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Structure of dominance relations, 5,68

1954

Bull. Mathematical Biophysics, 16 (1954), 131-140

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STRUCTURES OF DOMINANCE RELATIONS

ROBERT L. DAVIS*
UNIVERSITY OF MICHIGAN

Dominance relations and their structures have been studied in the context of animal sociology (Rapoport, 1949a, b; Landau, 1951a, b), and occur in many other theoretical models of the social and biological sciences. When Rapoport and Landau wrote, there was no method known for determining the number of dominance structures which could be defined on a set of n elements, for $n > 4$. The answer to this question can be important in certain further investigations of the structural properties of dominance. Using a general method developed elsewhere (Davis, 1953), this paper derives a formula to answer the question for any n, and gives an application of the method to treat a case not previously analysed.

1. Definitions and examples. A series of papers in this journal (Rapoport, 1949a, b; Landau, 1951a, b) called attention to the importance in social science and elsewhere of a kind of relation seldom previously recognized, viz., one which was asymmetric and "connected" [in Russell's (1919) sense] but not necessarily transitive. These were aptly termed *dominance relations*.

To be definite, let $N = \{1, \ldots, n\}$ be any set of n elements. Then we can describe any dyadic relation on N (to itself) in terms of a matrix $A = (a_{ij})$, where $a_{ij} = 1$ just when i stands in the given relation to j, and otherwise $a_{ij} = 0$.

Definition A. The dyadic relation A is a *dominance relation* if, and only if, (i) all $a_{ii} = 0$, and (ii) for all $i \neq j$, either $a_{ij} = 1$ or else $a_{ji} = 1$, but not both.

Rapoport and Landau examined these relations in a model for the "peck-right" among barnyard fowl, also mentioning their applicability to other situations commonly obtaining in small groups. Any tournament, for instance, in which each participant meets another just once (no draws) gives rise to a dominance relation; this example drives home the point that the relation may, or may not, be transitive. The "Method of Paired Comparisons" in psychological scaling gives rise (in the modern view, where intransitivities are not simply thrown out as "error") precise-

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ly to the structure of some dominance relation. And in genetics, the assumption of "dominance" in the one-locus diploid case amounts to asserting that there is some dominance relation defined on the set of alleles. Under the usual Hardy-Weinberg hypotheses, then, the relations between gene frequencies and phenotype frequencies are determined by the dominance *structure*.

2. *The structure problem.* In such questions an investigator is seldom concerned with the particular assignment of names to the objects or individuals being examined, but rather with the nature of some underlying structure. By structure is meant just what is left of the relation after the names are forgotten: the "map" (Russell, *loc. cit.*), or graph, of the relation. For a precise definition of structure, it is simplest to start with what it means for two relations to have the *same structure*.

We want to say that two relations A and B defined on the set N have the same structure, or are isomorphic, whenever there is some permutation π of N such that A has the same matrix with respect to N as has B with respect to $\pi(N)$. Technically, it is convenient to define, corresponding to each permutation π of the symmetric group S_n on n letters, a transformation t_π which operates on our relations:

$$t_\pi(A) = (a_{\pi(i)\pi(j)}), \text{ where } A = (a_{ij}). \qquad (1)$$

(Note that if P_π is the permutation matrix corresponding to π, then $t_\pi(A)$ can be thought of as $P_\pi A P_\pi^{-1}$.)

Definition B. Two relations A and B defined on N are *isomorphic* if, and only if, there is a permutation π in S_n such that $t_\pi(A) = B$.

Definition C. The *structure* of the relation A is the class of all relations with which it is isomorphic.

This is what Landau (1951a) called the *dominance structure*; he inserted the qualifier because Rapoport had used the word "structure" differently. To obtain a manageable set of invariants the latter (Rapoport, 1949a) had defined as "structure" something we may call the *structure sequence* of a relation A: this is just the set of row sums of the entries of A. (These are usually written down as a sequence, in non-increasing order; this amounts to disregarding their original order.) From its definition, it is clear that this structure sequence is invariant under the group of transformations t_π, i.e., that isomorphic relations will always have the same structure sequence. That the various such sequences do not form a "complete set" of invariants—in other words, that different structures may have the same structure sequence—can be seen already when $n = 5$ (Landau, 1951a).

Now it is an immediate consequence of the definition that the number of dominance *relations* on a set of n elements is $2^{n(n-1)/2}$. But how many of these are "essentially different" relations? That is, how many *dominance structures* can be defined on N?

The answer to this question may be useful in two ways: (1) The formula, though practically impossible to evaluate for sizeable n, may yet provide a guide in, e.g., comparing proposed approximations or estimating the error consequent upon various assumptions; (2) Perhaps more important is that methods used here furnish tools for analyzing structural properties of dominance which could be applied, without really undue difficulty, to give a complete treatment of the cases, say, $n \leq 12$. The cases previously so treated—$n \leq 4$—were exceptional, while some of the more variegated aspects of dominance set in for $n = 5$ and $n = 6$. Finally, it is not unlikely that in certain theories where the number of "individuals" involved is small by hypothesis (as in some concerning social groups) these accessible cases $n \leq 12$ would suffice for useful applications.

3. Algebraic background. Derivation of this formula rests on the theory of permutation groups and on a method developed (Davis, *loc. cit.*) to answer the more general question: How many non-isomorphic m-adic relations (of all kinds) are defined on a set of n elements? It would be bootless here to repeat the development of that paper. What can, perhaps, be profitably undertaken is: (1) to state those results in sufficient detail to show their application to the dominance question; (2) on that basis, to derive the main formula in such a way that anyone familiar with the other paper may verify it; (3) to illustrate the methods in an example; and (4) to give the more readily obtained numerical answers.

Suppose, then, that D_n is the set of all dominance relations on N. The symmetric group S_n "acts on" D_n in the sense that any transformation t_π corresponding to an element of S_n maps members of D_n into other members of D_n. Now, by an *orbit* in D_n we will mean a set consisting of some dominance relation, A, and all its images, $t_\pi(A)$, as π ranges over S_n. Hence to count all structures in D_n we have simply to count the orbits under S_n, since these orbits are precisely what *Definition C* says structures are. Further analysis now rests on the theorem that in any such case the number of orbits is equal to the average number (loosely speaking) of relations fixed per group element. More precisely, let $st(n)$ denote the number of dominance structures on n elements and for each π in S_n, let $f(\pi)$ be the number of dominance relations A such that $t_\pi(A) = A$. Then it can be shown that

$$st(n) = \frac{1}{n!} \sum_{\pi} f(\pi). \qquad (2)$$

We can further observe that if π and ϕ are *conjugate* elements of S_n (i.e., $\phi = \chi^{-1}\pi\chi$ for some χ), then $f(\phi) = f(\pi)$; this says f is what is known as a *class function*. Thus it is enough, for our purposes, to evaluate f on one representative of each conjugate class, $[\pi]$, and then multiply by the number, $c(\pi)$, of group elements in that class. The formula then becomes

$$s(n) = \frac{1}{n!} \sum_{[\pi]} c(\pi) f(\pi) \tag{3}$$

where the summation is now over all conjugate classes, $[\pi]$, in S_n.

In the symmetric group each conjugate class is determined by an n-tuple of non-negative integers $(p_1, \ldots, p_n)$—where p_k is the number of cycles of length k in the disjoint-cycle representation of any member of the class (consequently, $1p_1 + 2p_2 + \ldots + np_n = n$, the p_k's define a *partition* of the integer n). And now, rather than the coefficient $c(\pi)$, we will prefer to take the constant term inside the sum and thus use

$$n(\pi) = \frac{1}{n!} c(\pi) = \frac{1}{1^{p_1} p_1! \, 2^{p_2} p_2! \cdots n^{p_n} p_n!} \tag{4}$$

[making use of the formula for $c(\pi)$], so that the latest version of the formula we seek is

$$s(n) = \sum_{[\pi]} n(\pi) f(\pi) \tag{5}$$

Since for any n there is an explicit, though arduous, means of writing down all the partitions which determine our summands, and since we have determined $n(\pi)$ for each of these summands in (4), it remains only to find an expression for $f(\pi)$. This can be done by a method which is almost the same as those of theorems 4 and 5 (Davis, *loc. cit.*), deriving the number of structures for symmetric and asymmetric relations. The sketch which follows is to be taken as a proof only in conjunction with the other paper; readers more interested in application of the formula might well skip to the final section, pausing only to observe the results of section 4.

4. Derivation of the formula. There is one new wrinkle in the dominance case which makes later computations much easier.

Lemma. If the disjoint-cycle representation of π contains any cycle of even length, then $f(\pi) = 0$.

Proof. There is no loss of generality in assuming the given cycle to be [123 ... (2k)]. Now by formula (1), $T_\pi(A) = A$ will require, among other things, that

$$a_{1,k+1} = a_{2,k+2} = \cdots = a_{k,2k} = a_{k+1,1} = \cdots$$

But one of the entries $a_{1,k+1}$ or $a_{k+1,1}$ must be zero and the other must be one for A to be a dominance relation, so they could not thus be equated.

We now may as well banish these even-cycle cases altogether, replacing the coefficient $n(\pi)$ defined in (4) by

$$\bar{n}(\pi) = \begin{cases} 0, & \text{if } \pi \text{ has any cycle of even length} \\ n(\pi), & \text{otherwise}. \end{cases} \qquad (6)$$

As in several theorems of the former paper, the approach to evaluating $f(\pi)$ is through a concept that may roughly be described as "the number of degrees of freedom in a matrix scheme if the matrix is to be fixed under t_π." That is, for each permutation π we define a number $d(\pi)$ with the property that if you want to write down an arbitrary matrix with $t_\pi(A) = A$, $d(\pi)$ is the number of places in the matrix at which you can freely choose to write either a zero or a one. From this description of $d(\pi)$, it is evident that our $f(\pi) = 2^{d(\pi)}$. This gives us the final reduction

$$t(n) = \sum_{(\pi)} \bar{n}(\pi)\, 2^{d(\pi)}. \qquad (7)$$

Now it only remains to evaluate $d(\pi)$.

Theorem. The number of dominance structures on a set of n elements is given by formula (7) with

$$d(\pi) = \sum_{i=1}^{n} \frac{p_i}{2}(ip_i - 1) + \sum_{i<j} p_i p_j (i, j) \qquad (8)$$

where (i, j) is the greatest common divisor.

Proof. The argument here—we are only dealing with π whose cycles are all of odd length—parallels those of the theorems mentioned above. Its formulation rests on our choice, in each conjugate class, of a representative π for which it will be specially easy to compute $d(\pi)$, and further on our then splitting the matrix into "blocks" whose rows and columns will be permuted among themselves by individual cycles of π.

The right-hand sum in (8), then, is precisely the same as that part of the corresponding formulas in the theorems about symmetric and asymmetric relations: in all three cases this part arises from those blocks whose rows and columns are acted on by cycles of different lengths.

As for the left side: there are, for each i from 1 to n, $p_i(p_i - 1)$ blocks (previously called "near-diagonal") whose rows and columns are governed by different cycles of the same length. In each of these, when we write out the requirement that $t_\pi(A) = A$, we get i strings of equated entries (i entries per string), all of which can here be freely chosen so that A remains a dominance relation. But having made this set of choices for one such block, we have completely determined the choices in the "transpose block" (the reflection across the diagonal). Thus, from such blocks we get altogether $(ip_i/2)(p_i - 1)$ degrees of freedom.

Finally, the "diagonal blocks" are those whose rows and columns are acted on by the same cycle. Here a_{11}, for instance, is zero, while of the remaining entries in the first row we can freely choose just half before we completely determine the whole block. For $t_\pi(A) = A$ again gives i strings of equated entries, one consisting of diagonal terms and so permitting no choice, while of the remaining strings choice for any one determines that of its "transpose string." Thus for these blocks we have $p_i(i - 1)/2$ degrees of freedom; the sum of these expressions for near-diagonal and diagonal blocks then gives the left side of (8). A briefer and clearer form of (8) is easily seen to be

$$d(\pi) = \frac{1}{2}\left\{ \sum_{i,j=1}^{n} p_i p_j (i, j) - \sum_{i=1}^{n} p_i \right\}. \tag{9}$$

5. *An example.* Actual counting can be systematized in a tabular setting. Since conjugate classes in S_n correspond one-one to partitions of the integer n, we may list all these partitions in the first column (using the standard notation: e.g., (31^2) stands for the partition of 5 into three parts given by $5 = 3 + 1 + 1$). The second column repeats this information in terms of the non-zero cycle numbers. We next compute the denominator of the coefficient $s(\pi)$ as given by formula (1), while the fourth column gives $s(\pi)$ itself. The last column gives $d(\pi)$; this is computed on the basis of (8) in the lines beneath Table I. Below these we then write down the sum giving $st(n)$, as read off from formula (7) and this table. Note that the lemma above says we can discard all those partitions containing any even parts.

To compromise between complete transparency and sheer tedium, we cite in illustration the case $n = 5$; this is the first which was not previously analyzed by other means.

Granted there are 12 dominance structures on a set of five elements, can we say what they are like? Now while the methods above provide the only way known to the author of finding $st(n)$, they are not well adapted to determining which relations lie in what orbits. However, the theorem expressed in formula (2) rests on two hitherto unmentioned facts which simplify the task: (i) the set of all permutations leaving any relation, A, fixed is a group; (ii) the number of relations in the orbit of A is then given by the index of this subgroup in S_n.

Thus when $n = 5$ there are just three kinds of *fix-groups*, as we may call them: (i) those generated by 5-cycles, (ii) those generated by 3-cycles, and (iii) the group consisting of the identity permutation alone. Here, then, all orbits are of size 24, 40, or 120. (Note that members of the same orbit all have conjugate—but not identical—fix-groups, and also that the same group may fix relations in different orbits.)

The quick way now to a complete analysis of the structures is to use every available tool. In particular, the *structure sequences* described in section 2 afford valuable guideposts. Whenever we know that there is only

TABLE I

Partition	[illegible]	[illegible]	[illegible]	[illegible]
(5)	$p_5=1$	[illegible]	1/5	2
(41)	*	*	0	*
(32)	*	*	0	*
(31^2)	$p_3=1, p_1=2$	[illegible]	1/6	4
(2^21)	*	*	0	*
(21^3)	*	*	0	*
(1^5)	$p_1=5$	[illegible]	1/120	10

The sum is

$$d(5) = \tfrac{1}{2}(5-1) + 0 = 2\,,$$

$$d(31^2) = \tfrac{1}{2}(3-1) + \tfrac{2}{2}(2-1) + 2 = 4\,,$$

$$d(1^5) = \tfrac{5}{2}(5-1) + 0 = 10\,,$$

$$sd(5) = \tfrac{1}{5}(4) + \tfrac{1}{6}(16) + \tfrac{1}{120}(1024) = 12\,.$$

one orbit with a certain structure sequence, we will know all about the orbit: just write down a relation (or graph) with that sequence to represent the orbit and describe its structure. Now for $n = 5$ there are only nine structure sequences as against twelve orbits. However, combining structure-sequence arguments with those on fix-groups and orbits, we can derive Table II.

Thus the structure sequences are sufficient to determine seven of our twelve orbits unambiguously, and our description will be complete upon examination of the remaining cases. (Of course, this is the reverse of the *investigating* process; there you must first determine which are the ambiguous cases.)

Now, let

$$A_7=\begin{pmatrix}0&1&1&1&0\\0&0&1&1&1\\0&0&0&1&0\\0&0&0&0&1\\1&0&1&0&0\end{pmatrix}\text{ and }A_8=\begin{pmatrix}0&1&1&1&0\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\\1&0&0&0&0\end{pmatrix}.$$

Inspection shows both these matrices have structure sequence (3, 3, 2, 1, 1); further, since neither is fixed under any nontrivial permutation, both lie in 120-element orbits. That these must be different orbits is seen from the fact that a "man" dominating only one other dominates a "leader" in A_8, while this is not so for A_7. Hence we can take A_7 and A_8 to represent the seventh and eighth orbits.

If we further define

$$A_9=\begin{pmatrix}0&0&1&1&1\\1&0&1&0&0\\0&0&0&1&1\\0&1&0&0&1\\0&1&0&0&0\end{pmatrix},\quad A_{10}=\begin{pmatrix}0&1&1&0&1\\0&0&1&1&0\\0&0&0&1&1\\1&0&0&0&1\\0&1&0&0&0\end{pmatrix},\quad A_{11}=\begin{pmatrix}0&0&1&1&0\\1&0&0&1&0\\0&1&0&1&0\\0&0&0&0&1\\1&1&1&0&0\end{pmatrix},$$

we are faced with three relations whose sequence is (3, 2, 2, 2, 1). It can be argued as above that A_9 and A_{10} belong to 120-element orbits, A_{11}, on

TABLE II

Orbit	No. of Relations	Structure Sequence
1	120	(4, 3, 2, 1, 0)
2	40	(4, 3, 1, 1, 1)
3	40	(4, 2, 2, 2, 0)
4	120	(4, 2, 2, 1, 1)
5	40	(3, 3, 3, 1, 0)
6	120	(3, 3, 2, 2, 0)
7	120	(3, 3, 2, 1, 1)
8	120	
9	120	(3, 2, 2, 2, 1)
10	[illegible]	
11	40	
12	24	(2, 2, 2, 2, 2)

the other hand, is fixed under the group {1, (123), (132)} and so belongs to a 40-element orbit. (Thus A_{11} is more symmetric than the others in that all the "middle men" in A_{11} play similar roles.) That A_9 and A_{10} are further not isomorphic to each other is clearest, again, in simple combinatorial terms. In both the leader is dominated by one middle man but in A_9 that middle man picks up his other domination at the expense of a second middle man, whereas in A_{10} the middle man dominating the leader also dominates the subordinate, and does not dominate either of the other middle men. These three relations, then, can be taken to represent the orbits 9, 10, and 11.

Finally, Table III compares the number, Dom(n), of dominance rela-

TABLE III

n	Dom(n)	$st(n)$
2	2	1
3	8	2
4	64	4
5	1,024	12
6	32,768	56
7	2,097,152	456
8	268,435,456	6,880

tions with $st(n)$ for $n \leq 8$. These were quite easy pencil-and-paper computations, and despite the rapid increase of the number of partitions I believe their number could easily be doubled with a desk computer.

Landau's inequality (1951a),

$$st(n) \geq \frac{2^{n(n-1)/2}}{n!},$$

gives not only a lower bound but a very good approximation as soon as n is moderately large. For example, for $n = 8$, $st(8) = 6880$, while $2^{28}/8!$ is 6658 to the nearest integer. From the viewpoint of this paper, this approximation means using only the partition (1^n). If to this is added the term for the partition (31^{n-3}), we have a much better approximation, namely,

$$\frac{2^{[illegible]}}{n!} + \frac{2^{[illegible]}}{(n-3)!\,3}.$$

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