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R. Austin, R. K. Guy,

& R Nowakowski;

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A 629
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Richard Austin, Richard Guy and Richard Nowakowski
unpublished notes, 1987.

J. Schaer asked us how the sequence 1, 5, 25, ... continued. Luckily we had seen the next term, 149, on his blackboard, so were able to look in Sloane [ ] and find sequences #1622, #1623 and give him the further terms 1081, 9365, ... However, the references given there, Steffensen [ ] and Simmons [ ], did not mention the problem from which the sequence arose. P. Zvengrowski had asked how many simplices there were in a barycentric subdivision of a d-dimensional simplex. Figure 1 illustrates $0 \le d \le 3$.

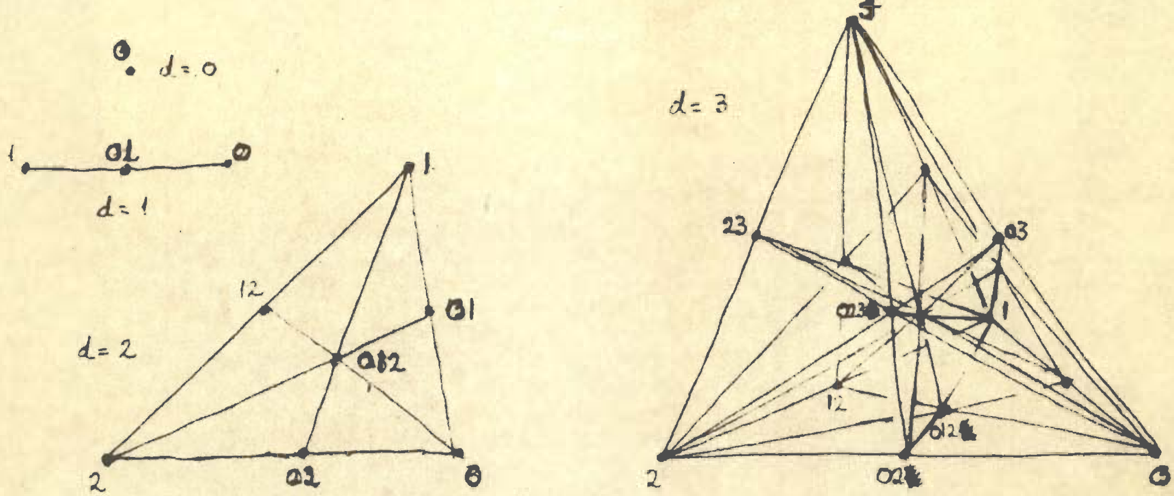

Figure 1

Denote by $s_{d,k}$ the number of k-dimensional simplices in a barycentric subdivision of a d-dimensional simplex. Then, by counting, we discover the following values of $s_{d,k}$ from Figure 1:

	k=0	k=1	k=2	k=3
d=0	1			
d=1	3	2		
d=2	7	12	6	
d=3	15	50	60	24

In order to extend this table we need a systematic way of counting, or better, a recurrence relation, such as

(1) $$s_{d,k} = (k+1)\,s_{d-1,k-1} + (k+2)\,s_{d-1,k}$$

We establish this by noting that when we add a vertex $\underline{d}$ to the ($\underline{d}$-1)-dimensional simplex with vertices $0,1,2,\dots,d-1$, to produce a $\underline{d}$-dimensional simplex, the $\underline{k}$-dimensional simplices that are formed are of two kinds. They are either simplices which already existed in the $(d-1)$-dimensional simplex, or have such a simplex as their projection from $\underline{d}$, or they are subdivisions of the k-D simplex formed by joining $\underline{d}$ to a ($\underline{k}$-1)-dimensional simplex in the original. There are $1+(k+1)$ of each of the former kind, and $k+1$ of each of the latter. For example, if $d=3$, $k=2$, we may start from any one of the six triangles for $d=2$, say 0, 02, 012, where 02 is the midpoint of the join of 0 to 2, 012 the centroid of the face through 0, 1 and 2, etc. Change the vertex with the longest label, 012, by adding the new dimension, giving 0123 and the triangle 0, 02, 0123. Next augment the next longest label to give 0, 023, 0123. Finally augment the remaining label to give 03, 023, 0123. Since a $\underline{k}$-D simplex has $k+1$ vertices, this process lists $k+1$ simplices in addition to the initial simplex. The second kind of triangle is exemplified by joining $\underline{d}=3$ to a (k-1)-D simplex, say 2, 012. This k-D simplex is made up of $k+1$ elementary simplices, which may be counted as before: 2, 012, 0123; 2, 23, 0123; 3, 23, 0123.

We can now use (1) to extend the table, and to calculate

(2) $$s_d = \sum_{k=0}^{d} s_{d,k},$$

the sequence originally asked for:

2050 1, 5, 25, 149, 1081, 9365, 94585, 1091669, 14174521, 204495125, 3245265145, 56183135189, 1053716696761, 21282685940885, ...

In order that (1) should hold for values of k other than $0 \le k \le d$, we define $s_{d,-1} = 1$ $(d \geq -1)$ and $s_{d,k} = 0$ for $k < -1$ or $k > d$.

1

1 1

1 3 2

1 7 12 6

1 15 50 60 24

1 31 180 390 360 120

1 63 602 2100 3360 2520 720

1 127 1932 10206 25200 31920 20160 5040

1 255 6050 46620 166824 317520 332640 181440 40320

1 511 18660 204630 1020600 2739240 4233600 3780000 1814400 362880

1 1023 57002 874500 5921520 21538440 46070640 59875200 46569600 19958400 3628800

The following formulas may be deduced from (1) :–

(3) $$s_{d,d} = (d+1)!$$

(4) $$s_{d,d-1} = \tfrac{1}{2}(d+2)!$$

(5) $$s_{d,d-2} = (d+2)!\,(3d+1)/24$$

(6) $$s_{d,d-3} = (d+2)!\,d(d+1)/48$$

(7) $$s_{d,d-4} = (d+2)!\,(15d^3-60d^2+65d-12)/5760$$

(8) $$s_{d,d-5} = (d+2)!\,(d-3)(d-2)(3d^2-11d+4)/11520$$

(9) $$s_{d,d-6} = (d+2)!\,(63d^5-945d^4+5355d^3-13951d^2+15806d-5304)/2903040$$

where (3) and (4) are #659 and #1179 in Sloane [ ]. We also have

$s_{d,d-j}$ is $(d+1)!$ times a polynomial of degree j in d, whose leading coefficient is $1/2^j j!$

(10) $$s_{d,0} = 2^{d+1}-1,$$

(11) $$s_{d,1} = 3^{d+1}-2^{d+2}+1,$$

(12) $$s_{d,2} = 4^{d+1}-3^{d+2}+3\cdot 2^{d+1}-1,$$

(13) $$s_{d,3} = 5^{d+1} - 4^{d+2} + 2.3^{d+2} - 2^{d+3} + 1,$$

where (10) are the Mersenne numbers, # 1059 in Sloane [ ]. J. Selfridge has pointed out that Schaer's original question is supposedly answered by

(14) $$5^{d} = s_{d,0} + s_{d,1} + s_{d,2} + s_{d-1,3}.$$

Formulae (10) to (13) are particular cases of

(15) $$s_{d,k} = \sum_{r=0}^{k+1} (-)^{r} \binom{k+1}{r} (k-r+2)^{d+1}.$$

In fact, the numbers $s_{d,k}$ are closely related to the Stirling numbers of the second kind:

(16) $$s_{d,k} = (k+1)!\, S_{d-2}^{(k+2)}.$$

These are well-known [ ] as occurring in various combinatorial contexts, in particular $S_{d-2}^{(k+2)}$ is the number of ways of partitioning a set of d-2 elements into k+2 non-empty subsets, and the formulae corresponding to (1) and (15) are

(17) $$S_{d-2}^{(k+2)} = (k+2) S_{d-3}^{(k+2)} + S_{d-3}^{(k+1)},$$

(18) $$S_{d-2}^{(k+2)} = \sum_{r=0}^{k+1} \frac{(-)^{r} (k-r+2)^{d+1}}{r!\,(k-r+1)!}.$$

~~$s_{n,j} = s_{d+1,k+1} = (k+2)s_{d,k} + (k+3)s_{d,k+1} = (j+1)s_{n,j-1} + (j+2)s_{n,j}$~~ ~~$n = d+1$, $j = k+1$~~

We can give a meaning to the definition of $s_{d,-1} = 1$ $(d \geq -1)$ by allowing it to stand for the empty set, of dimension $-\infty$. A similar device is used by Simmons [6] in considering a family of combination locks, where ~~none of $k+1$ switches out of $d+1$ are closed.~~ ~~[illegible]~~ $k+1$ switches out of a possible $n+1$ are closed, this being the case $k=-1$. It was his occasional inconsistency in including this one or excluding it that gave rise to the erroneous entry #1623 in Sloane; the sixth to the ninth terms (but not the tenth) should each be reduced by one; i.e. all terms are odd.

Simmons gives the following formula

$$(19) \qquad s_{d-1} + 1 = \left(\frac{1}{2e^{-x}-1}\right)^{(d)}_{x=0},$$

i.e. the dth derivative of $(2e^{-x}-1)^{-1}$ evaluated at $x=0$, from which we may obtain the generating function

$$\sum_{d=0}^{\infty} s_d x^d$$

$$= 1 + 5x + 25x^2 + 149x^3 + 1081x^4 + 9365x^5 + \cdots$$

$$= 2 + 6x + 26x^2 + 150x^3 + 1082x^4 + 9366x^5 + \cdots \quad - 1/(1-x)$$

?

$$(20) \qquad = \left(1/(2e^{-x}-1) - 1\right)\Big/x - 1/(1-x).$$

If we write $\left(\frac{1}{2e^{-x}-1}\right)^{(d)}$ as $\dfrac{a_{d,0} + a_{d,1}e^{-x} + a_{d,2}e^{-2x} + \cdots + a_{d,d}e^{-nx}}{(2e^{-x}-1)^{d+1}}$, then

$$(21) \qquad a_{d,k} = k\,a_{d-1,k} + 2(d+1-k)\,a_{d-1,k-1},$$

a relation similar to (1), which with appropriate boundary conditions, leads to the array

$2(d+1-k)$: 4, 6, 8, 10

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
$d=0$	1							$s_{-1}+1 = 1$
$d=1$	0	2						$s_0+1 = 2$
$d=2$	0	2	4					$s_1+1 = 6$
$d=3$	0	2	16	8				$s_2+1 = 26$
$d=4$	0	2	44	88	16			$s_3+1 = 150$
$d=5$	0	2	104	528	416	32		
$d=6$	0	2	228	2416	4832	1824	64	

It will be noted that

(22) $$a_{d,k} = 2^k c_{d,k}$$

where $c_{d,k}$ are the Eulerian numbers

	$k=0$	$k=1$	$k=2$	$k=3$						
$d=0$	1									1
$d=1$	0	1								1
$d=2$	0	1	1							2
$d=3$	0	1	4	1						6
$d=4$	0	1	11	11	1					24
$d=5$	0	1	26	66	26	1				120
$d=6$	0	1	57	302	302	57	1			720
$d=7$	0	1	120	1191	2416	1191	120	1		5040
$d=8$	0	1	247	4293	15619	15619	4293	247	1,	40320

sequences #1382, #2047, #2255, #2310, #2336, #2355, #2366 in Sloane [7], and

(23) $$c_{d,0} = 0 \quad (d \geq 1),$$

(24) $$c_{d,1} = c_{d,d} = 1 \quad (d \geq 1),$$

(25) $$c_{d,2} = c_{d,d-1} = 2^d - (d+1) \quad (d \geq 0)$$

(26) $$c_{d,3} = c_{d,d-2} = 3$$

(27) $$\sum_{k=1}^{d} c_{d,k} = d!$$

When the corrections are made to sequence # 1623 in Sloane, it is found to coincide with # 1622. Here the sequence arose in an actuarial context; Steffenson [6] wrote ~~(s_d as $\left[\begin{smallmatrix} d+2 \\ 1 \end{smallmatrix}\right]_{\theta=0}$ and s_d+1 as $\left[\begin{smallmatrix} d+2 \\ 0 \end{smallmatrix}\right]_{\theta=0}$, where the symbols $\left[\begin{smallmatrix} n \\ \nu \end{smallmatrix}\right]$ were defined by)~~

$$\begin{bmatrix} n \\ \nu \end{bmatrix} = \sum_{s=\nu}^{n-1} \frac{\left|\Delta^{s+1}\theta^n\right|}{(s+1)!}(s+\theta)^{s-\nu} . \tag{28}$$

These polynomials in θ occurred as coefficients in the evaluation of an integral connected with mortality tables. He calculated them by the recurrence

$$\begin{bmatrix} n \\ \nu-1 \end{bmatrix} = \frac{\left|\Delta^{\nu}\theta^n\right|}{\nu!} + (\nu+\theta)\begin{bmatrix} n \\ \nu \end{bmatrix} \tag{29}$$

with boundary condition $\begin{bmatrix} n \\ n-1 \end{bmatrix} = 1$. The sequence $\{s_d\}$ is the value of $\begin{bmatrix} d+2 \\ 1 \end{bmatrix}$ at $\theta = 0$, while $s_d + 1 = \begin{bmatrix} d+2 \\ 0 \end{bmatrix}_{\theta=0}$.

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5 John Riordan, An introduction to combinatorial analysis, Wiley, NY, 1958, 76, 91,

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8 J.F. Steffenson, On a class of polynomials and their application to actuarial problems, Skand. Aktuarietidskr., 11 (1928) 75–97.

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d	$s_{d,d-2}$ (cf #1873)	$s_{d,d-3}$ (cf #2143)	$s_{d,d-4}$ ($(d-3)!\cdot(\#2272)\,\mathcal{S}^{d-3}_{d-2}$)	$s_{d,d-5}$	$s_{d,d-6}$
1	1				
2	7	1			
3	50	15	1		
4	390	180	31	1	
5	3360	2100	602	63	1
6	31920	25200	10206	1932	127
7	332640	317520	166824	46620	6050
8	3780000	4233600	2739240	1020600	204630
9	46569600	59875200	46070640	21538440	5921520
10	618710400	898128000	801496080	451725120	158838240
11	8821612800	14270256000	14495120640	9574044480	4115105280
12	134399865600	239740300800	273158645760	207048441600	105398092800
13	2179457280000	4249941696000	5368729766400	4595022432000	2706620716800
14	37486665216000	79332244992000	110055327782400	105006251750400	70309810171200
15	681734237184000	1556132497920000	2351983118284800	2475732702643200	1858166876966400
16	13071512982528000	32011868528640000			
		689322235650048000			

5460	5461	5462	5463	5464
(5)	(6)	(7)	(8)	(9)
✓	✓	✓	✓	✓

? → NEW ASPECT OF

neither < ? → A ~~NOT~~ SLOANE SEQUENCE

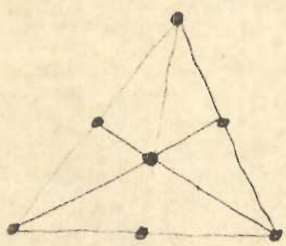
?

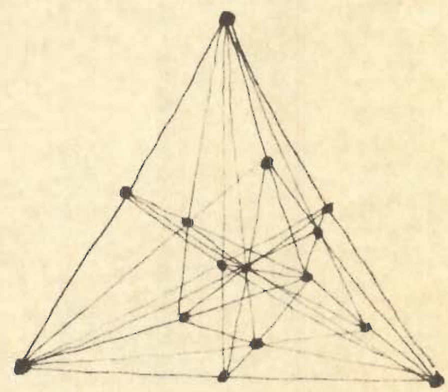
Richard Austin, Richard Guy and Richard Nowakowski

What is the number, s_d, of simplices in a barycentric subdivision of a d-dimensional simplex? Clearly

$$s_d = \sum_{k=0}^{d} s_{d,k}$$

where $s_{d,k}$ is the number of k-dimensional simplices in a barycentric subdivisional of a d-dimensional simplex.





Define $s_{-1,k}$ 1 $s_{d,d+1}$ 0

$s_{d,k}$ 0

$d < -1$

$k > d$

$s_{0,0}$

$s_{1,0}$ $s_{1,1}$

$s_{2,0}$ $s_{2,1}$ $s_{2,2}$

$s_{3,0}$ $s_{3,1}$ $s_{3,2}$ $s_{3,3}$

0 1

$\times 2$ $\times 3$

0 1 1 0 $\times 4$ 1

1 3 2 0 $\times 5$ 5

1 7 12 6 0 $\times 6$ 25

1 15 50 60 24 0 $\times 7$ 149

1 31 180 390 360 120 0 $\times 8$ 1081

1 63 602 2100 3360 2520 720 0 9365

1 127 1932 10206 25200 31920 20160 5040 0 94585

1 255 6050 46620 166824 317520 332640 181440 40320 0 1091669

1 511 18660 204630 1020600 2739240 4233600 3780000 1814400 362880 0 14174521

main seq A2050

add 1 A629

1, 2, 6, 26, 150, 1082, 9366, 94586, 1091670, 14174522

$$s_{d,k} = (d+1)\, s_{d-1,k-1} + (d+2)\, s_{d-1,k}$$

1023
57002
874500
5921520
21538440
46070640
59875200
46569600
19958400
3628800

204495125