THERE ARE 15973 SEMIGROUPS OF ORDER 6

Robert J. Plemmons

There are 15973 semigroups of order 6. Of these, 2143 are commutative; 1352 are regular; 208 are inverse. Also, 3107 of these have one idempotent (and only one); 1780 have two; 3093 have three; 4157 have four; 2961 have five; and 875 have six.

This brings the known exhaustive information on semigroups of small orders up to the point represented by the following table:

ORDER	1	2	3	4	5	6
NUMBER OF SEMIGROUPS	1	4	18	126	1160	15973
NUMBER OF COMMUTATIVE SEMIGROUPS	1	3	12	58	325	2143
NUMBER OF REGULAR SEMIGROUPS	1	3	9	42	206	1352
NUMBER OF INVERSE SEMIGROUPS	1	2	5	16	52	208
NUMBER OF SEMIGROUPS WITH ONE IDEMPOTENT	1	2	5	19	132	3107
NUMBER OF SEMIGROUPS WITH TWO IDEMPOTENTS		2	7	37	216	1780
NUMBER OF SEMIGROUPS WITH THREE IDEMPOTENTS			6	44	351	3093
NUMBER OF SEMIGROUPS WITH FOUR IDEMPOTENTS				26	326	4157
NUMBER OF SEMIGROUPS WITH FIVE IDEMPOTENTS					135	2961
NUMBER OF SEMIGROUPS WITH SIX IDEMPOTENTS						875

The above table was calculated by computer as described in [4]. Since the coding done for the purposes of [4] has, in general, a different form for each application discussed there, no source listings are presented. However, we here give a listing of the Cayley tables for all semigroups containing n elements for n ≤ 6. The results for order 1 through 5 agree completely with those given by [1] and [3]. Notice that the elements are designated by letters rather than numbers less than or equal to n. This is done to accentuate the abstractness of the binary operations. For example, the listing

```
A  A  A                        o | A  B  C
                               A | A  A  A        -
A  B  C     is the table       B | A  B  C
C  C  C                        C | C  C  C
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which represents the Cayley table for a semigroup of order 3 with binary operations
$\circ$ where $A \circ A = A$, $A \circ B = A$, $A \circ C = A$, $B \circ A = A$, $B \circ B = B$, etc. If the letter
R occurs above a table, then the semigroup represented by that table is regular.
Similarly, if the table has an I printed above it, then it represents an inverse
semigroup.

   (Note by the editor: In the original paper, 276 pages of computer printout
are appended at this point. In the interest of brevity, we reproduce here only
the complete results for orders 2, 3, and 4 and the first three pages and the last
three pages for orders 5 and 6. The full printouts may be obtained from the author,
who is currently at the University of Tennessee.)

3

A REMARK BY THE EDITOR

concerning Robert J. Plemmons' paper, THERE ARE 15,973 SEMIGROUPS OF ORDER 6

It has been brought to the editor's attention that, to those unfamiliar with semigroup theory, the statement that there are 15,973 semigroups of order 6 is slightly misleading. May we repeat some elementary semigroup theory for the benefit of such readers: To every semigroup S there corresponds an anti-isomorphic semigroup S', with the product xy of two elements in S' being defined as the product yx of those same two elements in S. A semigroup is commutative if and only if it is isomorphic, as well as anti-isomorphic, with this semigroup. Normally when we speak of the number of distinct algebraic systems of a certain type, we refer to "distinct" in the sense of "up to isomorphism"; but in semigroup theory, because of the obvious fact that every semigroup has a corresponding anti-isomorphic semigroup, we use "distinct" in the sense of "up to isomorphism or anti-isomorphism." It is obvious that the semigroup of order 2 defined (in Plemmons' terminology) by $\begin{smallmatrix} A B \\ A B \end{smallmatrix}$ is not isomorphic to any of the four semigroups of order 2 in Plemmons' tables; it is, however, anti-isomorphic to $\begin{smallmatrix} A A \\ B B \end{smallmatrix}$. There are, of course, _five_ semigroups of order 2, rather than four as Plemmons stated, if we understand this statement to mean "five semigroups, up to isomorphism"; they are in fact, given names in automata theory — N_2, U_1, L_2, Z_2, and R_2, in this order (the four given by Plemmons, followed by the extra one given here). N_2, U_1, and Z_2 are commutative, and L_2 and R_2 are anti-isomorphic. In the case of order 6, there are actually, up to isomorphism, 29,803 semigroups, consisting of the 2,143 commutative semigroups, the 13,830 semigroups of order 6 which are not commutative, and their 13,830 anti-isomorphic counterparts.

transpose the matrix