Computing the leading asymptotic of A001003

Starting from the OGF

$$D(z) = \frac{1+z-\sqrt{z^2-6z+1}}{4z}$$

we require

$$D_n = -[z^{n+1}]\frac{1}{4}\sqrt{z^2-6z+1} = -\frac{1}{4}\frac{1}{n+1}[z^n]\frac{z-3}{\sqrt{z^2-6z+1}}.$$

We start with

$$[z^n]\frac{1}{\sqrt{z^2-6z+1}} = [z^n]\frac{1}{\sqrt{(z-(3+2\sqrt{2}))(z-(3-2\sqrt{2}))}}.$$

The singularity that is closest to the origin is in the second square root term and we write

$$\frac{1}{(3-2\sqrt{2})^n}(3-2\sqrt{2})^n[z^n]\frac{1}{\sqrt{(z-(3+2\sqrt{2}))(z-(3-2\sqrt{2}))}}$$
$$= (3+2\sqrt{2})^n[z^n]\frac{1}{\sqrt{(z(3-2\sqrt{2})-(3+2\sqrt{2}))(z(3-2\sqrt{2})-(3-2\sqrt{2}))}}$$
$$= \frac{(3+2\sqrt{2})^n}{\sqrt{3-2\sqrt{2}}}[z^n]\frac{1}{\sqrt{(3+2\sqrt{2})-z(3-2\sqrt{2}))(1-z)}}.$$

Extracting the dominant asymptotics as on page 180 of Wilf's *generatingfunctionology* we get

$$(3+2\sqrt{2})^{n+1/2}\binom{n+1/2-1}{n}\frac{1}{\sqrt{4\sqrt{2}}} = \frac{1+\sqrt{2}}{2^{5/4}}(3+2\sqrt{2})^n\binom{n-1/2}{n}$$
$$= \frac{1+\sqrt{2}}{2^{5/4}}(3+2\sqrt{2})^n(-1)^n\binom{-1/2}{n} = \frac{1+\sqrt{2}}{2^{5/4}}(3+2\sqrt{2})^n[w^n]\frac{1}{\sqrt{1-w}}$$
$$= \frac{1+\sqrt{2}}{2^{5/4}}(3+2\sqrt{2})^n\frac{1}{4^n}\binom{2n}{n}.$$

Collecting the two contributions we find

$$(3+2\sqrt{2})^n\frac{1}{4^n}\binom{2n}{n}\left[-3+\frac{4}{3+2\sqrt{2}}\frac{n^2}{(2n)(2n-1)}\right].$$

The square bracketed constant is asymptotic to

$$-3+\frac{1}{3+2\sqrt{2}} = -3+3-2\sqrt{2} = -2^{3/2}.$$

It follows that the first term of the expansion is

$$\frac{1}{4}\frac{1}{n+1}2^{3/2}\frac{1+\sqrt{2}}{2^{5/4}}(3+2\sqrt{2})^n\frac{1}{4^n}\binom{2n}{n}.$$

Recall the central binomial coefficient has $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$ so that this becomes

$$\frac{1}{4}\frac{1}{n+1}2^{3/2}\frac{1+\sqrt{2}}{2^{5/4}}(3+2\sqrt{2})^n\frac{1}{\sqrt{\pi n}}.$$

The asymptotic of $1/(n+1)$ is $1/n - 1/n^2 + \cdots$ and $2 + 5/4 - 3/2 = 7/4$ so that we have at last

$$\boxed{D_n \sim \frac{1+\sqrt{2}}{\sqrt{\pi}2^{7/4}}(3+2\sqrt{2})^n\frac{1}{n^{3/2}}.}$$

This was [math.stackexchange.com problem 4703369](https://math.stackexchange.com/questions/4703369/).