slight modification of a triangulated polygon. Birkhoff and Lewis note in a footnote that this special subclass is counted by a Catalan number.

The author in his dissertation [2] called the number of elementary maps a hyper-Catalan number, because of the close involvement with the Catalan series. In Chapter 4, a complicated recursion is deduced (a different and also complex recursion had been earlier found by the author's father, Arthur F. Bernhart, but not published). A functional identity for the generating series was obtained by substituting into the series of the Catalan numbers. If that generating series is $y = y(x) = K_0 + K_1x + K_2x^2 + \ldots$ then the following bicubic is obtained. $K_n = \mu(n)$

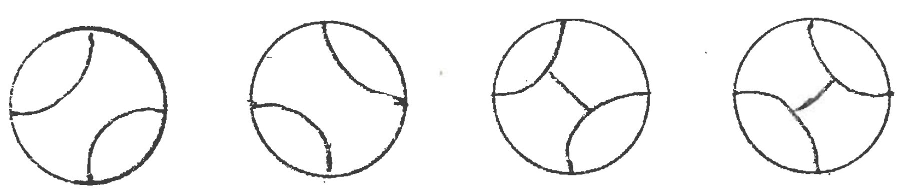
$$x^3y^3 - (x-1)y^2 + (x-2)y + 1 = 0$$

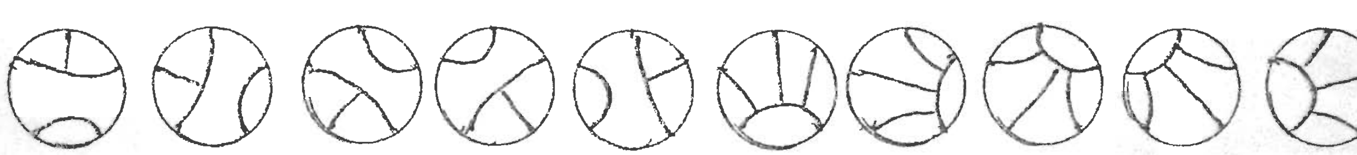
It was pointed out to me in a letter by Doug Rogers that Lagrange inversion could be applied to one of the stages of the argument, to obtain

$$K_n = \mu(n) = \sum_{r=1}^{[n/2]} \frac{\binom{n}{r-1}\binom{2n-3r-1}{n-2r}}{n-r} .$$

This checks with the table in [2], a table generated with a recursion based on the bicubic.

n	2	3	4	5	6	7	8	9	10	
$\mu(n)$	1	1	4	10	34	112	398	1443	5386	6343





The elementary maps are used to generate free chromatic polynomials. Each elementary map is inserted on one side of n-ring of a fixed map, and then the chromatic polynomial of the map is computed. It turns out that there is a basic set of $\nu(n)$ chromatic polynomials which determine all the others.

As a result of a discovery of W.T. Tutte regarding the form of the equations by which some of the chromatic polynomials depended on others, I was able to find a geometric solution of the basis problem. Start with a circuit in the plane of n vertices. Merge certain groups of vertices inside the circuit, but without violating the planarity or producing a loop. In other words, specify a partition of the elements $1,2,3,\ldots,n$ satisfying two conditions: (a) i and $i+1 \pmod n$ are not placed in the same part, and (b) if in the four element subsequence (i,j,k,ℓ) i and k are in the same part, and also j and ℓ are in the same part, then i,j,k,ℓ are all in the same part. We call these partitions the feasible planar partitions of an n-circuit. With condition (a) dropped, the answer is the Catalan numbers. Here we have p_i such that $p_{-1} = 0$, $p_0 = p_2 = p_3 = 1$, $p_4 = 3$, and

$$p_n = \sum_{i=0}^{n-1} p_i p_{n-i-1} + (-1)^n .$$

This gives us the table following.

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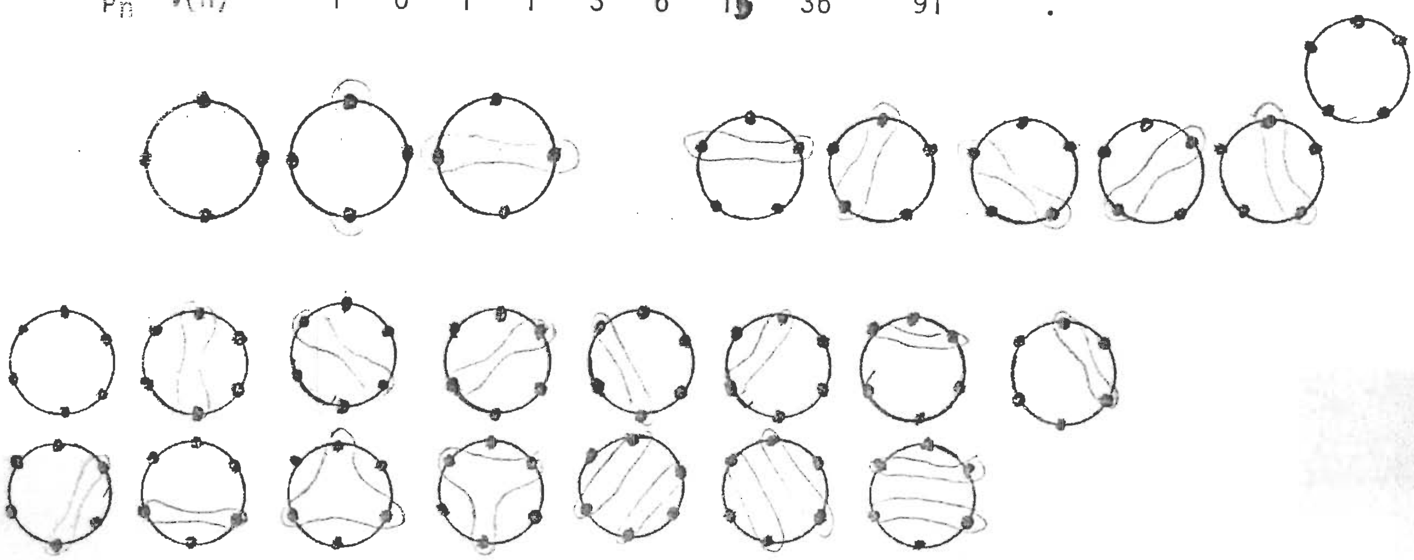
n	0	1	2	3	4	5	6	7	8
$p_n = \nu(n)$	1	0	1	1	3	6	15	36	91



We note that if in the definition, condition (a) be left out, we are only asking for planar partitions of an n-cycle, counted by the Catalan numbers. Leaving out (b) as well is of course counting all partitions (Bell numbers).

Now, from the recursion above one can find a functional relation for the generating function $y(x) = p_0 + p_1x + p_2x^2 + \ldots$. It is as follows.

$$y = 1/(1+x) + xy^2 .$$

Using Lagrange inversion on $y = z + xy^2$ we have $y = z + xz^2 + \ldots \frac{\binom{2n}{n}x^n z^{n+1}}{n+1} + \ldots$

and then putting $z = 1/(1+x)$ we finally find:

$$p_n = \sum_{i=0}^{n} \frac{\binom{2i}{i}\binom{n}{i}(-1)^i}{i+1} .$$

The recursion itself follows from some counting arguments suggested by Michael Rolle, a student at the University of Waterloo.

References

1. Appel, K. & Haken, W., Every planar map is four colorable: Part 1, Discharging, Part 2, Reducibility, Ill. J. Math. 21(No. 3), Sept. 77.

2. Bernhart, F.R., "Topics in Graph Theory Related to the Five Color Conjecture, Ph.D. Dissertation, Kansas State Univ., 1974.

3. Birkhoff, G.D., & Lewis, D.C., Chromatic Polynomials, Trans. Amer. Math. Soc. 60 (1946), 355-451.

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n	p(n)
0	1
1	0
2	1
3	1
4	3
5	6
6	15
7	36
8	91
9	232
10	603
11	1585
12	4213
13	11298
14	30537
15	83097
16	227475
17	625992

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LIST

 100 REM******* PARTITION NUMBERS *****
 110 DIM P(20)
 120 P(1)=1
 130 S=-1
 140 FOR N=1 TO 18
 150 IF N=1 THEN 220
 160 P(N)=0
 170 FOR I=1 TO N-1
 180 P(N)=P(N)+P(I)*P(N-I)
 190 NEXT I
 200 P(N)=P(N)+S
 210 S=-S
 220 PRINT N-1;TAB(8);P(N)
 230 NEXT N
*
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