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# ON THE NUMBER OF TOPOLOGIES DEFINABLE FOR A FINITE SET

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No general rule for determining the number $N(n)$ of topologies definable for a finite set of cardinal $n$ is known. In this note we relate $N(n)$ to a function $F_t(r_1, \cdots, r_{t+1})$ defined below which has a simple combinatorial interpretation. This relationship seems useful for the study of $N(n)$. In particular this can be used to calculate $N(n)$ for small values. For $n = 3, 4, 5, 6$ we find $N(3) = 29$, $N(4) = 355$, $N(5) = 7{,}181$, $N(6) = 145{,}807$.

Let $T$ be a topology on a finite set $E$. Let $S_1$ be the collection of all non-empty sets in $T$ which do not properly contain any non-empty set in $T$. It is clear that $S_1$ is a collection of disjoint subsets of $E$. If for any collection $K$ of sets $P_\cup(K)$ denotes the set of all non-empty unions of sets in $K$ then $P_\cup(S_1) \subseteqq T$. Let $\cup S_1$ be the union of all sets in $S_1$. Then every non-empty set in $T$ is of the form $U \cup V$ where $V \in P_\cup(S_1)$ and $U$ is a subset of $E - \cup S_1$. Let $T_1$ be the collection of all the sets $U$ and the null set. It can be easily proved that $T_1$ is a topology on $E - \cup S_1$. We shall refer to $S_1$ and $T_1$ as "nucleus" and "orbital topology" of the topology $T$, respectively.

By a "reduced base" of a topology on a finite set we shall mean a base such that no base set is a union of other base sets.

THEOREM. *Let $B_1$ be a reduced base for $T_1$. Then there is a unique single-valued mapping $f : B_1 \to P_\cup(S_1)$ such that $B = \{X_1 \cup X_1 f,\ X_1 \in B_1\} \cup S_1$ is a reduced base for $T$. Also, $f$ preserves the inclusion relation $\subseteqq$ for sets. Conversely if $S_1$ is a non-empty collection of disjoint non-empty subsets of $E$, $T_1$ is any topology on $E - \cup S_1$ and $f$ is a single-valued mapping from a reduced base $B_1$ for $T_1$ into $P_\cup(S_1)$ which preserves $\subseteqq$ then $B = \{X_1 \cup X_1 f,\ X_1 \in B_1\} \cup S_1$ is a reduced base for a topology $T$ on $E$ such that $S_1$, $T_1$ are respectively the nucleus and the orbital topology of $T$.*

PROOF. For any $X_1 \in B_1$, we define $X_1 f$ to be a member of $P_\cup(S_1)$ such that $X_1 \cup X_1 f \in T$ and $X_1 \cup V \notin T$ if $X_1 f \supset V$. $X_1 f$ exists because $T_1$ is the orbital topology of $T$. If $V^* \in P_\cup(S_1)$ has the property stated for $X_1 f$ then $V^* \supseteqq X_1 f$ and $X_1 f \supseteqq V^*$, so that $X_1 f = V^*$. Thus $f$ is a mapping from $B_1$ into $P_\cup(S_1)$. We show that $f$ is the mapping required by the first part of the theorem. Let $X_1 \subseteqq X'_1$; then

$$(X_1 \cup X_1 f) \cap (X'_1 \cup X'_1 f) = X_1 \cup (X_1 f \cap X'_1 f) \in T,$$

since $X_1$, $X'_1 f$ are disjoint for all $X_1$, $X'_1 \in B_1$. We conclude from the definition of $f$ that $X_1 f \cap X'_1 f = X_1 f$ so that $X_1 f \subseteqq X'_1 f$ and hence $f$ preserves $\subseteqq$. Next let $Y \in T$ and let $Y = U \cup V$, where $U \in T_1$, $V \in P_{\cup}(S_1)$. Since $B_1$ is a base for $T_1$ we can write $U = \cup B'_1$ for some subcollection $B'_1$ of $B_1$. If $U$ is empty, $Y$ is trivially a union of sets in

$$B = \{X_1 \cup X_1 f, X_1 \in B_1\} \cup S_1.$$

Hence we can suppose $B'_1$ non-empty. Then $X'_1 f \subseteqq V$ for every $X'_1 \in B'_1$; for

$$X'_1 \cup (V \cap X'_1 f) = (U \cup V) \cap (X'_1 \cup X'_1 f) \in T$$

and therefore $V \cap X'_1 f = X'_1 f$. Hence $Y = \cup \{X'_1 \cup X'_1 f, X'_1 \in B'_1\} \cup$ (union of sets in $S_1$). This proves that $B$ is a base for $T$. That $B$ is reduced follows directly from the definition of $f$ and the assumption that $B_1$ is reduced. To prove the uniqueness of the mapping $f$ suppose that $f^*$ is another mapping satisfying the first part of the theorem. Then, for some $X_1 \in B_1$, $X_1 f \subset X_1 f^*$. But $X_1 \cup X_1 f \in T$ and therefore is a union of sets in $B^* = \{Y_1 \cup Y_1 f^*, Y_1 \in B_1\} \cup S_1$. Since $B_1$ is reduced this is impossible in view of $X_1 f \subset X_1 f^*$.

For the converse, let $B$ be as defined in the theorem. Then $E = \cup B = (\cup B_1) \cup (\cup S_1)$. Let $Y$, $Y^*$ be any two members of $B$ and write $Y = X_1 \cup X_1 f$, $Y^* = X^*_1 \cup X^*_1 f$. Since $f$ preserves $\subseteqq$,

$$\begin{aligned} Y \cap Y^* &= (X_1 \cap X^*_1) \cup (X_1 f \cap X^*_1 f) \\ &= (X_1 \cap X^*_1) \cup (X_1 \cap X^*_1) f \cup \text{ (union of sets in } S_1). \end{aligned}$$

Now $X_1$, $X^*_1 \in B_1$ and $X_1 \cap X^*_1 = \cup B'_1$, where $B'_1$ is a subcollection of $B_1$. Since $Z'_1 f \subseteqq (X_1 \cap X^*_1) f$ for every $Z'_1 \in B'_1$, this gives

$$Y \cap Y^* = \cup \{Z'_1 \cup Z'_1 f, Z'_1 \in B'_1\} \cup \text{ (union of members of } S_1);$$

so that $Y \cap Y^*$ is a union of members of $B$. In case one or both of $Y$, $Y^*$ are members of $S_1$ and therefore not expressible in the form $X \cup Xf$, $Y \cap Y^*$ is trivially a union of sets in $B$. Hence the intersection of any two members of $B$ is a union of members of $B$ and therefore $B$ is a base for a topology $T$ on $E$. The rest of the theorem now follows directly.

For any topology $T$ on a finite set $E$ we can form the sequence $T_0 = T$, $(S_1, T_1)$, $(S_2, T_2)$, $\cdots$, $(S_t, T_t)$, $S_{t+1}$, where $S_k$, $T_k$ are respectively the nucleus and the orbital topology of $T_{k-1}$ for $t \geqq k \geqq 1$ and $S_{t+1}$ is a reduced base as well as the nucleus of $T_t$, so that $T_t = P_{\cup}(S_{t+1})$. By the above theorem there is a unique sequence of mappings $f_1, \cdots, f_t$ such that for

$1 \leqq i \leqq t$, $f_i$ maps $B_i$ into $P_\cup(S_i)$, where $B_i$ is a reduced base for $T_i$ and is defined by

$$B_t = S_{t+1},\ B_i = \{X_{i+1} \cup X_{i+1}f_{i+1},\ X_{i+1} \in B_{i+1}\} \cup S_{i+1},$$
$$\text{for } 0 \leqq i \leqq t.$$

By our theorem, every topology on $E$ can be obtained as follows: Partition $E$ into any number, say $r$, of disjoint and collectively exhaustive classes $E_1, \cdots, E_r$ and then partition, in an arbitrary way, the set $\{E_1, \cdots, E_r\}$ into disjoint and collectively exhaustive classes, say, $S_1, \cdots, S_{t+1}$. Let $f_1, \cdots, f_t$ be any mappings such that

(i) $f_t$ maps $B_t = S_{t+1}$ into $P_\cup(S_t)$,

(ii) $f_{t-i}$ maps $B_{t-i}$ into $P_\cup(S_{t-i})$ where

$$B_{t-i} = \{X \cup Xf_{t-i+1},\ X \in B_{t-i+1}\} \cup S_{t-i+1},$$

(iii) each of the mappings $f_1, \cdots, f_t$ preserves the inclusion relation $\subseteqq$ for sets.

Then $B = B_0 = \{X_1 \cup X_1f_1,\ X_1 \in B_1\} \cup S_1$ is a base for a topology on $E$ and every topology on $E$ is obtained in this way.

In view of this we can express the number $N(n)$ of topologies definable for a finite set of cardinal $n$ as follows:

$$(1)\qquad N(n) = \sum_{r=1}^{n}\left[M_{n,r}r! \sum_{r_1+\cdots r_{t+1}=r}\left\{[F_t(r_1, \cdots, r_{t+1})/r_1! \cdots r_{t+1}!]\right\}\right]$$

where $M_{n,r}$ is the number of ways a set of order $n$ can be partitioned into $r$ unordered classes and $F_t(r_1, \cdots, r_{t+1})$ is the number of sequences of mappings $f_1, \cdots, f_t$ described above when $S_1, \cdots, S_{t+1}$ have $r_1, \cdots, r_{t+1}$ members respectively. The summation in curly brackets extends over all finite sequences $r_1, \cdots, r_{t+1}$ of positive integers satisfying $r_1 + \cdots + r_{t+1} = r$.

The following recurrence relation holds for $M_{n,r}$:

$$(2)\qquad M_{n+1,r} = rM_{n,r} + M_{n,r-1}.$$

The function $F_t(r_1, \cdots, r_{t+1})$ has a simple combinatorial interpretation which we explain by taking $t = 3$ and by referring to the figure below.

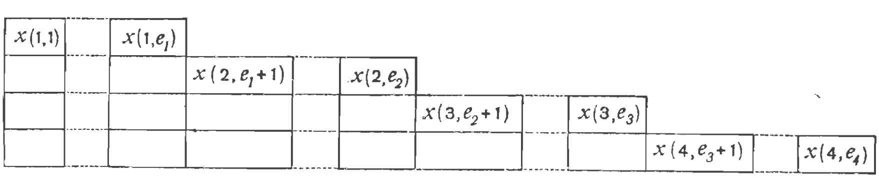


Figure 1

In this figure we have taken $e_1 = r_4$, $e_2 = r_3+r_4$, $e_3 = r_2+r_3+r_4$, $e_4 = r_1+r_2+r_3+r_4$. Every one of the $r_4$ squares in the first row is given to be occupied with just one of the symbols $x(1, 1), \cdots, x(1, e_1)$ that are labels for sets in $S_4$. In the second row only the last $r_3$ squares on the right are given to be initially occupied, each by just one of the $r_3$ symbols $x(2, e_1+1), \cdots, x(2, e_2)$ that similarly stand for sets in $S_3$; and so on. Let us refer to the $j$th square from the left in the $i$th row from the top as $\sigma(i, j)$. In what follows we shall not explicitly mention the restrictions on the ranges of the variables $i, j, k, \cdots$. Write $\Sigma(i, j)=\{x(i, j)\}$ if $\sigma(i, j)$ is not initially empty. The combinatorial problem now is to place in every *empty* square $\sigma(i, j)$ a non-empty set $\Sigma(i, j)$ of symbols such that

(iv) $\Sigma(i, j) \subseteqq \{x(i, e_{i-1}+1), \cdots, x(i, e_i)\}$,

(v) $x(i, k) \in \Sigma(i, j)$ implies $\Sigma(i+1, k) \subseteqq \Sigma(i+1, j)$.

Thus, for example, the conditions (iv), (v) compel us to place in the empty squares of the third row in Fig. 1 symbols chosen from $x(3, e_2+1), \cdots, x(3, e_3)$, and if $x(3, e_3)$ has been placed in $\sigma(3, e_2)$ (the square immediately below the one containing $x(2, e_2)$) then $x(3, e_3)$ will have to occur in any set of symbols to be placed in a square of the third row which comes directly under a square containing $x(2, e_2)$. Let $Y(i, k) = \bigcup_{l=1}^{i} \Sigma(l, k)$. Then it is easily seen that if we let $B_{4-i}$ be the set of all $Y(i, k)$ for fixed $i$ and write $Y(i, k)f_{4-i} = \Sigma(i+1, k)$ then $B_{4-i}$, $f_{4-i}$ satisfy (i), (ii), (iii) for $t = 3$.[1] It follows that $F_3(r_1, r_2, r_3, r_4)$ is the number of ways of placing the symbols $x(i, j)$ in the empty squares of Fig. 1 such that (iv) and (v) are satisfied.

We can use this interpretation of $F_t(r_1, \cdots, r_{t+1})$ to prove the following formulae.

(3) $F(r_1) = 1$,

(4) $F_1(r_1, r_2) = (2^{r_1}-1)^{r_2}$,

(5) $$F_2(r_1, 1, r_3) = \sum_{l=1}^{r_1} \binom{r_1}{l} 2^{(r_1-l)r_3},$$

(6) $$F_2(1, r_1, r_2) = \sum_{l=1}^{r_1} \sum_{m=1}^{r_2} 2^{r_2-m} \binom{r_1}{l} \binom{r_2}{m} (2^{r_2}-1)^{r_1-l}\{(2^m-1)^l - m(2^{m-1}-1)^l\},$$

(7) $$F_t(1, 1, \cdots, 1, r_{t+1}) = \sum_{j_1>0,\, j_1+\cdots+j_t \leq r_{t+1}} \binom{r_{t+1}}{j_1} \binom{r_{t+1}-j_1}{j_2} \cdots \binom{r_{t+1}-(j_1+\cdots+j_{t-1})}{j_t}.$$

[1] Strictly speaking, members of $B_{4-i}$ must be taken as the unions $\cup\, Y(i, k)$ of all sets represented by the $x$'s in $Y(i, k)$, but since $x$'s represent disjoint sets this will not effect our conclusion about $F_3(r_1, \cdots, r_4)$.

As an illustration we prove (5). We have to consider the number of ways some of the $x(i, j)$ can be placed in the empty squares in fig. 2 below such that (iv), (v) are satisfied.

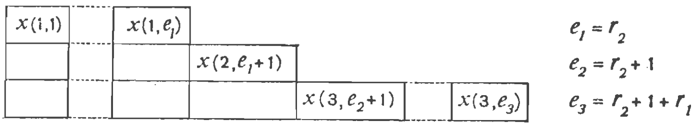


Figure 2

In every empty square of the second row of this figure we must put just $x(2, e_1+1)$. In the square $\sigma(3, e_1+1)$ under $x(2, e_1+1)$ we can place any subset $\Sigma(3, e_2+1)$ of $\{x(3, e_2+1), \cdots, x(3, e_3)\}$. In the remaining empty squares of the third row we must put every symbol in $\Sigma(3, e_2+1)$ in addition to some other symbols arbitrarily selected from

$$\{x(3, e_2+1), \cdots, (3\, e_3)\} - \Sigma(3, e_2+1).$$

The formula (5) is now obvious.

We have employed formulae (1)—(7) in calculating $N(n)$ for $n = 3, 4, 5, 6$.

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