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The Number of Graded Partially Ordered Sets

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ABSTRACT

We find an explicit formula for the number of graded partially ordered sets of rank h that can be defined on a set containing n elements. Also, we find the number of graded partially ordered sets of length h, and having a greatest and least element that can be defined on a set containing n elements. The first result provides a lower bound for $G^*(n)$, the number of posets that can be defined on an n-set; the second result provides an upper bound for the number of lattices satisfying the Jordan-Dedekind chain condition that can be defined on an n-set.

INTRODUCTION

The terminology we will use in connection with partially ordered sets is defined in detail in the first few pages of Birkhoff [1]; however, we need a few concepts not defined there. A *rank function* g maps the elements of a poset P into the chain of integers such that (i) $x > y$ implies $g(x) > g(y)$, and (ii) $g(x) = g(y) + 1$ if x covers y. A poset P is *graded* if at least one rank function can be defined on it; Birkhoff calls the pair (P, g) a graded poset, where g is a particular rank function defined on a poset P, so our definition differs from his. A *path* in a poset P is a sequence $(x_1, ..., x_n)$ such that x_i covers or is covered by x_{i+1}, for $i = 1, ..., n - 1$; P is *connected* if every pair of elements in P occur in some path in P. A *component* of P is a maximal subposet of P which is connected; thus, every poset P decomposes into unique non-empty components. Suppose $P = P_1 \cup \cdots \cup P_c$ is a graded poset with components $P_1, ..., P_c$, and suppose g and h are rank functions defined on P with g_i and h_i the restriction of g and h to P_i, respectively, for $i = 1, ..., c$. Now it is easy

not all posets can be graded

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to show that there exist integers j_i such that $j_i + g_i(x) = h_i(x)$, for all $x \in P_i$, $i = 1,..., c$. Thus,

$$R(P) = \max\{g(x) - g(y): x, y \in P_i, \qquad i = 1,..., c\}$$

is independent of the rank function g defined on P. The invarient $R(P)$ is the *rank* of a graded poset P.

An unsolved problem in combinatorial analysis asks for $G^*(n)$, the number of partial order relations that can be defined on an n-set; this problem is trivial unless it is understood that "enumeration" means we must describe how to calculate $G^*(1)\,x + G^*(2)\,x^2 + \cdots$ in terms of specified power series and operations defined in the ring of formal power series or some equivalent system. So far no reasonable bounds have been given for the magnitude of $G^*(n)$.[1] In the next section we determine the number of graded posets on a given n-set and thereby obtain a lower bound for $G^*(n)$. In the last section we find the number of graded posets on an n-set with a greatest and a least element such that exactly p elements cover the least element. This result gives an upper bound for the number of lattices that satisfy the Jordan-Dedekind chain condition that can be defined on an n-set, and it could be used to find an upper bound for the number of geometries having exactly p points.

In an earlier paper [2] we studied sums having the form

$$\sum f(n_1, n_2) f(n_2, n_3) \cdots f(n_{i-1}, n_i)\, g(n_i), \tag{1}$$

where $\{f(m, n)\}$ and $\{g(n)\}$ are given sets of numbers, and the sum extends over all compositions $(n_1,..., n_i)$ of n into an unrestricted number of positive parts. Sums having the form of (1) also appear here, but the generating functions of the appropriately defined sets $\{f(m, n)\}$ and $\{g(n)\}$ do not converge, so the theory developed involving an integral equation does not apply. However, the recurrence relations we gave in [2] can be used effectively to shorten the labor of computation involved in evaluating (1). With a simple change in notation the recurrence relations can be given as follows: Let

$$\{u(i, j):\ i, j = 0,\ 1,...\} \qquad \text{and} \qquad \{v(i):\ i = 0,\ 1,...\}$$

be given sets of numbers and define

$$t_h(p, n) = \sum \binom{n}{n_1, ..., n_h} u(n_1, n_2)\, u(n_2, n_3) \cdots u(n_{h-1}, n_h)\, v(n_h), \tag{2}$$

[1] D. Kleitman at Massachusetts Institute of Technology has evidently solved this problem, but so far his results are available only in the form of lecture notes.

where the sum extends over all compositions $(n_1, \ldots, n_h)$ of n into exactly h non-negative parts with $n_1 = p$. From this definition it follows that $t_1(p, n) = v(n)$ if $p = n$, $t_1(p, n) = 0$ if $p \neq n$, and

$$t_h(p, n) = \binom{n}{p} \sum_{k=0}^{n-p} u(p, k)\, t_{h-1}(k, n-p), \qquad h = 2, 3, \ldots . \tag{3}$$

Graded Posets

We are going to define a set $C(h, X)$ by constructing its elements and at the same time prove that the number of elements in the set is

$$c(h, n) = \sum \binom{n}{n_1, \ldots, n_h} 2^{n_1 n_2 + \cdots + n_{h-1} n_h}, \tag{4}$$

where the sum extends over all compositions $(n_1, \ldots, n_h)$ of n into exactly h non-negative parts. Note that this sum has the form of (2) if we sum the latter over p. Exhaustive disjoint subsets of $C(h, X)$ correspond to each of the compositions $(n_1, \ldots, n_h)$ of n into exactly h non-negative parts; the elements of these subsets of $C(h, X)$ are constructed as follows: Consider each of the $\binom{n}{n_1, \ldots, n_h}$ h-tuples $(X_1, \ldots, X_h)$ consisting of subsets of an n-set X such that $|X_i| = n_i$, and $X_i \cap X_j = \phi$ whenever $i \neq j$. There are

$$2^{n_1 n_2 + \cdots + n_{h-1} n_h}$$

subsets R of the set

$$\{(x, y): x \in X_i, y \in X_{i+1}, \qquad i = 1, \ldots, h-1\}$$

and each R together with the h-tuple $(X_1, \ldots, X_h)$ constitutes an element of $C(h, X)$; that is,

$$C(h, X) = \{(X_1, \ldots, X_h : R)\}.$$

It is evident from this construction that the number of elements in $C(h, X)$ is given by (4).

Let $B(h, X)$ be the set of all graded posets on an n-set X with rank less than h, for $h = 1, 2, \ldots$; also, let $b(h, n)$ be the number of elements in $B(h, X)$ for $h = 1, 2, \ldots$. (Thus, $b(h, 0) = 1$, and $b(h, n) = b(h', n)$ when $h, h' > n > 0$). Now an element $(X_1, \ldots, X_h : R)$ of $C(h, X)$ can be uniquely interpreted as an h-tuple $(S_1, \ldots, S_h)$ of posets $S_i \in B(i, Y_i)$, $|Y_i| = s_i$, where $S_i \cap X_i \neq \phi$ or $S_i = \phi$, $S_i \subseteq X_i \cup \cdots \cup X_h$ for $i = 1, \ldots, h$, and $(s_1, \ldots, s_h)$ is a composition of n into exactly h non-negative parts. Furthermore, each h-tuple $(S_1, \ldots, S_h)$ of posets $S_i \in B(i, Y_i)$ corresponds to some element of $C(h, X)$ in this way. In Figure 1, a 4-tuple of elements from $B(4, X_4)$, $B(3, X_3)$, $B(2, X_2)$, and $B(1, X_1)$, respectively,

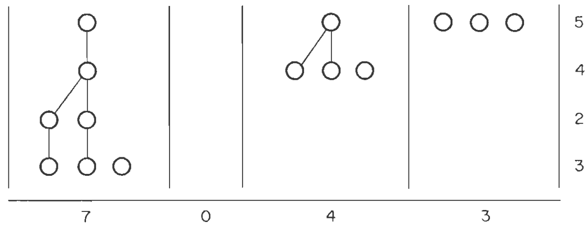

FIGURE 1

is shown with $|X_4| = 7$, $|X_3| = 0$, $|X_3| = 4$, $|X_1| = 3$; this 4-tuple arises from an element of $C(4, X)$, $X = X_1 \cup X_2 \cup X_3 \cup X_4$, which corresponds to the composition (3, 2, 4, 5) of 14. The labeling of the nodes has not been indicated in the diagram. Using the construction just described we find that

$$c(h, n) = \sum \binom{n}{s_1, \ldots, s_h} b(1, s_1)\, b(2, s_2) \cdots b(h, s_h), \tag{5}$$

where the sum extends over all compositions $(s_1, \ldots, s_h)$ of n into exactly h non-negative parts. A proof of (5) in terms of diagrams can be given as follows:

First, we describe how to construct diagrams representing elements of $C(h, X)$. Let $(i, 1)$, $(i, 2), \ldots$ be the i-th row of the plane, and suppose $X = \{1, \ldots, n\}$. Given an h-tuple $(X_1, \ldots, X_h)$ consisting of exhaustive, disjoint subsets of X, we label the points $(i, 1)$, $(i, 2), \ldots$ with the elements $x_{i1}, x_{i2}, \ldots$, respectively, where $X_i = \{x_{i1} < x_{i2} < \cdots\}$, for $i = 1, \ldots, h$. Now the elements of $C(h, X)$ corresponding to the h-tuple $(X_1, \ldots, X_h)$ may be represented as graphs obtained by connecting some of the labeled elements in the i-th row to some of the labeled elements in the $(i + 1)$-st row, for $i = 1, \ldots, h - 1$. Now suppose D is a diagram corresponding to an element of $C(h, X)$. The elements of D connected to elements in the first row have labels belonging to a subset Y_h of X, and D_h, the restriction of D to Y_h, is the Hasse diagram of an element of $B(h, Y_h)$. Let $D - D_h$ be the restriction of D to $X - Y_h$. The elements of $D - D_h$ connected to elements in the second row have labels belonging to a subset Y_{h-1} of $X - Y_h$, and D_{h-1}, the restriction of D to Y_{h-1}, is the Hasse diagram of an element of $B(h - 1, Y_{h-1})$. We can continue in this way to find that each element C of $C(h, X)$ corresponds in a unique way to an h-tuple

$(S_1, ..., S_h)$ with $S_i \in B(i, Y_i)$, where $(Y_1, ..., Y_h)$ is an h-tuple of exhaustive, disjoint subsets of X which depends on C. It is also clear that the Hasse diagrams of an h-tuple $(S_1, ..., S_h)$, with $S_i \in B(i, Y_i)$ and $(Y_1, ..., Y_h)$ an h-tuple of exhaustive, disjoint subsets of X, can be drawn in a prescribed way so that a unique element of $C(h, X)$ is obtained. This indicates a one-one correspondence between the elements of $C(h, X)$ and the set of all h-tuples $(S_1, ..., S_h)$ with $S_i \in B(i, Y_i)$ where $(Y_1, ..., Y_h)$ ranges over all h-tuples of exhaustive, disjoint subsets of X.

Now we invert the relation in (5) to find $b(h, n)$ in terms of the numbers $c(i, j)$. Let

$$B_h(x) = \sum_{n=0}^{\infty} \frac{b(h, n)}{n!} x^n, \qquad C_h(x) = \sum_{n=0}^{\infty} \frac{c(h, n)}{n!} x^n, \tag{6}$$

$$\frac{1}{C_h(x)} = \sum_{n=0}^{\infty} \frac{d(h, n)}{n!} x^n,$$

then (5) implies

$$C_h(x) = B_1(x) B_2(x) \cdots B_h(x). \tag{7}$$

Thus, $B_1(x) = C_1(x)$, and for $h > 1$,

$$B_h(x) = C_h(x)/C_{h-1}(x). \tag{8}$$

In the ring of formal power series it is possible to find $1/A(x)$ in terms of $A(x) = a(0) + a(1) x + \cdots$ provided $a(0) \neq 0$; in fact, if $a(0) = 1$,

$$\frac{1}{A(x)} = 1 + \sum_{n=1}^{\infty} \left\{ \sum (-1)^k a(n_1) a(n_2) \cdots a(n_k) \right\} x^n, \tag{9}$$

where the inner sum extends over all compositions $(n_1, ..., n_k)$ of n into an unrestricted number of positive parts. Using this result we have, for $h \geqslant 1$,

$$d(h, n) = \sum (-1)^k \binom{n}{n_1, ..., n_k} c(h, n_1) c(h, n_2) \cdots c(h, n_k), \tag{10}$$

type 1

where the index of summation is the same as that defined for the inner sum in (9). Now (8) implies

$$b(h, n) = \sum_{k=0}^{n} \binom{n}{k} c(h, k) d(h-1, n-k), \tag{11}$$

type 1

so that (4), (10), and (11) provide a formula for calculating the numbers

$b(i, j)$. By definition, $b(n, n)$ is the total number of graded posets that can be defined on an n-set since the rank of a partially ordered set of order n cannot exceed $n-1$. Also, $a(h, n) = b(h, n) - b(h-1, n)$ is the number of graded partially ordered sets of order n and rank $h-1$. Figure 2 indicates the elements of $B(4, X)$, $|X| = 4$.

h/n				$c(h, n)$		
1	0	0	0	0	0	0
1	1	1	1	1	1	1
1	2	6	26	162	1442	18306
1	3	13	81	721	9153	165313
1	4	22	166	1726	24814	494902
1	5	33	287	3309	50975	1058493
1	6	46	450	5650	91866	1957066
1	7	61	661	8953	152917	3334921

h/n				$b(h, n)$		
1	1	1	1	1	1	1
1	1	3	13	87	841	11643
1	1	3	19	195	3031	67263
1	1	3	19	219	3871	99063
1	1	3	19	219	3991	105903
1	1	3	19	219	3991	106623

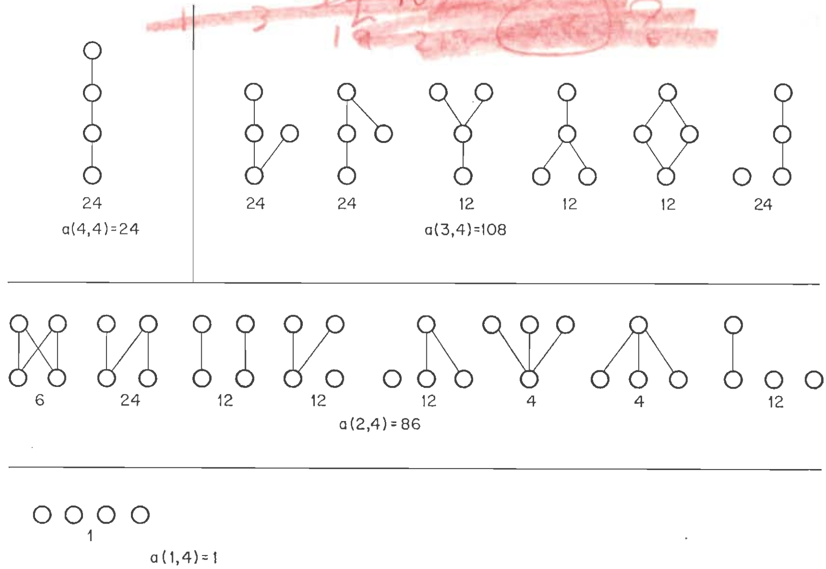

FIGURE 2

Graded Posets with a Greatest and Least Element

Let $G_h(a, X)$ be the set of all graded posets on an $(n + 2)$-set X with rank $h + 1$, a greatest and a least element, and exactly p elements covering the least element. We put $|G_h(p, X)| = g_h(p, n)$, and sometimes write

$$g_3(m, m + n) = \binom{m + n + 2}{1, m, n, 1} f(m, n).$$

We will prove that

$$f(m, n) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (2^k - 1)^n, \tag{12}$$

and for $h \geqslant 2$,

$$g_h(p, n) = (n + 2)(n + 1) \sum \binom{n}{n_1, \ldots, n_h} f(n_1, n_2) f(n_2, n_3) \cdots f(n_{h-1}, n_h), \tag{13}$$

where the sum extends over all compositions $(n_1, \ldots, n_h)$ of n into exactly h positive parts with $n_1 = p$. Note that the sum in (13) has the form of (2) with $u(i, j) = f(i, j)$ if $i, j \neq 0$, and $u(i, j) = 0$ otherwise, and $v(i) = 1$; thus, the recurrence relation (3) applies.

Suppose X and Y are sets containing m and n elements, respectively, and let $F(m, n)$ be the set of all bipartite graphs (X, Y) such that every element of X is joined to some element of Y and every element of Y is joined to some element of X. Suppose each element of $G_3(m, Z)$ is defined on an $(m + n + 2)$-set Z. Every element S of $G_3(m, Z)$ corresponds to a 4-tuple $(\{x\}, X, Y, \{y\})$ consisting of exhaustive disjoint subsets of Z with $|X| = m$, $|Y| = n$, where x and y are the least and greatest elements in S, respectively, and X and Y are the elements in S with rank 1 and 2, respectively. The partial order relation for S restricted to $X \cup Y$ has its Hasse diagram in $F(m, n)$. Conversely, every bipartite graph in $F(m, n)$ corresponds to an element of $G_3(m, Z)$. Thus, if $f(m, n) = |F(m, n)|$, we have

$$g_3(m, m + n) = \binom{m + n + 2}{1, m, n, 1} f(m, n).$$

Now we are going to find the number of elements in $F(m, n)$. Suppose X and Y are sets with m and n elements, respectively, and let $F^*(m, n)$ be the set of bipartite graphs (X, Y) such that every element of Y is joined to some element of X, but not necessarily vice versa. If $f^*(m, n) = |F^*(m, n)|$, we obviously have $f^*(m, n) = (2^m - 1)^n$, but the elements of $F^*(m, n)$ can be enumerated in a second way. The number of elements in $F^*(m, n)$ having exactly k isolated nodes in X is $\binom{m}{k} f(m-k, n)$; thus,

$$(2^m - 1)^n = f^*(m, n) = \sum_{k=0}^{m} \binom{m}{k} f(k, n). \tag{14}$$

Using the fact that if

$$b_n = \sum \binom{n}{k} a_k ,$$

then

$$a_n = \sum (-1)^{n-k} \binom{n}{k} b_k ,$$

we invert (14) to obtain (12). (Incidentally, by definition of $F(m, n)$, $f(m, n) = f(n, m)$, so that (12) implies an identity involving two sums.)

Suppose the elements of $G_h(p, Z)$ are defined on an $(n+2)$-set Z, then $G_h(p, Z)$ can be split into exhaustive, disjoint subsets corresponding to the compositions $(1, n_1, ..., n_h, 1)$ of $n+2$ into exactly $h+2$ positive parts with $n_1 = p$. Each of these subsets corresponds to an $(h+2)$-tuple $(\{z_1\}, Z_1, ..., Z_h, \{z_2\})$ with $|Z_i| = n_i$, and a sequence of bipartite graphs $(Z_i, Z_{i+1}) \in F(n_i, n_{i+1})$, for $i = 1, ..., h-1$. So the composition $(1, n_1, ..., n_h, 1)$ of $n+2$ corresponds to a subset of $G_h(p, Z)$ having

$$\binom{n+2}{1, n_1, ..., n_h, 1} f(n_1, n_2) f(n_2, n_3) \cdots f(n_{h-1}, n_h) \tag{15}$$

elements; summing over the appropriate compositions of $n+2$ gives (13).

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