

Pi in Golden Ratio Base Phi

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Abstract

π and ϕ (Golden Ratio) have fascinated mathematicians, scientists and artists for many centuries. Both are irrational numbers, but still related to integers in many ways. This article proves a new BBP-like formula for π in base ϕ .

1 Formula

$$\pi = \frac{4}{\phi} + \sum_{n=0}^{\infty} \frac{1}{\phi^{12n}} \left[\frac{8\phi^{-3}}{12n+3} + \frac{4\phi^{-5}}{12n+5} - \frac{4\phi^{-7}}{12n+7} - \frac{8\phi^{-9}}{12n+9} - \frac{4\phi^{-11}}{12n+11} + \frac{4\phi^{-13}}{12n+13} \right] \quad (1)$$

where

$$\phi = \frac{1 + \sqrt{5}}{2} \quad (2)$$

The equation 1 gives value of π sequentially from left to right providing increasing precision of 12 fractional places in base ϕ with each iteration.

2 Identity

equation 1 is derived from the following geometric relation between π and ϕ

$$\frac{\pi}{4} = \arctan\left(\frac{1}{\phi}\right) + \arctan\left(\frac{1}{\phi^3}\right) \quad (3)$$

We will first prove the equation 3 and then derive equation 1

3 Proof

Formula for addition of tan is

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)} \quad (4)$$

tan of right-hand side of equation 3 gives

$$= \frac{\frac{1}{\phi} + \frac{1}{\phi^3}}{1 - \frac{1}{\phi} \frac{1}{\phi^3}} \quad (5)$$

simplifying further

$$= \frac{\phi^3 + \phi}{\phi^4 - 1} \quad (6)$$

simplifying further

$$= \frac{\phi(\phi^2 + 1)}{(\phi^2 - 1)(\phi^2 + 1)} \quad (7)$$

since $\phi^2 - 1 = \phi$ hence

$$= \frac{\phi(\phi^2 + 1)}{\phi(\phi^2 + 1)} \quad (8)$$

simplifying further

$$= 1 = \tan\left(\frac{\pi}{4}\right) \quad (9)$$

which is left-hand side of equation 3. Hence proving equation 3.

4 Derivation

Taylor series for $\arctan(x)$ is

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1} \quad (10)$$

substituting $\frac{1}{\phi}$ and $\frac{1}{\phi^3}$ in equation 10 gives

$$\begin{aligned} \arctan\left(\frac{1}{\phi}\right) &= \frac{1}{\phi} - \frac{1}{3\phi^3} + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} + \frac{1}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} \\ &\quad - \frac{1}{15\phi^{15}} + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} + \frac{1}{21\phi^{21}} - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \dots \end{aligned} \quad (11)$$

$$\begin{aligned} \arctan\left(\frac{1}{\phi^3}\right) &= \frac{1}{\phi^3} - \frac{1}{3\phi^9} + \frac{1}{5\phi^{15}} - \frac{1}{7\phi^{21}} + \frac{1}{9\phi^{27}} - \frac{1}{11\phi^{33}} + \frac{1}{13\phi^{39}} \\ &\quad - \frac{1}{15\phi^{45}} + \frac{1}{17\phi^{51}} - \frac{1}{19\phi^{57}} + \frac{1}{21\phi^{63}} - \frac{1}{23\phi^{69}} + \frac{1}{25\phi^{75}} - \dots \end{aligned} \quad (12)$$

adding and arranging equation 11 and equation 12 gives

$$\begin{aligned} \frac{\pi}{4} &= \frac{1}{\phi} + \left(\frac{1}{\phi^3} - \frac{1}{3\phi^3}\right) + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} - \left(\frac{1}{3\phi^9} - \frac{1}{9\phi^9}\right) - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} \\ &\quad + \left(\frac{1}{5\phi^{15}} - \frac{1}{15\phi^{15}}\right) + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} - \left(\frac{1}{7\phi^{21}} - \frac{1}{21\phi^{21}}\right) - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \dots \end{aligned} \quad (13)$$

combining equal ϕ power terms from equation 13 gives

$$\begin{aligned} \frac{\pi}{4} &= \frac{1}{\phi} + \frac{2}{3\phi^3} + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} - \frac{2}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} \\ &\quad + \frac{2}{15\phi^{15}} + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} - \frac{2}{21\phi^{21}} - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \dots \end{aligned} \quad (14)$$

multiplying both sides by 4 and arranging equation 14 in groups of 6 terms gives

$$\begin{aligned} \pi &= \frac{4}{\phi} + \left(\frac{2}{3\phi^3} + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} - \frac{2}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}}\right) \\ &\quad + \frac{1}{\phi^{12}} \left(\frac{2}{15\phi^3} + \frac{1}{17\phi^5} - \frac{1}{19\phi^7} - \frac{2}{21\phi^9} - \frac{1}{23\phi^{11}} + \frac{1}{25\phi^{13}}\right) - \dots \end{aligned} \quad (15)$$

which represented as series gives us equation 1.

$$\pi = \frac{4}{\phi} + \sum_{n=0}^{\infty} \frac{1}{\phi^{12n}} \left[\frac{8\phi^{-3}}{12n+3} + \frac{4\phi^{-5}}{12n+5} - \frac{4\phi^{-7}}{12n+7} - \frac{8\phi^{-9}}{12n+9} - \frac{4\phi^{-11}}{12n+11} + \frac{4\phi^{-13}}{12n+13} \right] \quad (16)$$

5 References

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