~~v9n2~~ VB JRM v2n3 pgs 180-1 1974

A0142 A0165 A1813 A0085 A0898 A0903 A0900 A0902 A0901 A0407 A0899

S914

$$S_m = 1 + \binom{m}{2} + \frac{1}{2!}\binom{m}{2}\binom{m-2}{2} + \frac{1}{3!}\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2} + \cdots$$

It will be important later to know how many of the $n!$ rook arrangements are symmetric about both diagonals S and S'. Denote this number by $(S'')_n$ for the $n \times n$ board. Then

$$(S'')_{2m+1} = (S'')_{2m},$$

and

$$(S'')_{2m} = 2(S'')_{2m-2} + (2m-2)(S'')_{2m-4}.$$

(To see this, note that the rook in the first column may or may not occupy one of the two corner squares in that first column. Consider both cases.) See Table 1.

$\frac{(2n)!}{n!}$ $A_n = A_{n+1}$ $2A_{n-1}$ $2^n n!$ $+(n-1)A_{n-2}$ $+(n-1)A_{n-2}$

Table 1

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n	$n!$	$(R^2)_n$	R_n	S_n	$(S'')_n$	T_n
1	1	1		1		1
2	2	2		2	2	1
3	6	2		4	2	2
4	24	8	2	10	6	7
5	120	8	2	26	6	23
6	720	48		76	20	115
7	5,040	48		232	20	694
8	40,320	384	12	764	76	5,282
9	362,880	384	12	2,620	76	46,066
10	3,628,800	3,840		9,496	312	456,454
11	39,916,800	3,840		35,696	312	4,999,004
12	479,001,600	46,080	120	140,152	1,384	59,916,028
13	6,227,020,800	46,080	120	568,504	1,384	778,525,516
14	87,178,291,200	645,120		2,390,480	6,512	10,897,964,660
15	1,307,674,368,000	645,120		10,349,536	6,512	163,461,964,024

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In view of previous difficulties with this problem, it is interesting to look back to see why previous approaches did not work out so well. Kraitchik [2] and Madachy [4] indicate that the basic strategy heretofore was to divide the *essentially different solutions* into five mutually disjoint classes depending upon which *subgroups* of the group of symmetries of the square left them invariant. These classes were defined by:

A : Those invariant only under I,

B : Those invariant under R, (R^2 and R^3),

C : Those invariant under R^2, but not invariant under any other rotation or reflection,

D : Those invariant under S, S', and R^2, and

E : Those invariant under S or S', but not both.

If we let A_n, B_n, C_n, D_n, and E_n denote the number of elements in these sets, respectively, for the $n \times n$ case, we can see that

$$8A_n + 4C_n + 4E_n + 2B_n + 2D_n = n!$$

and

$$4C_n + 2B_n + 2D_n = 2^k k!, \text{ where } n = 2k \text{ or } 2k + 1$$

It was hoped that by playing with identities like this, one could simplify the problem. But such was not the case.

However, it is possible to work backward to obtain these numbers. It is easy to see that the following identities must hold:

$$E_n = [S_n - (S'')_n]/2; D_n = (S'')_n/2;$$

$$C_n = [(R^2)_n - R_n - (S'')_n]/4; \quad B_n = R_n/2;$$

$$A_n = T_n - (B_n + C_n + D_n + E_n).$$

Notice in Table 2 that as n becomes large, the ratio of A_n to T_n approaches one; that is to say, only a tiny proportion of the arrangements will possess symmetry. This, of course, is to be expected. Thus for large n, $T_n \approx n!/8$.

A0902

Table 2

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n	E_n	D_n	C_n	B_n	A_n	$\approx A_n/T_n$
1						
2		1				
3	1	1				
4	2	3		1	1	.142857
5	10	3		1	9	.391304
6	28	10	7		70	.608696
7	106	10	7		571	.822767
8	344	38	74	6	4,820	.912533
9	1,272	38	74	6	44,676	.969826
10	4,592	156	882		450,824	.987665
11	17,692	156	882		4,980,274	.996253
12	69,384	692	11,144	60	59,834,748	.998643
13	283,560	692	11,144	60	778,230,060	.999620
14	1,191,984	3,256	159,652		10,896,609,768	.999876
15	5,171,512	3,256	159,652		163,456,629,604	.999967

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Finally, consider the corresponding placement problem for bishops (note that giving the board a 45° turn transforms this problem into one of rooks on a diamond-shaped board). It is easily shown that a maximum of $2n - 2$ nonattacking bishops can be placed on an $n \times n$ board and the total number of ways this can be done is 2^n. To find how many of these are essentially different we may again apply the Burnside Theorem. For this problem (see Reference 4, pp. 43-46, for more detail),

$$F(I) = 2^n,$$

$$F(S) = F(S') = F(R) = F(R^2) = F(R^3) = 0.$$