Genocchi Numbers

1469.1

→ 144

Ref ✓1. MMAG 33 21 59–60
2. Lucas, Th. des Nos I p 250–263 (1891)
3. Bell, Messenger of Math, (2) 52 56–68
4. Nörlund, Vorles. üb. differenzrechng, 1924

Defn G_N:

$$\frac{2x}{e^x+1} = \sum_{N=1}^{\infty} G_N \frac{x^N}{N!}$$

$$G_N = -2(2^N-1) B_N \text{ (ib Bernoulli)}$$

$$C_{N-1} = 2^{N-1}\frac{G_N}{N}$$ Tangent Coeff^s of [4].

$$\frac{2x}{e^x+1} = \frac{2x}{2+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots} = \frac{x}{1+\frac{x}{2}+\frac{x^2}{2.2!}+\frac{x^3}{2.3!}+\cdots}$$

Now $$\frac{1}{1+x} = \frac{1-x}{1-x^2} = (1-x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$$
$$= (1-x)(1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+x^{14}+\cdots)$$
$$= 1-x+x^2-x^3+x^4-x^5+\cdots$$

$$\therefore = x\left(1-\frac{x}{2}-\frac{x^2}{4}-\frac{x^3}{12}-\frac{x^4}{48}-\frac{x^5}{240}\right)$$
$$\cdot\left(1+\frac{x^2}{4}+\frac{x^3}{4}+\frac{7}{48}x^4+\frac{1}{16}x^5\right)$$
$$\cdot\left(1+\frac{x^4}{16}+\frac{x^5}{8}\right)$$

$$= x\left(1-\frac{x}{2}+0x^2+\frac{1}{24}x^3-\frac{1}{16}x^4-\frac{47}{480}x^5\right)\left(1+\frac{x^4}{16}+\frac{x^5}{8}\right)$$

$$= x\left(1-\frac{x}{2}+\cancel{\frac{x^3}{24}}\ \frac{1}{24}\cdot x^3+0.x^4\ \cdots\right.$$

$$= 1.\frac{x}{1!}-1.\frac{x^2}{2!}+1.\frac{x^4}{4!}+O(x^6)$$

$$2x = (e^x+1)\sum_{n=1}^{\infty} \frac{G_n x^n}{n!} = (e^x+1)G(x) \qquad (A)$$

$$2 = (e^x+1)\sum_{n=0}^{\infty} G_{n+1}\frac{x^n}{n!} + e^x \sum_{n=1}^{\infty} \frac{G_n x^n}{n!}$$

$$= \frac{2x}{G(x)}\sum_{n=0}^{\infty} G_{n+1}\frac{x^n}{n!} + \quad 2x - G(x)$$

$$2G(x) = 2x\,G'(x) + 2x\,G(x) - G(x)'$$

$$(2x - 1)\,G'(x) + \quad 2(x-1)\,G(x) = 0 \qquad (B)$$

From (A): $$2x = \left(2 + \sum_{1}^{\infty}\frac{x^n}{n!}\right)\sum_{1}^{\infty}\frac{G_n x^n}{n!}$$

$$2x = 2\sum_{1}^{\infty}\frac{G_n x^n}{n!} + \sum_{2}^{\infty}\frac{\alpha_n x^n}{n!}$$

$$\frac{\alpha_n}{n!} = \sum_{r+s=n} \frac{}{r!}\; G_s \;\frac{}{s!}$$

$$\alpha_n = \sum_{r=1}^{n-1}\binom{n}{r} G_r$$

Equate %o : $G_1 = 1$, $G_0 = 0$, and

$$0 = 2G_n + \sum_{r=1}^{n-1}\binom{n}{r} G_r$$

$$\boxed{G_n = -\frac{1}{2}\sum_{r=1}^{n-1}\binom{n}{r} G_r} = -\frac{1}{2}\left\{n + \sum_{r=2}^{n-2}\binom{n}{r} G_r\right\}$$

{n must be even, ≥ 2}

$$G_2 = -\frac{1}{2}\binom{2}{1}G_1 = -1$$

$$G_3 = -\frac{1}{2}\left\{3.1 + 3(-1)\right\} = 0$$

$$G_4 = -\frac{1}{2}\left\{4.1 - 6 + 4.0\right\} = 1$$

$$G_5 = -\frac{1}{2}\left\{5.1 - 10.1 + \cancel{10.0} + 5.1\right\} = 0$$

$$G_6 = -\frac{1}{2}\left\{6.1 - 15.1 + \cancel{20} + .\,15.1 + \cancel{6}\right\} = -3$$

$$G_7 = -\frac{1}{2}\left\{7.1 - 21. + \cancel{35} + 35 + \cancel{21} - 3.7\right\} = 0$$

$$G_8 = -\frac{1}{2}\left\{8.1 - 28.1 + \cancel{56} + 70 + \cancel{56} - 3.28 + \cancel{8}\right\} = 17$$

78
112
34

n	T_{2n-1}	$2T_{2n-1}$		$2nT_{2n-1}$	$\div 2^{2n-1}$
1	1	2	2^1	2	1
2	2	4	2^2	8	-1
3	16	32	2^5	96	-3
4	272	544	2^5	$4.544 = 2^7.17$	17
5	7936	15872	2^9	$5.2^9.31$	155
6	353792	707584			

G_n

6
8) 544
4) 68
17

2) 15872
8) 7936
8) 992
4) 124
31

$$\boxed{G_n = \frac{2n}{2^{2n-1}}\,T_{2n-1}}$$