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Some contributions of computation to semigroups and groupoids†

TAKAYUKI TAMURA

REVIEWING the contribution of computers to the theory of semigroups, we note that G. E. Forsythe computed all semigroups of order 4 [2] in 1954, and T. S. Motzkin and J. L. Selfridge obtained all semigroups of order [redacted] 5 [4] in 1955. For the ten years from 1955 through 1965, nobody treated the computation of all semigroups of order 6. However, R. Plemmons did all semigroups of order 6 by IBM 7040 in 1965 [5]. [redacted] On the other hand the author and his students obtained the semigroups of order 3 in 1953 [8], of order 4 in 1954 [9] and of order 5 in 1955 [10] by hand, independently of those mentioned above. Beside these, certain special types of semigroups and groupoids of order 3, which are distributive to given semigroups of order 3, were computed by hand [12], [13], [14]. In 1965 we obtained the number of non-isomorphic, non-anti-isomorphic groupoids of order ≤ 4 which have a given permutation group as the automorphism group (§§ 1.4, 1.5). Although the result was presented at the meeting of the American Mathematical Society at Reno, 1965, it has not been published. Afterwards R. Plemmons checked the total number by computing machine and wrote to the author that our number was correct; the author wishes to thank Dr. Plemmons. Recently R. Dickinson analyzed the behavior of some operations on the binary relations by machine [17].

In this paper we announce the result concerning the automorphism groups and the total number of groupoids and additionally we introduce the significance of a new concept "general product", which uses a machine to get a suggestion for an important problem on the extension of semigroups, and further we show the result in a special case which easily computed without using a machine. The detailed proof of some theorems will be omitted because of pressure on space in these Proceedings, and the complete proof will be published elsewhere.

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PART I. GROUPOIDS AND THEIR AUTOMORPHISM GROUPS

1.1. Introduction. A groupoid G is a set S with a binary operation θ in which the product z of x and y of S is denoted by

$$z = x\theta y.$$

G is often denoted by $G = G(S, \theta)$. An automorphism α of G is a permutation of S (i.e. a one-to-one transformation of S onto S) such that

$$(x\theta y)\alpha = (x\alpha)\theta(y\alpha) \quad \text{for all} \quad x, y \in S.$$

The group of all automorphisms of G is called the automorphism group of G and denoted by $\mathfrak{A}(G)$ or $\mathfrak{A}(S, \theta)$. It is a subgroup of the symmetric group $\mathfrak{S}(S)$ over S. The following problem is raised:

Problem. Let S be a fixed set. Under what condition on $|S|$, for every subgroup $\mathfrak{H}$ of $\mathfrak{S}(S)$, does there exist $G = G(S, \theta)$ such that $\mathfrak{A}(G) = \mathfrak{H}$? This problem is a step towards the following problem.

Let $\mathfrak{H}$ be a subgroup of $\mathfrak{S}(S)$. Under what condition on $\mathfrak{H}$ does there exist a groupoid G such that $\mathfrak{A}(G) = \mathfrak{H}$?

However, we will consider only the first problem in this paper.

The answer to the problem is:

THEOREM 1.1. *For every subgroup $\mathfrak{H}$ of $\mathfrak{S}(S)$ there is at least a groupoid G defined on S such that $\mathfrak{A}(G) = \mathfrak{H}$ if and only if $|S| \leqslant 4$.*

In the next section we will sketch the outline of the proof. From now on we shall not distinguish in symbols S from G, that is, G shall denote a set as well as a groupoid defined on it. The groupoids with operations $\theta, \xi, \cdots$ are denoted by $(G, \theta), (G, \xi), \cdots$. The automorphism group $\mathfrak{A}(G, \theta)$ will be denoted by $\mathfrak{A}(G)$ or $\mathfrak{A}(\theta)$ if there is no fear of confusion as far as a set G is fixed. $\mathfrak{S}(G)$ is the symmetric group over a set G.

1.2. Outline of the proof of Theorem 1.1. The following theorem was given in 1963 **[16]**.

THEOREM 1.2 *Every permutation of a set G is an automorphism of a groupoid G if and only if G is either isomorphic or anti-isomorphic*† *onto one of the following:*

(1.1) *A right zero semigroup: $xy = y$ for all x, y.*
(1.2) *The idempotent quasigroup of order* 3.
(1.3) *The groupoid* $\{1, 2\}$ *of order* 2 *such that*

$$x \cdot 1 = 2, \qquad x \cdot 2 = 1 \qquad (x = 1, 2).$$

The following theorem partially contains Theorem 1.2.

THEOREM 1.3. *Let $|G| \geqslant 5$. The following statements are equivalent.*

† We will use "dually isomorphic" as synonymous to "anti-isomorphic".

(1.4) *A groupoid G is isomorphic onto either a right zero semigroup or a left zero semigroup.*

(1.5) $\mathfrak{A}(G) = \mathfrak{S}(G)$.

(1.6) *Every even permutation of G is contained in $\mathfrak{A}(G)$.*

(1.7) $\mathfrak{A}(G)$ *is triply transitive* (*i.e.* *3-ply transitive* (cf. [3])).

(1.8) $\mathfrak{A}(G)$ *is doubly transitive and there is an element $\varphi \in \mathfrak{A}(G)$ such that $a\varphi = a$, $b\varphi = b$ for some a, $b \in G$, $a \neq b$, but $x\varphi \neq x$ for all $x \neq a$, $x \neq b$.*

Proof. The proof will be done in the following direction:

$$\begin{tikzcd}
(1\cdot 7) \arrow[r] & (1\cdot 4) \arrow[dd] & \\
 & & (1\cdot 8) \arrow[ul] \\
(1\cdot 6) \arrow[uu] & (1\cdot 5) \arrow[l] \arrow[ur] &
\end{tikzcd}$$

(1.4) $\to$ (1.5) is given by Theorem 1.2; (1.5) $\to$ (1.6), (1.5) $\to$ (1.8) are obvious; (1.6) $\to$ (1.7) is easily proved by the fact that the alternating group is triply transitive if $|G| \geqq 5$. We need to prove only (1.7) $\to$ (1.4) and (1.8) $\to$ (1.4). The detailed proof is in [**20**].

Remark. We do not assume finiteness of G. The definitions of double and triple transitivity and even permutation are still effective.

Let $\mathfrak{H}$ be a proper subgroup of $\mathfrak{S}(G)$, $|(G)| \geqq 5$. If $\mathfrak{H}$ can be an automorphism group of a groupoid (G, θ) for some θ, then $\mathfrak{H}$ is neither triply transitive, nor the alternating group on a set G.

THEOREM 1.4. *Every permutation group on a set G, $|G| \leqq 4$, is the automorphism group of a groupoid (G, θ) for some θ.*

The proof of Theorem 1.4 is the main purpose of §§ 1.4, 1.5 below. In order to count the number of groupoids for each permutation group, we will experimentally verify the existence for each case.

§ 1.3 is the introduction of the basic concept for the preparation of §§ 1.4, 1.5.

1.3. Preparation. Let $\mathfrak{G}$ denote the set of all binary operations $\theta, \xi, \cdots$ defined on a set G. Let $\alpha, \beta, \cdots$ be elements of $\mathfrak{S}(G)$, i.e. permutations of G. To each α a unary operation $\bar{\alpha}$ on $\mathfrak{G}$, $\theta \to \theta^{\bar{\alpha}}$, corresponds in the following way:

$$x\theta^{\bar{\alpha}}y = \{(x\alpha^{-1})\theta(y\alpha^{-1})\}\alpha, \quad x, y \in G.$$

The groupoids (G, θ) and $(G, \theta^{\bar{\alpha}})$ are isomorphic since $(x\theta^{\bar{\alpha}}y)\alpha^{-1} = (x\alpha^{-1})\theta(y\alpha^{-1})$. Clearly α is an automorphism of (G, θ) if and only if $\theta^{\bar{\alpha}} = \theta$. The product $\bar{\alpha}\bar{\beta}$ of $\bar{\alpha}$ and $\bar{\beta}$ is defined in the usual way:

$$\theta^{\bar{\alpha}\bar{\beta}} = (\theta^{\bar{\alpha}})^{\bar{\beta}} \quad \text{for all} \quad \theta \in \mathfrak{G}.$$

It is easy to see that

$$\theta^{\bar{\alpha}\bar{\beta}} = \theta^{\overline{\alpha\beta}} \quad \text{for all} \quad \theta \in \mathfrak{G}.$$

$\theta^{\bar{\alpha}} = \theta^{\bar{\beta}}$ if and only if $\alpha\beta^{-1} \in \mathfrak{A}(\theta)$. Let $\bar{\mathfrak{S}} = \{\bar{\alpha};\ \alpha \in \mathfrak{S}(G)\}$. Then $\bar{\mathfrak{S}}$ is isomorphic onto $\mathfrak{S}(G)$ under $\alpha \to \bar{\alpha}$. Suppose $\bar{\alpha} = \bar{\beta}$. $\alpha\beta^{-1}$ is in $\mathfrak{A}(G, \theta)$ for all $\theta \in \mathfrak{G}$. On the other hand, there is $\theta_0 \in \mathfrak{G}$ such that $\mathfrak{A}(G, \theta_0)$ consists of the identical mapping ε alone (cf. **[20]**). Hence $\alpha\beta^{-1} = \varepsilon$ and so $\alpha = \beta$.

Define another unary operation $\theta \to \theta'$ on $\mathfrak{G}$ as follows:

$$x\theta' y = y\theta x.$$

Then clearly $(\theta')' = \theta$ and (G, θ) is anti-isomorphic onto (G, θ'); $\theta' = \theta$ if and only if (G, θ) is commutative. Also $(\theta')^{\bar{\alpha}} = (\theta^{\bar{\alpha}})'$ for all $\theta \in \mathfrak{G}$.

We denote $(\theta')^{\bar{\alpha}}$ by $\theta^{\bar{\alpha}'}$. Then α is an anti-automorphism of (G, θ) if and only if $\theta^{\bar{\alpha}'} = \theta$. We can easily prove

$$(\theta^{\bar{\alpha}})^{\bar{\beta}'} = \theta^{\overline{\alpha\beta}'}$$
$$(\theta^{\bar{\alpha}'})^{\bar{\beta}} = \theta^{\overline{\alpha\beta}'}$$
$$(\theta^{\bar{\alpha}'})^{\bar{\beta}'} = \theta^{\overline{\alpha\beta}}.$$

As defined in § 1.1, $\mathfrak{A}(\theta)$ is the automorphism group of (G, θ) while $\mathfrak{A}'(\theta)$ denotes the set of all anti-automorphisms of (G, θ).

We define

$$\mathfrak{B}(\theta) = \mathfrak{A}(\theta) \cup \mathfrak{A}'(\theta).$$

Then $\mathfrak{B}(\theta)$ is a subgroup of $\mathfrak{S}(G, \theta)$ and the index of $\mathfrak{A}(\theta)$ to $\mathfrak{B}(\theta)$ is 2.

Let $\beta \in \mathfrak{S}(G)$. Then

$$\mathfrak{A}(\theta^{\bar{\beta}}) = \beta^{-1} \cdot \mathfrak{A}(\theta) \cdot \beta, \qquad \mathfrak{A}'(\theta^{\bar{\beta}}) = \beta^{-1} \cdot \mathfrak{A}'(\boldsymbol{\theta}) \cdot \beta.$$

Let $\mathfrak{H} = \mathfrak{A}(G, \theta)$ and let $\alpha \in \mathfrak{S}(G)$. Then $\mathfrak{H} = \mathfrak{A}(\theta^{\bar{\alpha}})$ if and only if α is in the normalizer $\mathfrak{N}(\mathfrak{H})$ of $\mathfrak{H}$ in $\mathfrak{S}(G)$. Therefore $\theta^{\bar{\alpha}} = \theta^{\bar{\beta}}$ and $\mathfrak{A}(\theta^{\bar{\alpha}}) = \mathfrak{A}(\theta^{\bar{\beta}}) = \mathfrak{A}(\theta) = \mathfrak{H}$ if and only if $\alpha, \beta \in \mathfrak{N}(\mathfrak{H})$ and $\alpha \equiv \beta \pmod{\mathfrak{H}}$.

Let $\mathfrak{H}$ be a permutation group over a set G and suppose that $\mathfrak{H}$ is generated by a subset $\mathfrak{K} = \{\alpha_\lambda;\ \lambda \in X\}$ of $\mathfrak{H}$.

Let

$$G \times G = \{(x, y);\ x, y \in G\}.$$

A binary operation on G is understood to be a mapping θ of $G \times G$ into G. $\mathfrak{H}$ is contained in the automorphism group $\mathfrak{A}(G)$ of a groupoid G defined by θ if and only if, for $x, y \in G$,

$$[(x, y)\theta]\alpha = (x\alpha, y\alpha)\theta \quad \text{for all} \quad \alpha \in \mathfrak{H}.$$

We define an equivalence relation $\mathfrak{B}$ on $G \times G$ as follows:

$(x, y)\ \mathfrak{B}\ (z, u)$ if and only if $z = x\alpha$, $u = y\alpha$ for some $\alpha \in \mathfrak{H}$. Clearly $\mathfrak{B}$ is the transitive closure of a relation $\mathfrak{B}_1$, defined by

$(x, y)\ \mathfrak{B}_1\ (z, u)$ if and only if $z = x\alpha$, $u = y\alpha$ for some $\alpha \in \mathfrak{K}$.

If we let $c = (a, b)\theta$ and if $(x, y)\ \mathfrak{B}\ (a, b)$, then $(x, y)\theta$ is automatically determined by

$$(x, y)\theta = [(a, b)\theta]\alpha \quad \text{for some} \quad \alpha \in \mathfrak{H}.$$

Let $\{(a_\xi, b_\xi);\ \xi \in \Xi\}$ be a representative system from the equivalence classes modulo $\mathfrak{B}$. We may determine only $\{(a_\xi, b_\xi)\theta;\ \xi \in \Xi\}$. However, there is some restriction for choosing $(a_\xi, b_\xi)\theta$:

$$[(a_\xi, b_\xi)\theta]\alpha = (a_\xi\alpha, b_\xi\alpha)\theta.$$

For (a_ξ, b_ξ) define an equivalence relation $\underset{\xi}{\sim}$ on the set union $\mathfrak{R}\cup\mathfrak{R}^{-1}$ as follows:

$$\alpha \underset{\xi}{\sim} \beta \quad \text{if and only if} \quad (a_\xi\alpha, b_\xi\alpha) = (a_\xi\beta, b_\xi\beta).$$

For (a_ξ, b_ξ) we select an element c_ξ of G such that the following condition is satisfied:

$$\alpha \underset{\xi}{\sim} \beta \quad \text{implies} \quad c_\xi\alpha = c_\xi\beta.$$

1.4. Groupoids of order $\leqslant 3$. First of all we explain the notation and the abbreviations appearing below:

$\mathfrak{H}$	Automorphism group $\mathfrak{H}$.
S_i	The symmetric group of degree i.
c	The number of conjugates of $\mathfrak{H}$, i.e. $c = \dfrac{\lvert S_i \rvert}{\lvert \text{Normalizer} \rvert}$.
n	The index of $\mathfrak{H}$ to its normalizer, $n = \dfrac{\lvert \text{Normalizer} \rvert}{\lvert \mathfrak{H} \rvert}$.
up to iso	Up to isomorphism.
up to dual	Up to dual-isomorphism (i.e. anti-isomorphism).
self-dual	Anti-isomorphic to itself.
comm	Commutative.

First we have the following table for groupoids of order 2. Since the case is simple, we omit the explanation.

TABLE 1. *Groupoids of Order 2*

$\mathfrak{H}$	c	n	Comm, up to iso	Self-dual, non-comm, up to iso	Non-self-dual, up to iso, up to dual	Total up to iso, up to dual	Total up to iso	Semi-groups up to iso, up to dual
S_2	1	1	0	0	2	2	4	1
$\{e\}$	1	2	4	0	1	5	6	3
Total			4	0	3	7	10	4

For S_2 $\boxed{\begin{matrix}1&1\\2&2\end{matrix}}$ † $\boxed{\begin{matrix}2&1\\2&1\end{matrix}}$

For $\{e\}$ $\boxed{\begin{matrix}1&1\\1&1\end{matrix}}$ $\boxed{\begin{matrix}1&1\\1&2\end{matrix}}$ $\boxed{\begin{matrix}1&2\\2&1\end{matrix}}$ $\boxed{\begin{matrix}2&1\\1&1\end{matrix}}$ $\boxed{\begin{matrix}1&1\\2&1\end{matrix}}$

In the following table [(1, 2, 3)] is the permutation group generated by a 3-cycle $(1, 2, 3)$. [(1, 2)] is one generated by a 2-cycle or substitution $(1, 2)$.

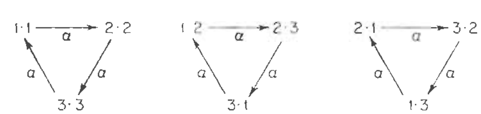
TABLE 2. *Groupoids of Order 3*

$\mathfrak{H}$	c	n	Comm, up to iso	Self-dual, non-comm, up to iso	Non-self-dual, up to iso, up to dual	Total up to iso, up to dual	Total up to iso	Semi-groups up to iso, up to dual
S_3	1	1	1	0	1	2	3	1
[(1, 2, 3)]	1	2	4	0	4	8	12	0
[(1, 2)]	3	1	8	0	35	43	78	5
$\{e\}$	1	6	116	9	1556	1681	3237	12
Total			129	9	1596	1734	3330	18

By Theorem 1, if $\mathfrak{H} = S_3$, we have two isomorphically, dual-isomorphically distinct groupoids:

$$\boxed{\begin{matrix}1&2&3\\1&2&3\\1&2&3\end{matrix}} \qquad \boxed{\begin{matrix}1&3&2\\3&2&1\\2&1&3\end{matrix}}$$

Case $\mathfrak{H} = [(1, 2, 3)]$. Let $\alpha = (1, 2, 3)$. $\mathfrak{V}$-classes:



† $\boxed{\begin{matrix}1&1\\2&2\end{matrix}}$ denotes the multiplication table $\begin{array}{c|cc} & 1 & 2\\ \hline 1 & 1 & 1\\ 2 & 2 & 2\end{array}$

Since there is no restriction to choosing $\{(1\cdot1)\theta, (1\cdot2)\theta, (2\cdot1)\theta\}$, we have 27 groupoids G such that

$$[\alpha] \subseteq \mathfrak{A}(G).$$

However, the set of the 27 groupoids contains the 3 groupoids in which S_3 is the automorphism group; the number of isomorphically distinct groupoids G for $\mathfrak{H} = [\alpha]$ is

$$\tfrac{1}{2}(27-3) = 12 \quad \text{where} \quad n = 2 \quad \text{in Table 2.}$$

If G has dual-automorphisms, $\beta = (1, 2)$ must be a dual-automorphism. In this case, since

$$(3\cdot3)\beta = 3\cdot3, \quad (1\cdot2)\beta = 1\cdot2, \quad (2\cdot1)\beta = 2\cdot1,$$

we must have $(3\cdot3)\theta = (1\cdot2)\theta = (2\cdot1)\theta = 3$. Therefore if $\mathfrak{H} \subseteq \mathfrak{A}(G)$ and if $(1, 2)$ is a dual-automorphism of G, then G is

1	3	2
3	2	1
2	1	3

while this G is already obtained for $\mathfrak{H} = S_3$, and $(1, 2)$ is an automorphism. In the present case there is no self-dual, non-commutative groupoid. There are formally 9 commutative groupoids; but excluding one we have non-isomorphic commutative groupoids:

$$\tfrac{1}{2}(9-1) = 4$$

and $\quad 12-4 = 8 \quad$ non-self-dual G's,

$\quad 8 \div 2 = 4 \quad$ isomorphically distinct non-self-dual G's.

Therefore we have $4+4 = 8$ non-isomorphic, non-dual-isomorphic G's.

Case $\mathfrak{H} = [(1, 2)]$. Let $\alpha = (1, 2)$. There are 5 $\mathfrak{B}$-classes among which a class consists of only $3\cdot3$. Clearly $(3\cdot3)\,\theta = 3$. The number of non-isomorphic G's is

$$81-3 = 78.$$

If G is commutative then $(1\cdot2)\alpha = 1\cdot2$, hence $(1\cdot2)\theta = 3$. The number of non-isomorphic commutative G's is

$$9-1 = 8.$$

The number of non-self-dual G's is $78-8 = 70$, and hence the number of those, up to isomorphism, is

$$70 \div 2 = 35.$$

Therefore the number of non-isomorphic, non-dual-isomorphic G's is

$$35+8 = 43.$$

We remark that there is no non-commutative self-dual groupoid for $\mathfrak{H}$ because S_3 has no subgroup of order 4.

Case $\mathfrak{H} = \{e\}$. First, we find the number of self-dual, non-commutative groupoids G for $\{e\}$. Let $\beta = (1, 2)$ be a dual-automorphism of G. Then we can easily see that

$$(1\cdot 2)\beta = (1\cdot 2),\quad (2\cdot 1)\beta = (2\cdot 1),\quad (3\cdot 3)\beta = (3\cdot 3)$$
$$(1\cdot 1)\beta = (2\cdot 2),\quad (1\cdot 3)\beta = (3\cdot 2),\quad (3\cdot 1)\beta = (2\cdot 3),$$

and hence

$$(1\cdot 2)\theta = (2\cdot 1)\theta = (3\cdot 3)\theta = 3.$$

The 27 G's contain the 9 G's which appeared in the previous cases. We have that the number of isomorphically distinct G's is

$$\tfrac{1}{2}(27-9) = 9$$

since we recall that the normalizer of $[(1, 2)]$ is itself.

The number $y = 116$ of all non-isomorphic commutative groupoids whose automorphism group is $\{e\}$ is the solution of

$$6y+4\times 2+8\times 3+1 = 3^6.$$

The number $x = 3237$ of all non-isomorphic groupoids corresponding to $\{e\}$ is the solution of

$$6x+78\times 3+12\times 2+3 = 3^9.$$

The number of non-self-dual G's, up to isomorphism and dual-isomorphism, is

$$\tfrac{1}{2}(3237-(116+9)) = 1556.$$

The total number of G's up to isomorphism and dual-isomorphism is the sum

$$116+9+1556 = 1681.$$

For $[(1, 2, 3)]$, let $\alpha = (1, 2, 3)$:

x	y	z
$z\alpha$	$x\alpha$	$y\alpha$
$y\alpha^2$	$z\alpha^2$	$x\alpha^2$

where (x, y, z) is

$(1, 2, 1), (2, 2, 1), (2, 3, 2), (2, 1, 3)$ commutative,

$(1, 1, 2), (2, 1, 1), (2, 1, 2), (2, 3, 1)$ non-self-dual.

For [(1, 2)], let $\beta = (1, 2)$:

commutative

x	3	z
3	$x\beta$	$z\beta$
z	$z\beta$	3

where (x, z) is

(1, 1), (1, 3), (2, 1), (2, 2),
(2, 3), (3, 1), (3, 2), (3, 3).

Non-self-dual:

x	y	z
$y\beta$	$x\beta$	$z\beta$
u	$u\beta$	3

where (x, y, z, u) is

(1, 1, 1, 1), (1, 1, 3, 1), (1, 3, 2, 3), (2, 1, 2, 2), (2, 3, 1, 2), (3, 1, 1, 3), (3, 1, 3, 2),
(1, 1, 1, 2), (1, 1, 3, 2), (2, 1, 1, 1), (2, 1, 2, 3), (2, 3, 1, 3), (3, 1, 2, 1), (3, 1, 3, 3),
(1, 1, 2, 1), (1, 1, 3, 3), (2, 1, 1, 2), (2, 1, 3, 1), (2, 3, 2, 3), (3, 1, 2, 2), (3, 3, 1, 2),
(1, 1, 2, 2), (1, 3, 1, 2), (2, 1, 1, 3), (2, 1, 3, 2), (3, 1, 1, 1), (3, 1, 2, 3), (3, 3, 1, 3),
(1, 1, 2, 3), (1, 3, 1, 3), (2, 1, 2, 1), (2, 1, 3, 3), (3, 1, 1, 2), (3, 1, 3, 1), (3, 3, 2, 3).

For $\{e\}$:

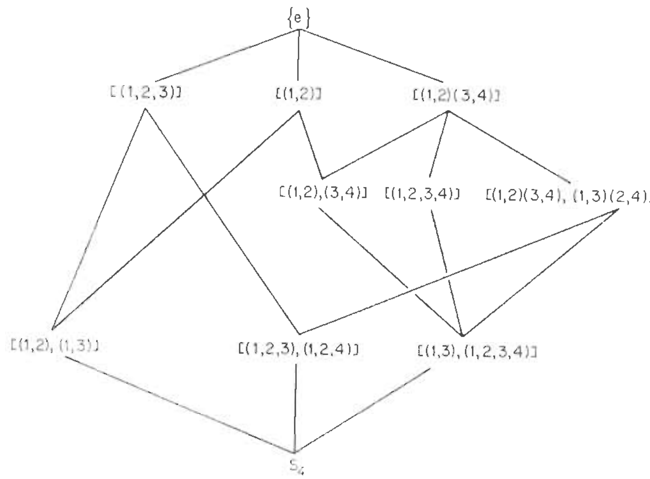
x	3	y
3	$x\beta$	$z\beta$
z	$y\beta$	3

where (x, y, z) is

(1, 1, 2), (2, 1, 2), (3, 1, 2),
(1, 1, 3), (2, 1, 3), (3, 1, 3),
(1, 2, 3), (2, 2, 3), (3, 2, 3).

1.5. Groupoids of order 4.

TABLE 3. *Subgroups of S_4*

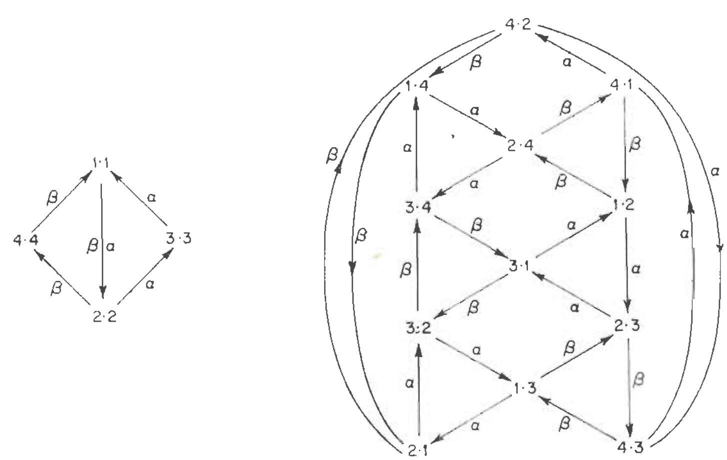


This diagram shows that A is lower than B and is connected with B by a segment if and only if some conjugate of A contains some conjugate of B.

If $\mathfrak{H} = S_4$, G is isomorphic to either a right zero or a left zero semigroup, by Theorem 1.1.

(1) $\mathfrak{H} = [(1, 2, 3), (1, 2, 4)]$ (alternating group).

Let $\alpha = (1, 2, 3)$, $\beta = (1, 2, 4)$. We have the $\mathfrak{B}$-classes:



$(3\cdot 3)\beta = 3\cdot 3$ implies $3\cdot 3 = 3$, hence G has to be idempotent. G is a right zero semigroup if $4\cdot 2 = 2$; a left zero semigroup if $4\cdot 2 = 4$. Let G_i be a groupoid determined by $4 \cdot 2 = i$ ($i = 1, 2$). G_1 is isomorphic to G_2 under a transposition $(1, 2)$ and also G_1 is anti-isomorphic to G_2.

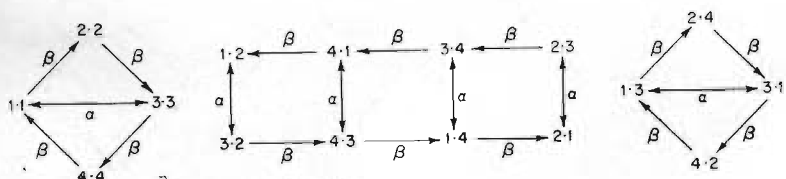
$$G_1 \quad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 3 & 4 & 2 \\ 2 & 4 & 2 & 1 & 3 \\ 3 & 2 & 4 & 3 & 1 \\ 4 & 3 & 1 & 2 & 4 \end{array} \quad \cong \quad G_2 \quad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 4 & 2 & 3 \\ 2 & 3 & 2 & 4 & 1 \\ 3 & 4 & 1 & 3 & 2 \\ 4 & 2 & 3 & 1 & 4 \end{array}$$

G_1 is characterized by the groupoid, which is neither a left nor a right zero semigroup, such that any permutation is either an automorphism or an anti-automorphism.

(2) $\mathfrak{H} = [(1, 3), (1, 2, 3, 4)]$.

The normalizer of $\mathfrak{H}$ is $\mathfrak{H}$ itself, $|\mathfrak{H}| = 8$, $n = 1$, $c = 24/8 = 3$.

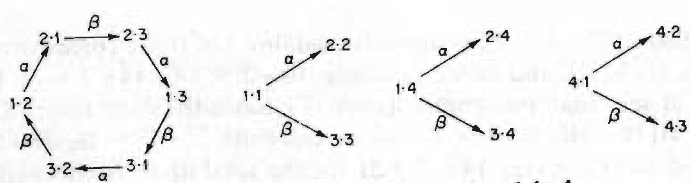
Let $\alpha = (1, 3)$, $\beta = (1, 2, 3, 4)$. We have the $\mathfrak{B}$-classes:



The calculation $2^2 \times 4 = 16$, $16 - 2 = 14$ gives the number of non-isomorphic G's. Suppose the groupoids have a dual automorphism. Since no subgroup is of order 16, every element of $\mathfrak{H}$ is a dual automorphism. For a dual automorphism α, $(1 \cdot 3)\alpha = 1 \cdot 3$, hence $1 \cdot 3 = 2$ or 4. For an automorphism β, $(1 \cdot 3)\beta = 2 \cdot 4 = 2$ or 4 because $(2 \cdot 4)\alpha = 2 \cdot 4$. However, this is a contradiction to $2\beta = 3$, $4\beta = 1$. Hence there are no self-dual G's. The number of non-isomorphic, non-anti-isomorphic G's is $14 \div 2 = 7$.

(3) $\mathfrak{H} = [(1, 2), (1, 3)]$.

$|\mathfrak{H}| = 6$, $n = 1$, $c = 24/6 = 4$. We have the $\mathfrak{B}$-classes:



The calculation $4\times 2^3 = 32$, $32-2 = 30$ gives the number of non-isomorphic G's. If a dual automorphism exists, then it is in $\mathfrak{H}$. Accordingly if G is self-dual, it must be commutative.

We find the commutative G's:

$$(1\cdot 2)\alpha = 2\cdot 1 = 1\cdot 2, \quad 1\cdot 2 = 3 \text{ or } 4.$$

We have 8 non-isomorphic commutative G's, and the calculation $30-8 = 22$, $22\div 2 = 11$ gives the number of non-self-dual G's, and we have a total of $8+11 = 19$ non-isomorphic and non-anti-isomorphic G's.

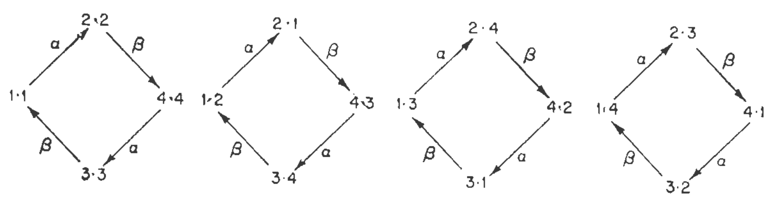
(4) $\mathfrak{H} = [(1, 2)(3, 4), (1, 3)(2, 4)]$.

$|\mathfrak{H}| = 4$, $\mathfrak{H}$ is normal, $n = 24/4 = 6$, $c = 1$,

The number of groupoids G with $\mathfrak{H} \subset \mathfrak{A}(G)$ which appeared in the previous cases is

$$14\times 3+2+2 = 46.$$

Let $\alpha = (1, 2)(3, 4)$, $\beta = (1, 3)(2, 4)$. We have the $\mathfrak{B}$-classes:

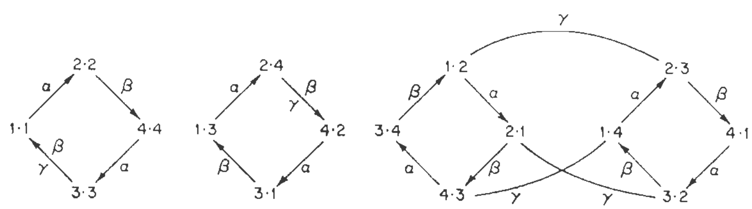


The calculation $4^4-46 = 210$, $210\div 6 = 35$ gives the number of non-isomorphic G's. There is no commutative G in the 35 G's, because

$$(1\cdot 2)\alpha = 1\cdot 2$$

and we have no value $1\cdot 2$.

Suppose $\gamma = (1, 3)$ is a dual automorphism. We then have the $\mathfrak{B}$-classes:



2·2, 2·4 = 2 or 4

The class of $2^2\times 4 = 16$ groupoids contains 2 of those corresponding to $[(1, 2, 3), (1, 2, 4)]$, and so we calculate $16-2 = 14$, $14\div 2 = 7$, for the number of self-dual, non-commutative G's. Since the normalizer of $[(1, 3), (1, 2, 3, 4)]$ is itself, $8\div 4 = 2$, and we calculate $35-7 = 28$, $28\div 2 = 14$ non-self-dual G's, giving $14+7 = 21$ for the total up to isomorphism and dual-isomorphism.

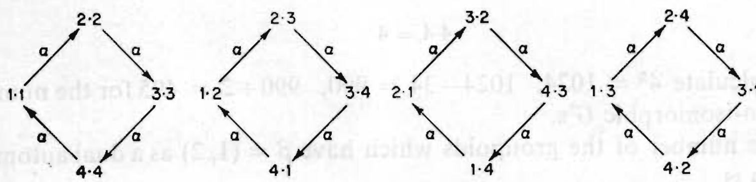
(5) $\mathfrak{H} = [(1, 2, 3, 4)]$.

$|\mathfrak{H}| = 4$. The normalizer of $\mathfrak{H}$ is $[(1,3), (1,2,3,4)]$ of order 8, $n = 8/4 = 2$, $c = 24/8 = 3$.

The number of the groupoids corresponding to the groups which contain $\mathfrak{H}$ is

$$14+2 = 16.$$

Let $\alpha = (1,2,3,4)$. We have the $\mathfrak{B}$-classes:



We calculate $4^4-16 = 240$, $240 \div 2 = 120$ for the number of non-isomorphic G's. We have $[(1,2,3,4)] \subset [(1,3),(1,2,3,4)]$. Suppose $\gamma = (1,3)$ is a dual-automorphism. Then $(1{\cdot}2)\alpha = (1{\cdot}2)\gamma$, but there is no value $1{\cdot}2$ which satisfies this. Suppose some G is commutative. Then $(1{\cdot}3)\alpha^2 = 3{\cdot}1 = 1{\cdot}3$, but there is no $1{\cdot}3$ fixed by α^2. Consequently there is no self-dual G in this case, and we have only the

$$120 \div 2 = 60 \text{ non-self-dual } G\text{'s}.$$

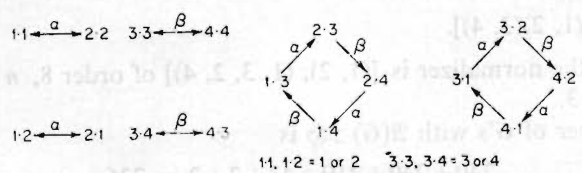
(6) $\mathfrak{H} = [(1, 2), (3, 4)]$.

$|\mathfrak{H}| = 4$. Its normalizer is $[(1, 2), (1,3,2,4)]$ of order 8, $n = 8/4 = 2$, $c = 24/8 = 3$.

The number of the groupoids corresponding to the groups $\supset \mathfrak{H}$ is

$$14+2 = 16.$$

Let $\alpha = (1, 2)$, $\beta = (3, 4)$. We have the $\mathfrak{B}$-classes:



We calculate $2^4 \times 4^2 = 256$, $256-16 = 240$, $240 \div 2 = 120$ for the number of non-isomorphic G's. We can prove that there is no self-dual G, so we have

$$120 \div 2 = 60 \text{ non-self-dual } G\text{'s}.$$

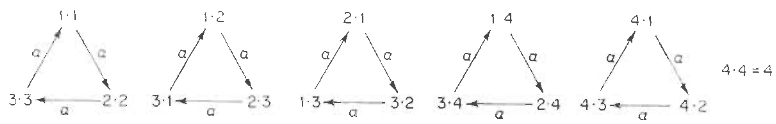
(7) $\mathfrak{H} = [(1, 2, 3)]$.

$|\mathfrak{H}| = 3$. The normalizer is $[(1, 2), (1, 3)]$ of order 6, $n = 6/3 = 2$, $c = 24/6 = 4$.

The number of groupoids corresponding to the groups bigger than $\mathfrak{H}$ is

$$30+2+2 = 34.$$

Let $\alpha = (1, 2, 3)$.



$$4\cdot4 = 4$$

We calculate $4^5 = 1024$, $1024-34 = 990$, $990\div2 = 495$ for the number of non-isomorphic G's.

The number of the groupoids which have $\beta = (1, 2)$ as a dual automorphism is

$$2^3\times4 = 32$$

since $1\cdot1 = 1$ or 4, $2\cdot3 = 1$ or 4, $3\cdot2 = 1$ or 4.

Among the 1024 groupoids, there are 64 commutative ones. Eight of the 64 correspond to $[(1, 2), (1, 3)]$, and so $32-8 = 24$ is the number of non-commutative G's which have $(1, 2)$ as a dual-automorphism. Two of these 24 G's correspond to $[(1, 2, 3), (1, 2, 4)]$, leaving

$$24-2 = 22.$$

The number of commutative G's is $(64-8)\div2 = 28$ (up to isomorphism). The number of non-commutative self-dual G's is

$$22\div2 = 11 \text{ (up to isomorphism)}.$$

To count the total number up to isomorphism and dual-isomorphism, we calculate

$495-(28+11) = 456$, $456\div2 = 228$, $228+39 = 267$ for this number.

(8) $\mathfrak{H} = [(1, 2)(3, 4)]$.

$|\mathfrak{H}| = 2$, the normalizer is $[(1, 2), (1, 3, 2, 4)]$ of order 8, $n = 8/2 = 4$, $c = 24/8 = 3$.

The number of G's with $\mathfrak{A}(G)\supset\mathfrak{H}$ is

$$240+240+210+42+2+2 = 736.$$

Under $\alpha = (1, 2)(3, 4)$, we have the $\mathfrak{B}$-classes:

$$1\cdot1 \xleftrightarrow{\alpha} 2\cdot2 \qquad 3\cdot3 \xleftrightarrow{\alpha} 4\cdot4 \qquad 1\cdot2 \xleftrightarrow{\alpha} 2\cdot1 \qquad 3\cdot4 \xleftrightarrow{\alpha} 4\cdot3$$

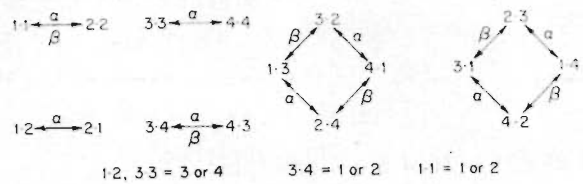
$$1\cdot4 \xleftrightarrow{\alpha} 2\cdot3 \qquad 4\cdot1 \xleftrightarrow{\alpha} 3\cdot2 \qquad 1\cdot3 \xleftrightarrow{\alpha} 2\cdot4 \qquad 3\cdot1 \xleftrightarrow{\alpha} 4\cdot2$$

The calculation $4^8-736 = 64{,}800$, $64{,}800\div4 = 16{,}200$ gives the number of non-isomorphic G's.

Considering the self-dual G's, we have

$$[(1, 2)(3, 4)] \subset [(1, 2), (3, 4)],$$
$$[(1, 2)(3, 4)] \subset [(1, 3, 2, 4)],$$
$$[(1, 2)(3, 4)] \subset [(1, 2)(3, 4), (1, 3)(2, 4)].$$

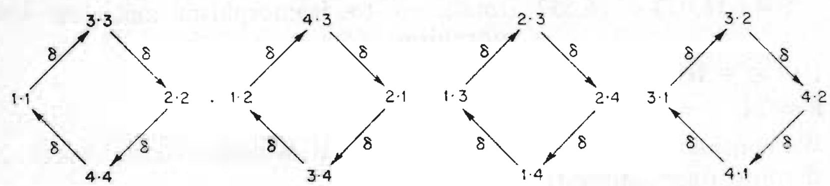
If $\beta = (1, 2)$ is a dual-automorphism, we have the $\mathfrak{B}$-classes:



Here we have $2^4 \times 4^2 = 256$ non-isomorphic G's.

If $\gamma = (1, 3)(2, 4)$, is a dual-automorphism, $(1\cdot 3)\gamma = 1\cdot 3$, but no element is fixed by γ. This case is impossible, therefore we see that there is no commutative G.

If $\delta = (1, 3, 2, 4)$ is a dual-automorphism, under δ, we have the $\mathfrak{B}$-classes:



We have $4^4 = 256$ non-isomorphic G's in this case. The two cases contain 16 groupoids in common among which 14 correspond to $[(1, 2)(3, 4), (1, 3)(2, 4)]$ and 2 to $[(1, 2, 3), (1, 2, 4)]$, and we calculate $256-16 = 240$, $240 \times 2 = 480$, $480 \div 4 = 120$ self-dual non-commutative G's, and further calculation gives $16{,}200-120 = 16{,}080$, $16{,}080 \div 2 = 8040$, $8040+120 = 8160$ for the number up to isomorphism and dual-isomorphism.

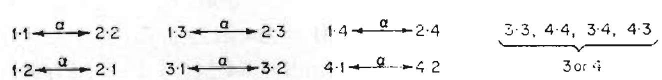
(9) $\mathfrak{H} = [(1, 2)]$.

$|\mathfrak{H}| = 2$, the normalizer is $[(1, 2), (3, 4)]$ of order 4, $n = 4/2 = 2$, $c = 24/4 = 6$.

The number of G's with $\mathfrak{A}(G) \supset \mathfrak{H}$ is

$$60+240+14+2 = 316.$$

Under $\alpha = (1, 2)$ we have the $\mathfrak{B}$-classes:



We calculate $4^6 \times 2^4 = 65{,}536$, $65{,}536 - 316 = 65{,}220$, $65{,}220 \div 2 = 32{,}610$, for the number of non-isomorphic G's.

$$[(1, 2)] \subset [(1, 2), (3, 4)].$$

Suppose $\beta = (3, 4)$ is a dual-automorphism. Then

$$(3 \cdot 4)\alpha = (3 \cdot 4)\beta = 3 \cdot 4.$$

This is impossible since no element is fixed by both α and β. Therefore there is no self-dual non-commutative G. To find all commutative G's, we have the $\mathfrak{B}$-classes:

$$1\cdot 1 \overset{\alpha}{\longleftrightarrow} 2\cdot 2 \qquad 1\cdot 2 \overset{\alpha}{\longleftrightarrow} 2\cdot 1 \qquad 1\cdot 3 \overset{\alpha}{\longleftrightarrow} 2\cdot 3 \qquad 1\cdot 4 \overset{\alpha}{\longleftrightarrow} 2\cdot 4, \qquad 3\cdot 4 = 4\cdot 3$$

$$1\cdot 2,\ 3\cdot 3,\ 3\cdot 4,\ 4\cdot 4 = 3 \text{ or } 4,$$

and we find $4^3 \times 2^4 = 1024$ non-isomorphic G's.

The 32 commutative groupoids correspond to $[(1, 2), (1, 3)]$. Of these, 16 are contained in the 1024 groupoids, and we calculate:

$1024 - 16 = 1008$,
$1008 \div 2 = 504$ (commutative, up to isomorphism),
$32{,}610 - 504 = 32{,}106$,
$32{,}106 \div 2 = 16{,}053$ (non-commutative, up to isomorphism),
$504 + 16{,}053 = 16{,}557$ (total, up to isomorphism and dual isomorphism).

(10) $\mathfrak{H} = \{e\}$.

$n = 24$, $c = 1$.

We consider G's. with dual-automorphisms. We may assume that $(1, 2)$ is the only dual-automorphism.

The $\mathfrak{R}$-classes are:

$$\begin{array}{lll} 1\cdot 1 \longleftrightarrow 2\cdot 2 & 1\cdot 4 \longleftrightarrow 4\cdot 2 & 3\cdot 3;\ 4\cdot 4,\ 1\cdot 2,\ 2\cdot 1 = 3 \text{ or } 4 \\ 1\cdot 3 \longleftrightarrow 3\cdot 2 & 4\cdot 1 \longleftrightarrow 2\cdot 4 & \\ 3\cdot 1 \longleftrightarrow 2\cdot 3 & 3\cdot 4 \longleftrightarrow 4\cdot 3 & \end{array}$$

We find $4^6 \times 2^4 = 65{,}536$ non-isomorphic self-dual G's.

Among them there are $4^3 \times 2^4 = 1024$ commutative G's, leaving $65{,}536 - 1024 = 64{,}512$ non-commutative self-dual G's.

The number of self-dual, non-commutative G's with dual-automorphism $(1, 2)$ already counted is:

for $[(1, 2, 3), (1, 2, 4)]$	$1 \times 2 = 2$
for $[(1, 2)(3, 4), (1, 3)(2, 4)]$	$7 \times 2 = 14$
for $[(1, 2, 3)]$	$11 \times 4 = 44$
for $[(1, 2)(3, 4)]$	$120 \times 2 = 240$
totalling	300
leaving	$64{,}512 - 300 = 64{,}212$.

The self-dual non-commutative G's number $64{,}212 \div 4 = 16{,}053$.

Next we count the number of commutative G's, comprising the already counted commutative G's:

$$\begin{aligned} 8\times 4 &= 32 \\ 28\times 8 &= 224 \\ 504\times 12 &= 6048 \end{aligned}$$

totalling 6304.

Solving $24x+6304 = 4^{10} = 1{,}048{,}576$, we obtain

$$x = 43{,}428.$$

To count the total number y of non-isomorphic G's, we may subtract the following sum from 4^{16}:

$$2\times 1+1\times 2+14\times 3+30\times 4+35\times 6+120\times 6$$

$$+120\times 6+495\times 8+16{,}200\times 12+32{,}610\times 12;$$

then we have $y = 178{,}932{,}325$.

To count the number z of non-self-dual G's, we have:

$$2z+43{,}428+16{,}053 = 178{,}932{,}325,$$

$$z = 89{,}436{,}422.$$

The number w of non-isomorphic, non-anti-isomorphic G's is

$$w = 43{,}428+16{,}053+89{,}436{,}422 = 89{,}495{,}903.$$

Table 4 shows the summary.

Addendum. We would like to mention the following propositions.

THEOREM 1.5. *Let G be a finite set. For every permutation group $\mathfrak{H}$ on G (i.e. $\mathfrak{H} \subseteq \mathfrak{S}(G)$), there is at least a groupoid G with $\mathfrak{H} \subseteq \mathfrak{A}(G)$.*

Let $N(\mathfrak{H})$ denote the number of all groupoids G with $\mathfrak{H} \subseteq \mathfrak{A}(G)$ and $M(\mathfrak{H})$ the number of all groupoids G with $\mathfrak{H} = \mathfrak{A}(G)$. $N(\mathfrak{H})$ and $M(\mathfrak{H})$ are the numbers which count seemingly distinct G (containing isomorphic or anti-isomorphic G's).

The following theorem is obvious.

THEOREM 1.6. *Let $\mathfrak{H}$ be a proper subgroup of $\mathfrak{S}(G)$. There exists a groupoid G with $\mathfrak{H} = \mathfrak{A}(G)$ if and only if*

$$N(\mathfrak{H}) > \sum_{\mathfrak{H} \subset \mathfrak{K}} M(\mathfrak{K}).$$

Problem. Let $|G| = 5$. Under what condition on the properties (for example transitivity) on $\mathfrak{H}$, do there exist groupoids G, $|G| = 5$, such that $\mathfrak{H} = \mathfrak{A}(G)$?

TABLE 4. *Groupoids of Order 4 and their Automorphism Groups*

$\mathfrak{H}$	c	n	Comm, up to iso	Self-dual, non-comm, up to iso	Non-self-dual, up to iso	Total up to iso, up to dual iso	Total up to iso	Semigroups up to iso, up to dual iso
S_4	1	1	0	0	1	1	2	1
[(1, 2, 3), (1, 2, 4)]	1	2	0	1	0	1	1	0
[(1, 3), (1, 2, 3, 4)]	3	1	0	0	7	7	14	0
[(1, 2), (1, 3)]	4	1	8	0	11	19	30	5
[(1, 2)(3, 4), (1,3)(2, 4)]	1	6	0	7	14	21	35	1
[(1, 2, 3, 4)]	3	2	0	0	60	60	120	0
[(1, 2), (3, 4)]	3	2	0	0	60	60	120	2
[(1, 2, 3)]	4	2	28	11	228	267	495	0
[(1, 2) (3, 4)]	3	4	0	120	8040	8160	16,200	4
[(1, 2)]	6	2	504	0	16,053	16,557	32,610	40
[e]	1	24	43,428	16,053	89,436,422	89,495,903	178,932,325	73
Total			43,968	16,192	89,460,896	89,512,864	178,981,952	126

PART II. SYSTEM OF OPERATIONS AND EXTENSION THEORY

2.1. Introduction. Let T be a right zero semigroup, i.e. $\alpha\beta = \beta$ for all $\alpha, \beta \in T$, and $\{D_\alpha; \alpha \in T\}$ be a system of semigroups with same cardinality $|D_\alpha| = m$. The problem at the present time is to construct a semigroup D such that D is a set union of D_α, $\alpha \in T$, and

$$D_\alpha D_\beta \subseteq D_\beta \quad \text{for all} \quad \alpha, \beta \in T.$$

D does not necessarily exist for an arbitrary system of semigroups. For example, let

D_1: right zero semigroup of order 2.

	a	b
a	a	b
b	a	b

D_2: a group of order 2.

	c	d
c	c	d
d	d	c

Here $D_1 \cap D_2 = \emptyset$. Then there is no semigroup D satisfying

$$D = D_1 \cup D_2, \quad D_1 D_2 \subseteq D_2, \quad D_2 D_1 \subseteq D_1.$$

So our question is this:

Under what condition on $\{D_\alpha; \alpha \in T\}$ does there exist such a semigroup D?

How can we determine all D for given T and $\{D_\alpha; \alpha \in T\}$?

The problem in some special cases was studied by R. Yoshida **[18]**, **[19]** in which he did not assume the same cardinality of D_α. In this paper we look at the problem from the more general point of view; we will introduce the concept of a general product of a set by a semigroup using the system of groupoids. Finally we will show the computing results on a certain special case. The detailed proof will be published elsewhere.

2.2. The system of operations.† Let E be a set and $\mathfrak{B}_E$ be the set of all binary operations (not necessarily associative) defined on E. Let $x, y \in E$, $\theta \in \mathfrak{B}_E$ and let $x\theta y$ denote the product of x and y by θ. A groupoid with θ defined on E is denoted by $E(\theta)$. The equality of elements of $\mathfrak{B}_E$ is defined in the natural sense:

$$\theta = \eta \text{ if and only if } x\theta y = x\eta y \text{ for all } x, y \in E.$$

Let $a \in E$ be fixed. For a we define two binary operations ${}_a\!*$ and $*_a$ as follows:

$$x(\theta \;{}_a\!*\; \eta)y = (x\theta a)\eta y, \tag{2.1}$$

$$x(\theta *_a \eta)y = x\theta(a\eta y). \tag{2.2}$$

† The system of semigroup operation was studied in [7].

Immediately we have:

PROPOSITION 2.1. *$\mathfrak{B}_E$ is a semigroup with respect to ${}_a*$ and $*_a$ for all $a \in E$.*

The semigroups $\mathfrak{B}_E$ with ${}_a*$ and $*_a$ are denoted by $\mathfrak{B}_E\,({}_a*)$ and $\mathfrak{B}_E\,(*_a)$ respectively.

$E(\theta)$ is associative if and only if $\theta \;{}_a* \;\theta = \theta *_a \;\theta$ for all $a \in E$.

Let φ be a permutation of E. For $\theta \in \mathfrak{B}_E$, $\theta\varphi$ is defined by

$$x(\theta\varphi)y = [(x\varphi^{-1})\theta(y\varphi^{-1})]\varphi. \tag{2.3}$$

Thus φ induces a permutation of $\mathfrak{B}_E$. For $\theta \in \mathfrak{B}_E$ another operation θ' is defined by

$$x\theta' y = y\theta x.$$

LEMMA.

$$(\theta \;{}_a* \eta)\varphi = (\theta\varphi) \;{}_{a\varphi}* (\eta\varphi),$$
$$(\theta *_a \eta)\varphi = (\theta\varphi) *_{a\varphi} (\eta\varphi),$$
$$(\theta \;{}_a* \eta)' = \eta' *_a \theta',$$
$$(\theta *_a \eta)' = \eta' \;{}_a* \theta'.$$

PROPOSITION 2.2. *$\mathfrak{B}_E({}_a*)$ is isomorphic with $\mathfrak{B}_E({}_b*)$ and is anti-isomorphic with $\mathfrak{B}_E(*_b)$ for all $a, b \in E$.*

2.3. General product. Let S be a set and T be a semigroup. Suppose that a mapping Θ of $T\times T$ into $\mathfrak{B}_S$, $(\alpha, \beta)\Theta = \theta_{\alpha,\beta}$, satisfies

$$\theta_{\alpha,\beta} \;{}_a* \;\theta_{\alpha\beta,\gamma} = \theta_{\alpha,\beta\gamma} *_a \theta_{\beta,\gamma} \quad \text{for all } \alpha, \beta, \gamma \in T \text{ and all } a \in S. \tag{2.4}$$

Consider the product set

$$S\times T = \{(x, \alpha);\, x \in S, \alpha \in T\}$$

in which $(x, \alpha) = (y, \beta)$ if and only if $x = y$, $\alpha = \beta$.

Given S, T, Θ, a binary operation is defined on $S\times T$ as follows:

$$(x, \alpha)\,(y, \beta) = (x\theta_{\alpha,\beta}\,y,\, \alpha\beta). \tag{2.5}$$

PROPOSITION 2.3. *$S\times T$ is a semigroup with respect to the operation* (2.5), *and it is homomorphic onto T under the projection $(x, \alpha)\to\alpha$.*

Definition. The semigroup, $S\times T$ with (2.5), is called a *general product* of a set S by a semigroup T with respect to Θ, and is denoted by

$$S\overline{\times}_\Theta T.$$

If it is not necessary to specify Θ it is denoted by

$$S\overline{\times}T.$$

PROPOSITION 2.4. *Suppose that T is isomorphic with T' under a mapping ψ and $|S| = |S'|$; let φ be a bijection $S\to S'$. Then*

$$S\overline{\times}_\Theta T \cong S'\overline{\times}_{\Theta'} T'$$

where $\quad \Theta = \{\theta_{\alpha,\beta};\, (\alpha, \beta) \in T\times T\}$, $\Theta' = \{\theta'_{\alpha\psi,\beta\psi};\, (\alpha\psi, \beta\psi) \in T'\times T'\}$

and $\quad x\theta'_{\alpha\psi,\beta\psi}y = [(x\varphi^{-1})\theta_{\alpha,\beta}(y\varphi^{-1})]\varphi, \quad x, y \in S'.$

In this case we say that Θ in S is equivalent to Θ' in S'.

We understand that $S\overline{\times}_{\Theta}T$ is determined by T, $|S|$ and the equivalence of Θ in the above sense.

Definition. If a semigroup D is isomorphic onto some $S\overline{\times}_{\Theta}T$, then D is called *general product decomposable* (*gp-decomposable*). If $|S|>1$ and $|T|>1$, then D is called *properly gp-decomposable.*

Definition. Let g be a homomorphism of a semigroup D onto a semigroup T: $D=\bigcup_{\alpha\in T}D_\alpha$, $D_\alpha g=\alpha$. If $|D_\alpha|=|D_\beta|$ for all α, $\beta\in T$, then g is called a *homogeneous homomorphism* (*h-homomorphism*) of D, or D is said to be *h-homomorphic* onto T. If $|D_\alpha|>1$ and $|T|>1$, then g is called a *proper h-homomorphism.*

THEOREM 2.1. *A semigroup D is gp-decomposable if and only if D has an h-homomorphism.*

In other words, $D\cong S\overline{\times}_{\Theta}T$, $|S|>1$, $|T|>1$, *for some* Θ *if and only if D is properly h-homomorphic onto T.*

Proof. Suppose that D is h-homomorphic onto T under g.

$$D=\bigcup_{\alpha\in T}D_\alpha,\quad D_\alpha g=\alpha.$$

Let S be a set with $|S|=|D_\alpha|$ for all $\alpha\in T$, and let f_α be a bijection of D_α to S. Fixing $\{f_\alpha;\ \alpha\in T\}$, for each $(\alpha,\beta)\in T\times T$ we define a binary operation $\theta_{\alpha,\beta}$ on S as follows. Let $x, y\in S$:

$$x\theta_{\alpha,\beta}y=[(xf_\alpha^{-1})(xf_\beta^{-1})]f_{\alpha\beta}.$$

Let a be any element of D, hence $a\in D_\alpha$ for some $\alpha\in T$. We define a mapping ψ of D onto $S\times T$ as follows:

$$a\xrightarrow{\psi}(af_\alpha,\alpha).$$

Then ψ is an isomorphism of D onto $S\overline{\times}_{\Theta}T$. The proof of the converse is easy.

Even if D, S, T are given, Θ depends on the choice of $\{f_\alpha;\ \alpha\in T\}$. However, Θ is unique in some sense. To explain this situation we shall define a terminology.

Definition. Let g and g' be homomorphisms of semigroups A and B onto a semigroup C respectively. An isomorphism h of A into (onto) B is called a *restricted isomorphism* of A into (onto) B with respect to g and g' or we say A is *restrictedly isomorphic* into (onto) B with respect to g and g' if there is an automorphism k of C such that $h\cdot g'=g\cdot k$:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[g']{} & C \end{array}$$

Definition. Let $G(\theta)$ and $G'(\theta')$ be groupoids with binary operations θ, θ' respectively. If there are three bijections h, q, r of $G(\theta)$ to $G'(\theta')$ such that

$$(x\theta y)r = (xh)\theta'(yq) \quad \text{for all} \quad x, y \in G(\theta),$$

then we say that $G(\theta)$ is *isotopic* to $G'(\theta')$. If it is necessary to specify h, q, r, we say $G(\theta)$ is (h, q, r)*-isotopic* to $G'(\theta')$. We denote it by

$$G(\theta) \underset{(h,q,r)}{\approx} G'(\theta') \quad \text{or} \quad G(\theta) \approx G'(\theta').$$

THEOREM 2.2. *Let S and T be a fixed set and a semigroup respectively. Let $(\alpha,\beta)\Theta = \theta_{\alpha,\beta}$, $(\alpha,\beta)\Theta' = \theta'_{\alpha,\beta}$, $\alpha, \beta \in T$. $S\overline{\times}_{\Theta}T$ is restrictedly isomorphic onto $S\overline{\times}_{\Theta'}T$ with respect to the projections of $S\overline{\times}_{\Theta}T$ and $S\overline{\times}_{\Theta'}T$ to T if and only if there is an automorphism $\alpha \to \alpha'$ of T and a system $\{f_\alpha;\ \alpha \in T\}$ of permutations of S such that a groupoid $S(\theta_{\alpha,\beta})$ is $(f_\alpha, f_\beta, f_{\alpha\beta})$-isotopic to $S(\theta'_{\alpha',\beta'})$ for all $\alpha, \beta \in T$.*

Let ϱ and σ be relations on a semigroup D. As usual the product $\varrho\cdot\sigma$ of ϱ and σ is defined by

$$\varrho\cdot\sigma = \{(x, y);\ (x, z) \in \varrho, \quad (z, y) \in \sigma \quad \text{for some} \quad z \in D\}.$$

Let $\omega = D\times D$, $\iota = \{(x, x);\ x \in D\}$.

THEOREM 2.3. *A semigroup D is gp-decomposable if and only if there is a congruence ϱ on D and an equivalence σ on D such that*

$$\varrho\cdot\sigma = \omega, \tag{2.6}$$

$$\varrho \cap \sigma = \iota, \tag{2.7}$$

in which (2.6) *can be replaced by*

$$\sigma\cdot\varrho = \omega. \tag{2.6'}$$

Then $D \cong (D/\sigma)\overline{\times}(D/\varrho)$ where D/ϱ is the factor semigroup of D modulo ϱ and D/σ is the factor set of D modulo σ.

We know many examples of general products: Direct product, semidirect product **[3]**, **[6]**, group extension **[3]**, Rees' regular representation of completely simple semigroups **[1]**, the representation of commutative archimedean cancellative semigroups without idempotent **[11]**, $\mathfrak{H}$-semigroups **[15]**, and so on.

2.4. Left general product. As a special case of a general product, we make the

Definition. A general product $S\overline{\times}_{\Theta}T$ is called a *left general product* of S by T if and only if

$$(\alpha, \beta)\Theta = (\alpha, \gamma)\Theta \quad \text{for all} \quad \alpha, \beta, \gamma \in T. \tag{2.8}$$

$S\overline{\times}_{\Theta}T$ is called a *right general product* of S by T if and only if

$$(\alpha, \beta)\Theta = (\gamma, \beta)\Theta \quad \text{for all} \quad \alpha, \beta, \gamma \in T. \tag{2.8'}$$

In case (2.8), $\theta_{\alpha,\beta}$ depends on only α, so $\theta_{\alpha,\beta}$ is denoted by $\theta_{\alpha}.$. Then (2.4) is rewritten:

$$\theta_{\alpha}. \; {}_a\!\ast \; \theta_{\alpha\beta}\cdot = \theta_{\alpha}. \; \ast_a \; \theta_{\beta}. \quad \text{for all } \alpha, \beta \in T, \text{ all } a \in S. \tag{2.9}$$

In case (2.8′), $\theta_{\alpha,\beta}$ is independent of α, and $\theta_{\alpha,\beta}$ is denoted by $\theta_{.\beta}$ and (2.4) is

$$\theta_{.\alpha} \; {}_a\!\ast \; \theta_{.\beta} = \theta_{.\alpha\beta} \; \ast_a \; \theta_{.\beta} \quad \text{for all } \alpha, \beta \in T, \text{ all } a \in S. \tag{2.9'}$$

A left congruence is a left compatible equivalence, namely an equivalence σ satisfying

$$x \,\sigma\, y \Rightarrow zx \,\sigma\, zy \quad \text{for all} \quad z.$$

THEOREM 2.4. *Let D be a semigroup. D is isomorphic onto a left general product of a set S by a semigroup T if and only if there is a congruence ϱ on D and a left congruence σ on D such that*

$$D/\varrho \cong T, \quad |D/\sigma| = |S|$$

and

$$\varrho\cdot\sigma = \omega \quad (\textit{equivalently } \sigma\cdot\varrho = \omega),$$
$$\varrho \cap \sigma = \iota.$$

EXAMPLE. Let T be a semigroup, F a set, and let x denote a mapping of F into T:

$$\lambda x = \alpha_\lambda \quad \text{where} \quad \lambda \in F, \; \alpha_\lambda \in T.$$

The set of all mappings x of F into T is denoted by S. For $\beta \in T$ and $x \in S$ we define an element $\beta\cdot x$ as follows:

$$\lambda x = \alpha_\lambda \Rightarrow \lambda(\beta\cdot x) = \beta\alpha_\lambda.$$

Then

$$(\beta\gamma)\cdot x = \beta\cdot(\gamma\cdot x).$$

A binary operation is defined on $G = S \times T$ as follows:

$$(x, \alpha)(y, \beta) = (\alpha\cdot y, \alpha\beta). \tag{2.10}$$

Then G is a semigroup with respect to (2.10) and it is a left general product of S by T. Further the semigroup G with (2.10) is completely determined by a semigroup T and a cardinal number $m = |F|$, and G is denoted by

$$G = \mathfrak{S}\mathfrak{D}_m(T).$$

We can describe the structure of $\mathfrak{B}_E({}_a\!\ast)$ in terms of the semigroup of this kind.

THEOREM 2.5. *Let $m = |E|-1$ and $\mathfrak{T}_E$ be the full transformation semigroup over E (cf.* [1]*). $\mathfrak{B}_E({}_a\!\ast)$ is isomorphic onto $\mathfrak{S}\mathfrak{D}_m(\mathfrak{T}_E)$.*

2.5. Sub-general product. In § 2.3 we found that the two concepts, h-homomorphism and general product, are equivalent. What relationship does there exist between general products and homomorphisms?

Let U be a subset of $S\overline{\times}T$, and define

$$P_{rj_T}(U) = \{\alpha \in T;\quad (x, \alpha) \in U\}.$$

Definition. If U is a subsemigroup of $S\overline{\times}_\Theta T$ and if $p_{rj_T}(U) = T$, then U is called a *sub-general product* of $S\overline{\times}_\Theta T$.

In the following theorem, the latter statement makes the theorem have sense.

THEOREM 2.6. *If a semigroup D is homomorphic onto a semigroup T under a mapping g, then D is restrictedly isomorphic into $S\overline{\times}_\Theta T$ with respect to g and the projection of $S\overline{\times}_\Theta T$ to T for some S. Furthermore there exists an S_0 among the above S such that $|S_0|$ is either the minimum of $|S|$ or possibly the minimum plus one.*

Proof. Let $D = \bigcup_{\alpha \in T} D_\alpha$, $D_\alpha g = \alpha$, Clearly $|D_\alpha| \leqq |D|$ for all $\alpha \in T$. The set $\{|D_\alpha|; \alpha \in T\}$ has a least upper bound. (For this the well-ordered principle is used.) Let

$$m = 1 + \text{l.u.b. } \{|D_\alpha|;\quad \alpha \in T\}$$

and take a system of sets S_α of symbols such that

$$|S_\alpha| = m \quad \text{for all} \quad \alpha \in T$$

and a set S_0 with $|S_0| = m$. Further we assume that $D_\alpha \subsetneqq S_\alpha$ and S_α contains a special symbol 0_α,

$$0_\alpha \notin D_\alpha,$$

and S_0 contains a special symbol 0. Now let f_α be a bijection of S to S_α such that

$$0f_\alpha = 0_\alpha.$$

We define a binary operation on $G = S \times T$ as follows:

$$(x, \alpha)(y, \beta) = \begin{cases} ((xf_\alpha \cdot yf_\beta)f_{\alpha\beta}^{-1}, \alpha\beta) & xf_\alpha \in D_\alpha,\ yf_\beta \in D_\beta \\ (0, \alpha\beta) & \text{otherwise.} \end{cases}$$

Then we can prove that $G = S\overline{\times}_\Theta T$ where

$$x\theta_{\alpha,\beta}y = \begin{cases} ((xf_\alpha)(yf_\beta))f_{\alpha\beta}^{-1}, & xf_\alpha \in D_\beta,\ yf_\beta \in D_\beta \\ 0 & \text{otherwise.} \end{cases}$$

Let $D' = \{(x, \alpha); xf_\alpha \in D_\alpha, \alpha \in T\}$. Then $P_{rj_T}(D') = T$ and $D' \cong D$ under $(x, \alpha) \to xf_\alpha$, $\alpha \in T$.

2.6. Construction of some general products. As a simplest interesting example of general product, we will construct all general left products of a set S by a right zero semigroup T.

Let $T = \{\alpha, \beta\}$, $\quad \begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & \alpha & \beta \\ \beta & \alpha & \beta \end{array}$

The equations (2.9) are

$$\left.\begin{aligned}
\theta_{\alpha}.\ {}_a{*}\ \theta_{\beta}. &= \theta_{\alpha}.\ {*}_a\ \theta_{\beta}.\\
\theta_{\beta}.\ {}_a{*}\ \theta_{\alpha}. &= \theta_{\beta}.\ {*}_a\ \theta_{\alpha}.\\
\theta_{\alpha}.\ {}_a{*}\ \theta_{\alpha}. &= \theta_{\alpha}.\ {*}_a\ \theta_{\alpha}.\\
\theta_{\beta}.\ {}_a{*}\ \theta_{\beta}. &= \theta_{\beta}.\ {*}_a\ \theta_{\beta}.
\end{aligned}\right\} \tag{2.11}$$

$\theta_{\alpha}.$ and $\theta_{\beta}.$ are semigroup operations. In order to construct all left general products $G = S\overline{\times}_{\theta}T$ we may find all ordered pairs of semigroup operations on S:

$$(\theta_{\alpha}., \theta_{\beta}.)$$

which corresponds to

$$G = G_{\alpha}(\theta_{\alpha}.) \cup G_{\beta}(\theta_{\beta}.), \quad |G_{\alpha}| = |G_{\beta}|.$$

For fixed T and S, G is denoted by $G(\theta_{\alpha}., \theta_{\beta}.)$. Clearly

$$G(\theta_{\alpha}., \theta_{\beta}.) \cong G(\theta_{\beta}., \theta_{\alpha}.).$$

Instead of ordered pairs it is sufficient to find pairs $(\theta_{\alpha}., \theta_{\beta}.)$ regardless of order.

Let $\mathfrak{S}_S$ denote the set of all semigroup operations defined on S. ($\mathfrak{S}_S$ contains isomorphic ones.) We define a relation $\sim$ on $\mathfrak{S}_S$ as follows:

$\theta \sim \eta$ if and only if $\theta\ {}_a{*}\eta = \theta {*}_a \eta$ and $\eta\ {}_a{*}\theta = \eta\ {*}_a\theta$ for all $a \in S$.

The relation $\sim$ is reflexive symmetric.

Let φ_x^{θ}, ψ_x^{θ} be transformations of S defined by

$$z\varphi_x^{\theta} = z\theta x, \quad z\psi_x^{\theta} = x\theta z$$

respectively. Then

$$\theta\ {}_a{*}\eta = \theta {*}_a \eta \text{ for all } a \in S, \text{ if and only if}$$
$$\psi_x^{\theta}\varphi_y^{\eta} = \varphi_y^{\eta}\psi_x^{\theta} \text{ for all } x, y \in S.$$

As special cases we will determine the relation $\sim$ on $\mathfrak{S}_S$ in the case $|S| \leqslant 3$.

I. *Left general product of S, $|S| = 2$, by right zero semigroup T.*

Let $S = \{a, b\}$. $\boxed{\begin{matrix} x & y \\ z & u \end{matrix}}$, $x, y, z, u = a$ or b, is the table $\begin{array}{c|cc} & a & b \\ \hline a & x & y \\ b & z & u \end{array}$

Explanation of the notations which will be used later: For example

4 denotes the semigroup $\boxed{\begin{matrix} a & b \\ b & a \end{matrix}}$, i.e. $4_0 = 4$.

4_1 denotes $\boxed{\begin{matrix} b & a \\ a & b \end{matrix}}$ which is the isomorphic image of 4 under 1 $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

TABLE 5. *Semigroups of Order 2*

	$0\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$1\begin{pmatrix} a & b \\ b & a \end{pmatrix}$
1	$\boxed{\begin{matrix} a & b \\ a & b \end{matrix}}$	0
2	$\boxed{\begin{matrix} a & a \\ a & a \end{matrix}}$	$\boxed{\begin{matrix} b & b \\ b & b \end{matrix}}$
3	$\boxed{\begin{matrix} a & a \\ a & b \end{matrix}}$	$\boxed{\begin{matrix} a & b \\ b & b \end{matrix}}$
4	$\boxed{\begin{matrix} a & b \\ b & a \end{matrix}}$	$\boxed{\begin{matrix} b & a \\ a & b \end{matrix}}$

1_1 is exactly the same as 1, i.e. $1_0 = 1_1$.

$1'$ denotes $\boxed{\begin{matrix} a & a \\ b & b \end{matrix}}$, omitted from Table 5.

Table 6 shows all η such that $\theta_0 \sim \eta$. We may pick θ_0 from all non-isomorphic semigroups, but must select η from all semigroups. Generally the following holds:

$$\theta \sim \eta \text{ implies } \theta\varphi \sim \eta\varphi \text{ for all permutations } \varphi \text{ of } S \text{ (see § 2.2)}, \tag{2.12}$$

$$\theta \sim \eta \text{ implies } \eta' \sim \theta'. \tag{2.13}$$

TABLE 6

$\theta_0 \sim \eta$	
θ_0	η
1	1
$1'$	$1'$
2	2, 3
3	2, 3
4	4, 4_1

From the table we also have

$$\theta_0 = 2_1, \quad \eta = 2_1, 3_1,$$
$$\theta_0 = 3_1, \quad \eta = 2_1, 3_1,$$
$$\theta_0 = 4_1, \quad \eta = 4,\ 4_1.$$

Table 7 shows all non-isomorphic left general products D of S, $|S| = 2$, by a right zero semigroup of order 2.

TABLE 7

θ_{α}.	θ_{β}.
1	1
1′	1′
2	2
2	3
3	3
4	4

As an application of the above results, we have

THEOREM 2.7. *Let S be a set, $|S| = 2$, and T be a right zero semigroup of order n. A left general product D of S by T is isomorphic onto either the direct product of a semigroup S of order 2 and a right zero semigroup T of order n*

$$D \cong S \times T, \quad |S| = 2, \quad |T| = n$$

or the union of the two direct products

$$D = (S_1 \times T_1) \cup (S_2 \times T_2),$$

where T_1 and T_2 are right zero semigroups, $|T_1| + |T_2| = n$ and S_1 is a null semigroup of order 2 and S_2 is a semilattice of order 2.

II. *Left general product of S, $|S| = 3$, by right zero semigroup.*

Let $T = \{\alpha, \beta\}$, $\begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & \alpha & \beta \\ \beta & \alpha & \beta \end{array}$.

The method is the same as in case I, and we use the same notation. Let $\mathfrak{S}_3$ denote the set of all semigroups defined on S, $|S| = 3$. Table 8 shows $\mathfrak{S}_3$ except the dual forms. Those were copied from **[8]**, **[10]**. Table 9 shows all η for given θ_0 such that $\theta_0 \sim \eta$.

This table shows, for example, that $2 = 2_0 = 2_1$, $2_2 = 2_3$, $2_4 = 2_5$.

In the following family $\mathfrak{T}$ of ten subsets of $\mathfrak{S}_3$, each set satisfies the property: Any two elements of each set are $\sim$-equivalent, and each set is a maximal set with this property.

$$\mathfrak{T}\begin{cases} \{1\},\ \{2, 3, 15\},\ \{4, 5_2, 16\},\ 6, 6_2, 6_4\},\ \{7, 7_2, 11\}, \\ \{7, 12_2\},\ \{2, 8, 14', 14'_1\},\ \{2, 9, 18\},\ \{2, 10, 10_1\},\ \{2, 13, 13_1\ 17\}. \end{cases}$$

Let $\mathfrak{T}'$ denote the family obtained from $\mathfrak{T}$ by replacing $\{6, 6_2, 6_4,\}$ by $\{6\}$ and $\{2, 10, 10_1\}$ by $\{2, 10\}$ and leaving the remaining sets unchanged.

TABLE 8. *All Semigroups of Order 3 up to Isomorphism and Dual-isomorphism*

	$0\begin{pmatrix}a&b&c\\a&b&c\end{pmatrix}$	$1\begin{pmatrix}a&b&c\\a&c&b\end{pmatrix}$	$2\begin{pmatrix}a&b&c\\b&a&c\end{pmatrix}$	$3\begin{pmatrix}a&b&c\\b&c&a\end{pmatrix}$	$4\begin{pmatrix}a&b&c\\c&a&b\end{pmatrix}$	$5\begin{pmatrix}a&b&c\\c&b&a\end{pmatrix}$
1	$\boxed{\begin{matrix}a&b&c\\a&b&c\\a&b&c\end{matrix}}$	0	0	0	0	0
2	$\boxed{\begin{matrix}a&a&a\\a&a&a\\a&a&a\end{matrix}}$	0	$\boxed{\begin{matrix}b&b&b\\b&b&b\\b&b&b\end{matrix}}$	2	$\boxed{\begin{matrix}c&c&c\\c&c&c\\c&c&c\end{matrix}}$	4
3	$\boxed{\begin{matrix}a&a&a\\a&a&a\\a&a&b\end{matrix}}$	$\boxed{\begin{matrix}a&a&a\\a&c&a\\a&a&a\end{matrix}}$	$\boxed{\begin{matrix}b&b&b\\b&b&b\\b&b&a\end{matrix}}$	$\boxed{\begin{matrix}c&b&b\\b&b&b\\b&b&b\end{matrix}}$	$\boxed{\begin{matrix}c&c&c\\c&a&c\\c&c&c\end{matrix}}$	$\boxed{\begin{matrix}b&c&c\\c&c&c\\c&c&c\end{matrix}}$
4	$\boxed{\begin{matrix}a&b&a\\b&a&b\\a&b&a\end{matrix}}$	$\boxed{\begin{matrix}a&a&c\\a&a&c\\c&c&a\end{matrix}}$	$\boxed{\begin{matrix}b&a&a\\a&b&b\\a&b&b\end{matrix}}$	$\boxed{\begin{matrix}b&b&c\\b&b&c\\c&c&b\end{matrix}}$	$\boxed{\begin{matrix}c&a&a\\a&c&c\\a&c&c\end{matrix}}$	$\boxed{\begin{matrix}c&b&c\\b&c&b\\c&b&c\end{matrix}}$
5	$\boxed{\begin{matrix}a&b&b\\b&a&a\\b&a&a\end{matrix}}$	$\boxed{\begin{matrix}a&c&c\\c&a&a\\c&a&a\end{matrix}}$	$\boxed{\begin{matrix}b&a&b\\a&b&a\\b&a&b\end{matrix}}$	$\boxed{\begin{matrix}b&c&b\\c&b&c\\b&c&b\end{matrix}}$	$\boxed{\begin{matrix}c&c&a\\c&c&a\\a&a&c\end{matrix}}$	$\boxed{\begin{matrix}c&c&b\\c&c&b\\b&b&c\end{matrix}}$
6	$\boxed{\begin{matrix}a&b&c\\b&c&a\\c&a&b\end{matrix}}$	0	$\boxed{\begin{matrix}c&a&b\\a&b&c\\b&c&a\end{matrix}}$	2	$\boxed{\begin{matrix}b&c&a\\c&a&b\\a&b&c\end{matrix}}$	4
7	$\boxed{\begin{matrix}a&b&a\\a&b&a\\a&b&a\end{matrix}}$	$\boxed{\begin{matrix}a&a&c\\a&a&c\\a&a&c\end{matrix}}$	$\boxed{\begin{matrix}a&b&b\\a&b&b\\a&b&b\end{matrix}}$	$\boxed{\begin{matrix}b&b&c\\b&b&c\\b&b&c\end{matrix}}$	$\boxed{\begin{matrix}a&c&c\\a&c&c\\a&c&c\end{matrix}}$	$\boxed{\begin{matrix}c&b&c\\c&b&c\\c&b&c\end{matrix}}$
8	$\boxed{\begin{matrix}a&a&a\\a&b&c\\a&b&c\end{matrix}}$	0	$\boxed{\begin{matrix}a&b&c\\b&b&b\\a&b&c\end{matrix}}$	2	$\boxed{\begin{matrix}a&b&c\\a&b&c\\c&c&c\end{matrix}}$	4
9	$\boxed{\begin{matrix}a&a&a\\a&b&b\\a&b&b\end{matrix}}$	$\boxed{\begin{matrix}a&a&a\\a&c&c\\a&c&c\end{matrix}}$	$\boxed{\begin{matrix}a&b&a\\b&b&b\\a&b&a\end{matrix}}$	$\boxed{\begin{matrix}c&b&c\\b&b&b\\c&b&c\end{matrix}}$	$\boxed{\begin{matrix}a&a&c\\a&a&c\\c&c&c\end{matrix}}$	$\boxed{\begin{matrix}b&b&c\\b&b&c\\c&c&c\end{matrix}}$
10	$\boxed{\begin{matrix}a&a&a\\a&b&c\\a&c&b\end{matrix}}$	$\boxed{\begin{matrix}a&a&a\\a&c&b\\a&b&c\end{matrix}}$	$\boxed{\begin{matrix}a&b&c\\b&b&b\\c&b&a\end{matrix}}$	$\boxed{\begin{matrix}c&b&a\\b&b&b\\a&b&c\end{matrix}}$	$\boxed{\begin{matrix}a&b&c\\b&a&c\\c&c&c\end{matrix}}$	$\boxed{\begin{matrix}b&a&c\\a&b&c\\c&c&c\end{matrix}}$

TABLE 8 (*continued*)

	$0\begin{pmatrix}a&b&c\\a&b&c\end{pmatrix}$	$1\begin{pmatrix}a&b&c\\a&c&b\end{pmatrix}$	$2\begin{pmatrix}a&b&c\\b&a&c\end{pmatrix}$	$3\begin{pmatrix}a&b&c\\b&c&a\end{pmatrix}$	$4\begin{pmatrix}a&b&c\\c&a&b\end{pmatrix}$	$5\begin{pmatrix}a&b&c\\c&b&a\end{pmatrix}$
11	*a b a* *a b b* *a b c*	*a a c* *a b c* *a c c*	0	*a b c* *b b c* *c b c*	1	3
12	*a b b* *a b b* *a b c*	*a c c* *a b c* *a c c*	*a b a* *a b a* *a b c*	*a b c* *c b c* *c b c*	*a a c* *a b c* *a a c*	*a b c* *b b c* *b b c*
13	*a a a* *a a a* *a a c*	*a a a* *a b a* *a a a*	*b b b* *b b b* *b b c*	*a b b* *b b b* *b b b*	*c c c* *c b c* *c c c*	*a c c* *c c c* *c c c*
14	*a a a* *a a b* *a a c*	*a a a* *a b a* *a c a*	*b b a* *b b b* *b b c*	*a b b* *b b b* *c b b*	*c a c* *c b c* *c c c*	*a c c* *b c c* *c c c*
15	*a a a* *a a b* *a b c*	*a a a* *a b c* *a c a*	*b b a* *b b b* *a b c*	*a b c* *b b b* *c b b*	*c a c* *a b c* *c c c*	*a b c* *b c c* *c c c*
16	*a b a* *b a b* *a b c*	*a a c* *a b c* *c c a*	*b a a* *a b b* *a b c*	*a b c* *b b c* *c c b*	*c a a* *a b c* *a c c*	*a b c* *b c b* *c b c*
17	*a a a* *a b a* *a a c*	0	*a b b* *b b b* *b b c*	2	*a c c* *c b c* *c c c*	4
18	*a a a* *a b b* *a b c*	*a a a* *a b c* *a c c*	*a b a* *b b b* *a b c*	*a b c* *b b b* *c b c*	*a a c* *a b c* *c c c*	*a b c* *b b c* *c c c*

We have the following theorem, in which we do not assume T is finite:

THEOREM 2.8. *A left general product D of S, $|S| = 3$, by a right zero semigroup T is determined by a mapping π of the set T into one of the sets belonging to $\mathfrak{T}'$ in such a way that $\theta_{\alpha\cdot} = \pi(\alpha)$, $\alpha \in T$. Every left general product D of S, $|S| = 3$, by T is isomorphic or anti-isomorphic onto one of those thus obtained. Accordingly D is the disjoint union of at most four* *distinct (but not necessarily isomorphically distinct) direct products, i.e.*

$$D = \bigcup_{i=1}^{m} (S(\theta_i) \times T_i), \quad m \leqq 4,$$

where $T = \bigcup_{i=1}^{m} T_i$, T_i' *s are right zero semigroups and either* $\theta_i = \pi(T_i)$ $(i = 1, \ldots, m)$ *or* $\theta_i' = \pi(T_i)$ $(i = 1, \ldots, m)$.

III. *Right general product of S by right zero semigroup.*

First let $T = \{\alpha, \beta\}$, $\begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & \alpha & \beta \\ \beta & \alpha & \beta \end{array}$.

The equations (2.9′) are

$$\left.\begin{array}{l} \theta_{\cdot\alpha} \ {}_a{*}\ \theta_{\cdot\beta} = \theta_{\cdot\beta} \ {*}_a\ \theta_{\cdot\beta} \\ \theta_{\cdot\beta} \ {}_a{*}\ \theta_{\cdot\alpha} = \theta_{\cdot\alpha} \ {*}_a\ \theta_{\cdot\alpha} \\ \theta_{\cdot\alpha} \ {}_a{*}\ \theta_{\cdot\alpha} = \theta_{\cdot\alpha} \ {*}_a\ \theta_{\cdot\alpha} \\ \theta_{\cdot\beta} \ {}_a{*}\ \theta_{\cdot\beta} = \theta_{\cdot\beta} \ {*}_a\ \theta_{\cdot\beta} \end{array}\right\} \tag{2.14}$$

A relation $\approx$ is defined on $\mathfrak{S}_S$ as follows:

$\theta \approx \eta$ if and only if

$$\left.\begin{array}{l} \theta \ {}_a{*}\ \eta = \eta \ {*}_a\ \eta \\ \eta \ {}_a{*}\ \theta = \theta \ {*}_a\ \theta \end{array}\right. \qquad \text{for all } a \in S.$$

Recall that

$$z\varphi_x^\theta = z\theta x, \quad z\psi_x^\theta = x\theta z.$$

Using these notations,

$\theta \ {}_a{*}\ \eta = \eta \ {*}_a\ \eta$ for all $a \in S$ if and only if $\psi^{\eta}_{x\theta y} = \psi^{\eta}_y\psi^{\eta}_x$ for all $x, y \in S$.

Therefore $x \to \psi^{\eta}_x$ is an anti-homomorphism of a semigroup $S(\theta)$ into the left regular representation of a semigroup $S(\eta)$.

We have obtained all non-isomorphic right general products of S, $|S| \leqq 3$, by a right zero semigroup of order 2. The results will be published elsewhere.

IV. *General product of S by a right zero semigroup T of order* 2.

Let $T = \{\alpha, \beta\}$, $\begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & \alpha & \beta \\ \beta & \alpha & \beta \end{array}$.

TABLE 9†

$\theta_0 \sim \eta$

θ_0	η
1	1
2	2, 3, 3_1, 8, 8′, 9, 9_1, 10, 10_1, 13, 13_1, 14, 14_1, 14′, $14'_1$, 15, 15_1, 17, 18, 18_1
3	2, 3, 15
4	4, 5_2, 16
5	4_2, 5, 16_2
6	6, 6_2, 6_4
7	7, 7_2, 11, 12_2
8	2, 8, 14′, $14'_1$
9	2, 9, 18
10	2, 10, 10_1
11	7, 7_2, 11
12	7_2, 12
13	2, 13, 13_1, 17
14	2, 8′, 14, 14_1
15	2, 3, 15
16	4, 5_2, 16
17	2, 13, 13_1, 17
18	2, 9, 18

† These were computed by P. Dubois, J. Youngs, T. Okamoto, R. Kaneiwa, and A. Ohta under the author's direction.

To find $(\theta_{\alpha,\alpha}, \theta_{\alpha,\beta}, \theta_{\beta,\alpha}, \theta_{\beta,\beta})$ we may solve the following equations:

$$\left.\begin{aligned}
&\theta_{\alpha,\alpha}\ {}_a{*}\ \theta_{\alpha,\beta} = \theta_{\alpha,\beta} *_a \theta_{\alpha,\beta}, && \theta_{\beta,\beta}\ {}_a{*}\ \theta_{\beta,\alpha} = \theta_{\beta,\alpha} *_a \theta_{\beta,\alpha},\\
&\theta_{\alpha,\beta}\ {}_a{*}\ \theta_{\beta,\alpha} = \theta_{\alpha,\alpha} *_a \theta_{\beta,\alpha}, && \theta_{\beta,\alpha}\ {}_a{*}\ \theta_{\alpha,\beta} = \theta_{\beta,\beta} *_a \theta_{\alpha,\beta},\\
&\theta_{\alpha,\beta}\ {}_a{*}\ \theta_{\beta,\beta} = \theta_{\alpha,\beta} *_a \theta_{\beta,\beta}, && \theta_{\beta,\alpha}\ {}_a{*}\ \theta_{\alpha,\alpha} = \theta_{\beta,\alpha} *_a \theta_{\alpha,\alpha},\\
&\theta_{\alpha,\alpha}\ {}_a{*}\ \theta_{\alpha,\alpha} = \theta_{\alpha,\alpha} *_a \theta_{\alpha,\alpha}, && \theta_{\beta,\beta}\ {}_a{*}\ \theta_{\beta,\beta} = \theta_{\beta,\beta} *_a \theta_{\beta,\beta}.
\end{aligned}\right\} \quad (2.15)$$

These are equivalent to:

$$\left.\begin{array}{l} \psi_x^{\alpha,\alpha}\varphi_y^{\alpha,\beta} = \varphi_y^{\alpha,\beta}\psi_x^{\alpha,\beta}, \quad \psi_x^{\beta,\beta}\varphi_y^{\beta,\alpha} = \varphi_y^{\beta,\alpha}\psi_x^{\beta,\alpha} \\ \psi_x^{\alpha,\beta}\varphi_y^{\beta,\alpha} = \varphi_y^{\beta,\alpha}\psi_x^{\alpha,\alpha}, \quad \psi_x^{\beta,\alpha}\varphi_y^{\alpha,\beta} = \varphi_y^{\alpha,\beta}\psi_x^{\beta,\beta} \\ \psi_x^{\alpha,\beta}\varphi_y^{\beta,\beta} = \varphi_y^{\beta,\beta}\psi_x^{\alpha,\beta}, \quad \psi_x^{\beta,\alpha}\varphi_y^{\alpha,\alpha} = \varphi_y^{\alpha,\alpha}\psi_x^{\beta,\alpha} \\ \theta_{\alpha,\alpha} \text{ and } \theta_{\beta,\beta} \text{ are semigroups.} \end{array}\right\} \tag{2.16}$$

The author and R. Dickinson have computed all non-isomorphic general products of S, $|S| \leq 3$, by a right zero semigroup of order 2 using a CDC 6600. The results will be published elsewhere.

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