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Moon & Goldberg

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THEOREM 3. *If $Z \circ S = S \circ Z$ for non-trivial tournaments S and Z, then S and Z are both transitive or $S = H^u$ and $Z = H^v$ for some tournament H.*

Proof. Suppose that $z = qs + r$ where $s = |S| \le z = |Z|$ and $0 \le r < s$. If $s = z$, then $S = Z$ by Theorem 1a. If $s < z$ and $r = 0$, then $Z = C \circ S$ for some nontrivial tournament C; hence, $C \circ S = S \circ C$, by Corollary 1.1, and the result for this case now follows by induction on $m = \max(|S|, |Z|)$.

If $s < z$ and $r > 0$, then $Z = V \circ T_{z/(s,r)} = T_{z/c} \circ C$ and $S = T_{s/c} \circ C = V \circ T_{s/(s,r)}$ for some tournaments V and C by Theorem 1c; therefore,

$$T_{z/c} \circ C \circ V \circ T_{s/(s,r)} = Z \circ S = S \circ Z = T_{s/c} \circ C \circ V \circ T_{z/(s,r)} .$$

Since $s \neq z$, it follows from Lemma 4.2 that $C \circ V$ is transitive. But then C and V must both be transitive and, consequently, both S and Z are transitive. This completes the proof of the theorem.

The set of all dominance-preserving permutations of the nodes of a tournament R forms a group $G(R)$, the *automorphism group* of R. It is shown in [1] that $G(R \circ S)$ is equal to the composition (or wreath product) of $G(R)$ with $G(S)$. The main step is proving this is showing that any automorphism of $R \circ S$ maps copies of S onto copies of S. That this is the case follows almost immediately from the proof of Theorem 1.

6. The number of prime tournaments. Let us say that tournament B is a multiple of tournament A if $B = X \circ A$ for some tournament X. It follows from Theorem 2 that every nontrivial tournament H has a unique representation of the type $H = R \circ S$ where exactly one of the following alternatives holds:

(a) S is a nontransitive prime tournament, or
(b) S is a nontrivial transitive tournament, and R is not a multiple of a nontrivial transitive tournament.

This observation can be used to determine the number of prime tournaments in terms of the total number of tournaments.

Let $t(n)$ denote the number of nonisomorphic tournaments with n nodes (Davis [2] has determined the value of $t(n)$ for $n \le 8$; his general formula for $t(n)$ involves a sum over partitions of n). Also, let $r(n)$ denote the number of nonisomorphic tournaments with n nodes that are not multiples of a nontrivial transitive tournament (notice that $r(1) = 1$ but that the transitive tournament T_n is not among the tournaments counted by $r(n)$ if $n > 1$). It follows from the above observation that $t(n) = \sum_{d/n} r(n/d)$. Therefore,

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$t(n)$
$r(n)$

$$r(n) = \sum_{d/n} \mu(d) t(n/d),$$

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where μ denotes the Mobius function, so we may regard the numbers $r(n)$ as known.

If $p(n)$ denotes the number of nonisomorphic nontransitive prime tournaments with n nodes, then $p(n)$ equals $r(n)$ minus the number of nonisomorphic

$p(n)$
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nontransitive tournaments with n nodes that are nontrivial multiples of nontransitive prime tournaments when $n > 1$. Therefore,

$$p(n) = r(n) - \Sigma' p(d)t(n/d),$$

when $n > 1$, where the sum is over-all divisors of n other than 1 and n. The formulas for $r(n)$ and $p(n)$ may be combined to yield the following result which permits the numbers $p(n)$ to be determined recursively.

THEOREM 4. *If $n > 1$, then*

$$\{p(n) - \mu(n)\} = t(n) - \Sigma'\{p(d) - \mu(d)\}t(n/d),$$

where the sum is over all divisors of n other than 1 and n.

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The total number of nonisomorphic prime tournaments with n nodes is $p(n) + 1$ or $p(n)$, according as n is or is not a prime number, since the transitive tournament T_n is prime if and only if n is prime. (The first few values of $t(n)$ and $p(n) - \mu(n)$ are given in the table below.) The following result, which we state without proof, follows from the formula Davis [2] gave for $t(n)$ and Theorem 4.

THEOREM 5. *$p(n) \sim t(n) \sim 2^{\binom{n}{2}}/n!$, as $n \to \infty$.*

n	$t(n)$	$p(n) - \mu(n)$	n	$t(n)$	$p(n) - \mu(n)$
1	1	−1	7	456	456
2	1	1	8	6,880	6,873
3	2	2	9	191,536	191,532
4	4	3	10	9,733,056	9,733,032
5	12	12	11	903,753,248	903,753,248
6	56	52	12	154,108,311,168	154,108,311,046

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