STRUCTURES

American Math. Soc., 3 (1952).

Glasgow Math. Assoc., 6 (1963).

s sur la théorie des treillis, des riques (Gauthier-Villars, Paris.

ON THE COEFFICIENTS OF THE POWERS OF DEDEKIND'S MODULAR FORM

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1. Let

$$f(x) = \prod_{j=1}^{\infty} (1-x^j)$$

$$= 1+\sum_{j=1}^{\infty} (-1)^j \{x^{j(3j-1)/2}+x^{j(3j+1)/2}\}. \tag{1.1}$$

Then for any integer r, we define $p_r(n)$ by the relation

$$\{f(x)\}^r = \sum_{n=0}^{\infty} p_r(n)\, x^n. \tag{1.2}$$

We take $p_r(\alpha) = 0$ except when α is a non-negative integer.

Evidently

$$p_r(0) = 1 \text{ for every } r, \tag{1.3}$$

and

$$p_0(n) = 0 \text{ for every } n \geqslant 1. \tag{1.4}$$

It is easy to show that

$$np_r(n) = -r \sum_{j=1}^{n} \sigma(j)\, p_r(n-j), \tag{1.5}$$

where

$$\sigma(j) = \sum_{d \mid j} d.$$

Using (1.5), Newman [1] has shown that $p_r(n)$ is a polynomial in r of degree n. He has actually given expressions for $n!\,p_r(n)$ for values of $n \leqslant 10$. Here we express $p_r(n)$ in terms of combinatory functions $\binom{r}{k}$. The coefficients involved are naturally smaller than those in Newman's formulae. While (1.5) requires the use of all the n terms on the right, the formula obtained here, which is a simple consequence of (1.1), requires comparatively very few.

2. We have

$$f(x) = 1+y,$$

where

$$y = \sum_{j=1}^{\infty} (-1)^j \{x^{j(3j-1)/2}+x^{j(3j+1)/2}\},$$

$$= -x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots. \tag{2.1}$$

Let

$$y^k = \sum_{n=k}^{\infty} (-1)^n C_k(n)\, x^n, \quad k \geqslant 0; \tag{2.2}$$

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where $C_0(0) = 1,\ C_0(n) = 0$ for every $n \geqslant 1$.

Then, identifying coefficients of x^n on the two sides of

$$y^k = y.y^{k-1}, \quad k \geqslant 1;$$

we obtain the recurrence

$$C_k(n) = \sum_{j=1}^{v}\left\{(-1)^{j(j-1)/2} C_{k-1}\left(n-\frac{j(3j-1)}{2}\right)+(-1)^{j(j+1)/2} C_{k-1}\left(n-\frac{j(3j+1)}{2}\right)\right\} \tag{2.3}$$

where

$$v = [\tfrac{1}{6}(\sqrt{24n+1}+1)],$$

and we take

$$C_t(m) = 0 \text{ whenever } m < t.$$

Now (1.2) gives

$$(-1)^n p_r(n) = \sum_{k=0}^{n} C_k(n)\binom{r}{k},$$

for every $n \geqslant 0$. (2.4)

The table that follows gives the values of $C_k(n)$ for

$$0 < k \leqslant n \leqslant 50.$$

The results have been checked for $r = -1$ against the partition table and for $n = 50$, with $r = -2$, modulo 13. This leaves no doubt in my mind that there is no mistake in the calculations.

As an illustration of the use of the table, we have

$$p_r(10) = \binom{r}{2}+6\binom{r}{3}-16\binom{r}{4}+19\binom{r}{5}+9\binom{r}{6}-35\binom{r}{7}+28\binom{r}{8}-9\binom{r}{9}+\binom{r}{10};$$

in place of Newman's

$$10!\, p_r(10) = r(r-1)(r^8-134r^7+6496r^6-147854r^5+1709659r^4 -10035116r^3+28014804r^2-29758896r+6531840).$$

3. My thanks are due to Professor D. H. Lehmer for his useful suggestions towards the improvement of this paper.

Reference

1. M. Newman, "An identity for the coefficients of certain modular forms", *Journal London Math. Soc.*, 30 (1955), 488–493.

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n \ k	1	2	3
1	1		
2	−1	1	
3	0	−2	1
4	0	1	−3
5	−1	0	3
6	0	−2	−1
7	−1	2	−3
8	0	−2	6
9	0	2	−6
10	0	1	6
11	0	0	0
12	−1	2	−3
13	0	−2	6
14	0	3	−9
15	1	0	8
16	0	2	−6
17	0	0	0
18	0	0	0
19	0	2	−6
20	0	−2	6
21	0	0	−13
22	1	−2	3
23	0	2	−6
24	0	−1	3
25	0	0	0
26	1	0	−3
27	0	−2	6
28	0	−2	−9
29	0	−2	6
30	0	1	−3
31	0	−2	6
32	0	0	0
33	0	−2	6
34	0	−2	6
35	1	0	−3
36	0	2	11
37	0	0	0
38	0	−2	6
39	0	0	0
40	−1	−2	9
41	0	0	0
42	0	0	0
43	0	0	0
44	0	1	−3
45	0	2	13
46	0	0	0
47	0	0	0
48	0	2	−6
49	0	0	0
50	0	2	−6

Coefficients of a modular form Type 2 - Ref JLMS 39, ..., 64

47638 contd

1815 47639 1482 1483 1484 1485 1486 1487 1488

n \ k	1	2	3	4	5	6	7	8	9	10
1	1									
2	−1	1								
3	0	−2	1							
4	0	1	−3	1						
5	−1	0	3	−4	1					
6	0	−2	−1	6	−5	1				
7	−1	2	−3	−4	10	−6	1			
8	0	−2	6	−3	−10	15	−7	1		
9	0	2	−6	12	0	−20	21	−8	1	
10	0	1	6	−16	19	9	−35	28	−9	1
11	0	0	0	16	−35	24	28	−56	36	−10
12	−1	2	−3	−6	40	−65	21	62	−84	45
13	0	−2	6	−8	−25	90	−105	0	117	−120
14	0	3	−9	18	−10	−75	181	−148	−54	200
15	1	0	8	−28	45	6	−189	328	−177	−162
16	0	2	−6	26	−75	90	77	−419	540	−160
17	0	0	0	−20	80	−180	140	280	−837	810
18	0	0	0	2	−60	220	−385	140	755	−1530
19	0	2	−6	12	15	−180	546	−728	−54	1730
20	0	−2	6	−23	45	66	−511	1232	−1197	−749
21	0	0	−13	32	−85	110	252	−1336	2535	−1630
22	1	−2	3	−36	115	−264	203	848	−3204	4755
23	0	2	−6	28	−115	360	−693	224	2520	−7070
24	0	−1	3	−6	90	−365	1029	−1582	−246	6700
25	0	0	0	4	−21	264	−1092	2688	−3150	−2450
26	1	0	−3	22	−35	−66	798	−3072	6426	−5295
27	0	−2	6	−20	95	−178	−203	2408	−8106	14070
28	0	−2	−9	39	−130	375	−581	−742	7011	−20010
29	0	−2	6	−32	135	−510	1281	−1568	−2844	19350
30	0	1	−3	32	−135	496	−1708	3836	−3549	−10157
31	0	−2	6	−12	70	−414	1687	−5264	10359	−6290
32	0	0	0	2	−35	180	−1232	5306	−15120	25515
33	0	−2	6	16	−65	60	413	−3744	15804	−40660
34	0	−2	6	−12	105	−330	602	924	−11403	44940
35	1	0	−3	24	−146	570	−1485	2576	2574	−34268
36	0	2	11	−40	120	−622	2233	−5686	8610	9180
37	0	0	0	28	−150	582	−2366	7792	−18972	24510
38	0	−2	6	−34	90	−390	2009	−8092	25425	−57195
39	0	0	0	0	−65	220	−1099	6272	−25824	78060
40	−1	−2	9	−6	−25	96	14	−2751	18954	−79087
41	0	0	0	−16	90	−300	1099	−1848	−6165	56610
42	0	0	0	0	−115	621	−2072	6008	−10080	−13935
43	0	0	0	−40	150	−630	2667	−9296	25101	−39600
44	0	1	−3	6	−125	705	−2807	10556	−35262	89805
45	0	2	13	−36	130	−492	2254	−9800	37799	−1 21638
46	0	0	0	26	−45	300	−1477	6692	−31374	1 25405
47	0	0	0	−32	80	0	0	−2240	17379	−96440
48	0	2	−6	−5	35	−235	1057	−3206	1929	40350
49	0	0	0	0	−5	420	−2346	8168	−21906	31820
50	0	2	−6	−20	160	−570	2744	−11524	39114	−1 02989

47649

1489 1490 1491 1492 1493

47639

k / n	11	12	13	14	15
11	1				
12	−11	1			
13	55	−12	1		
14	−165	66	−13	1	
15	319	−220	78	−14	1
16	−352	483	−286	91	−15
17	−44	−660	702	−364	105
18	1100	252	−1131	987	−455
19	−2585	1320	845	−1820	1350
20	3542	−4059	1300	1897	−2793
21	−2519	6644	−5928	754	3625
22	−1530	−6336	11583	−8008	−765
23	8085	240	−13715	18928	−9840
24	−14410	12255	5915	−26845	29120
25	16170	−27192	15834	19460	−48657
26	−9460	35850	−47477	15015	47370
27	−6644	−27972	73658	−76272	1680
28	28105	−2343	−71201	1 41065	−1 11060
29	−46145	50568	20436	−1 63072	2 52555
30	50248	−99286	79391	90727	−3 43526
31	−32802	1 22496	−1 98796	99386	2 67540
32	−6193	−96162	2 80345	−3 68277	63210
33	57200	11584	−2 58557	6 02616	−6 23510
34	−1 02575	1 15116	92807	−6 43734	12 16425
35	1 21968	−2 42616	2 00850	3 58190	−14 95173
36	−1 00397	3 15216	−5 36341	2 74547	10 93210
37	35123	−2 83800	7 73916	−11 01100	1 66425
38	60390	1 28304	−7 68222	18 01086	−20 73645
39	−1 58840	1 26280	4 32705	−19 82330	39 63260
40	2 26413	−4 09398	2 04477	13 44525	−48 64839
41	−2 34344	6 22644	−9 79628	1 48316	38 72295
42	1 68773	−6 71550	16 26196	−21 63590	−6 18310
43	−37070	5 01468	−18 56569	40 32756	−43 45470
44	−1 31175	−1 22508	14 71184	−49 38843	94 77960
45	2 90851	−3 82360	−4 52192	42 16576	−126 11991
46	−3 91402	8 74116	−9 93200	−16 60841	117 07605
47	3 95813	−11 91960	24 63331	−22 82280	−57 64605
48	−2 89355	12 05291	−34 64318	65 34619	−44 67435
49	88550	−8 59188	35 62559	−96 46208	163 19355
50	1 62162	1 99716	−25 44815	102 39502	−257 58849

k / n	16
16	1
17	−16
18	120
19	−560
20	1804
21	−4128
22	6312
23	−3920
24	−10530
25	42208
26	−82752
27	99584
28	−39460
29	−1 41200
30	4 22568
31	−6 73936
32	6 60941
33	−1 44720
34	−9 38840
35	23 01568
36	−32 57188
37	29 16592
38	−6 28040
39	−34 92160
40	82 17536
41	−113 41568
42	104 08280
43	−38 85040
44	−76 68720
45	210 33408
46	−308 48064
47	313 16096
48	−186 69150
49	−65 20640
50	382 40784

k / n	44	45
44	1	
45	−44	
46	946	
47	−13244	
48	1 35707	−1
49	−10 84116	1 4
50	70 19276	−12 1

n \ k	16	17	18	19
16	1			
17	−16	1		
18	120	−17	1	
19	−560	136	−18	1
20	1804	−680	153	−19
21	−4128	2363	−816	171
22	6312	−5916	3042	−969
23	−3920	10319	−8262	3857
24	−10530	−9656	16098	−11286
25	42208	−8534	−19278	24206
26	−82752	57426	−1377	−34542
27	99584	−1 33076	72556	14706
28	−39460	1 90383	−2 03184	83011
29	−1 41200	−1 34810	3 39030	−2 94880
30	4 22568	−1 40148	−3 26961	5 69753
31	−6 73936	6 57611	−53244	−6 80694
32	6 60941	−12 40116	9 40050	2 20286
33	−1 44720	14 61337	−21 47916	11 98672
34	−9 38840	−7 70917	29 75391	−35 02612
35	23 01568	−11 71504	−22 93488	56 61867
36	−32 57188	40 61946	−9 11369	−55 71579
37	29 16592	−66 78161	66 16332	7 91350
38	−6 28040	70 71269	−129 06162	97 21976
39	−34 92160	−33 76863	158 83884	−234 94393
40	82 17536	−49 39180	−109 36899	334 15357
41	−113 41568	159 63612	−46 60974	−292 25230
42	104 08280	−250 98443	287 58849	23 52751
43	−38 85040	262 65408	−526 60134	470 86598
44	−76 68720	−145 13461	625 18248	−1045 17176
45	210 33408	−108 10368	−445 01988	1408 34118
46	−308 48064	437 92034	−74 65464	−1212 55530
47	313 16096	−721 09835	845 65242	224 28018
48	−186 69150	804 26796	−1606 79988	1481 16932
49	−65 20640	−564 84574	1979 33022	−3400 98974
50	382 40784	−21 20240	−1604 28456	4662 00720

n \ k	44	45	46	47	48	49	50
44	1						
45	−44	1					
46	946	−45	1				
47	−13244	990	−46	1			
48	1 35707	−14190	1035	−47	1		
49	−10 84116	1 48950	−15180	1081	−48	1	
50	70 19276	−12 19779	1 63139	−16215	1128	−49	1

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n \ k	20	21	22	23
20	1			
21	−20	1		
22	190	−21	1	
23	−1140	210	−22	1
24	4825	−1330	231	−23
25	−15124	5964	−1540	253
26	35320	−19929	7293	−1771
27	−57760	50253	−25872	8832
28	45220	−91920	69971	−33143
29	80560	97965	−1 40822	95611
30	−4 05954	51604	1 83711	−2 09231
31	9 10460	−5 26659	−25102	3 17009
32	−12 89340	13 89297	−6 34480	−1 81401
33	8 52340	−22 80320	20 27804	−6 86642
34	12 59530	21 18690	−38 17814	28 28977
35	−53 57924	7 69065	44 39116	−60 99278
36	101 51510	−76 13319	−9 19600	84 22623
37	−120 48660	172 20042	−98 29270	−49 06406
38	58 83350	−239 99430	276 60479	−109 19687
39	121 86960	180 24405	−447 79042	419 68146
40	−401 35713	107 48850	436 32974	−789 77952
41	662 44280	−637 78953	−18 98820	932 97545
42	−696 48870	1241 34772	−925 18261	−403 51223
43	281 91460	−1527 93270	2199 61214	−1172 65247
44	669 20755	990 72120	−3134 63842	3675 81446
45	−1953 66168	717 22224	2674 48104	−6065 62624
46	3008 81530	−3410 62407	157 57973	6313 82751
47	−3032 74960	6100 85721	−5470 42056	−2078 79980
48	1325 56920	−7078 55890	11730 33400	−7779 07725
49	2235 77940	4518 09285	−15561 99392	21320 43121
50	−6801 25006	2470 04499	12654 11466	−32397 93271

n \ k	40	41	42	43
40	1			
41	−40	1		
42	780	−41	1	
43	−9880	820	−42	1
44	91350	−10660	861	−43
45	−6 56448	1 01229	−11480	903
46	38 08700	−7 47758	1 11888	−12341
47	−182 76440	44 64367	−8 48946	1 23367
48	735 85785	−220 75220	52 11304	−9 60792
49	−2500 72680	917 70095	−265 28886	60 59388
50	7144 15468	−3229 92137	1137 43266	−317 28668

n \ k	24
24	1
25	−24
26	276
27	−2024
28	10602
29	−41952
30	1 28500
31	−3 03048
32	5 17155
33	−4 63496
34	−6 09684
35	37 57992
36	−93 40852
37	149 12280
38	−129 57624
39	−86 69712
40	597 07149
41	−1322 95080
42	1834 99244
43	−1315 01856
44	−1136 98752
45	5752 21744
46	−11119 21752
47	13631 92680
48	−8244 06065
49	−8895 13752
50	36385 65960

n \ k	36
36	1
37	−36
38	630
39	−7140
40	58869
41	−3 75732
42	19 26336
43	−81 10800
44	283 54590
45	−822 42328
46	1942 22826
47	−3508 48620
48	3758 72910
49	2905 13340
50	−27573 29550

n	22	23
22	1	
23	−22	1
24	231	−23
25	−1540	253
26	7293	−1771
27	−25872	8832
28	69971	−33143
29	−1 40822	95611
30	1 83711	−2 09231
31	−25102	3 17009
32	−6 34480	−1 81401
33	20 27804	−6 86642
34	−38 17814	28 28977
35	44 39116	−60 99278
36	−9 19600	84 22623
37	−98 29270	−49 06406
38	276 60479	−109 19687
39	−447 79042	419 68146
40	[illegible]36 32974	−789 77952
41	−18 98820	932 97545
42	−925 18261	−403 51223
43	2199 61214	−1172 65247
44	−3134 63842	3675 81446
45	2674 48104	−6065 62624
46	157 57973	6313 82751
47	−5470 42056	−2078 79980
48	11730 33400	−7779 07725
49	−15561 99392	21320 43121
50	12654 11466	−32397 93271

n	42	43
42	1	
43	−42	1
44	861	−43
45	−11480	903
46	1 11888	−12341
47	−8 48946	1 23367
48	52 11304	−9 60792
49	−265 28886	60 59388
50	1137 43266	−317 28668

6665

k / n	24	25	26	27
24	1			
25	−24	1		
26	276	−25	1	
27	−2024	300	−26	1
28	10602	−2300	325	−27
29	−41952	12625	−2600	351
30	1 28500	−52530	14924	−2925
31	−3 03048	1 70175	−65130	17523
32	5 17155	−4 29500	2 22404	−80028
33	−4 63496	8 09325	−5 97350	2 87208
34	−6 09684	−9 36675	12 25900	−8 17128
35	37 57992	−2 85290	−16 91170	18 08001
36	−93 40852	47 12675	4 68585	−28 49340
37	149 12280	−137 44475	54 79994	19 22427
38	−129 57624	250 19400	−194 36625	56 73888
39	−86 69712	−277 75350	401 29750	−263 64897
40	597 07149	9 82790	−533 16484	618 54570
41	−1322 95080	783 69625	249 69490	−951 38118
42	1834 99244	−2108 03025	914 69430	744 82200
43	−1315 01856	3384 06450	−3190 77200	855 33435
44	−1136 98752	−3218 02775	5911 71425	−4563 81378
45	5752 21744	−283 54425	−6870 73270	9836 14008
46	−11119 21752	8301 91925	2439 60925	−13446 34965
47	13631 92680	−19335 90750	10682 22376	8947 94823
48	−8244 06065	27477 89000	−31844 62125	11173 03227
49	−8895 13752	−22729 23225	52267 93650	−49395 79476
50	36385 65960	−4921 01356	−53560 71240	94351 52025

k / n	36	37	38	39
36	1			
37	−36	1		
38	630	−37	1	
39	−7140	666	−38	1
40	58869	−7770	703	−39
41	−3 75732	66008	−8436	741
42	19 26336	−4 34565	73777	−9139
43	−81 10800	23 01437	−5 00536	82212
44	283 54590	−100 29960	27 35335	−5 74275
45	−822 42328	364 05891	−123 23590	32 35167
46	1942 22826	−1102 14046	463 69177	−150 50451
47	−3508 48620	2744 79949	−1461 77602	586 18287
48	3758 72910	−5365 84915	3823 26550	−1920 37807
49	2905 13340	6964 40640	−8006 00340	5256 95040
50	−27573 29550	49 92928	11943 18264	−11701 02765

n \ k	28	29	30	31
28	1			
29	−28	1		
30	378	−29	1	
31	−3276	406	−30	1
32	20447	−3654	435	−31
33	−97524	23722	−4060	465
34	3 66884	−1 17943	27375	−4495
35	−11 01384	4 64029	−1 41636	31434
36	26 07255	−14 64964	5 81565	−1 68981
37	−45 74388	36 87814	−19 25310	7 22765
38	44 67036	−70 80843	51 28650	−25 02785
39	46 48616	86 54093	−106 47060	70 26150
40	−341 33967	13 80487	152 47185	−156 30014
41	918 85584	−417 56694	−56 90700	252 85305
42	−1607 20190	1316 94945	−472 88995	−188 70940
43	1667 78220	−2596 76788	1819 97040	−473 08480
44	355 74084	3274 03707	−4037 09985	2418 57970
45	−6097 33404	−1014 39303	5927 57100	−6060 78799
46	15623 33052	−7422 11848	−3956 22060	10127 77440
47	−24664 73772	23680 42444	−7743 15660	−9561 95279
48	22521 71019	−42901 75407	34131 76460	−5555 85565
49	6016 68028	48487 32169	−71211 22260	46404 43785
50	−71298 96060	−12131 16226	95075 68539	−1 13143 23146

n \ k	32	33	34	35
32	1			
33	−32	1		
34	496	−33	1	
35	−4960	528	−34	1
36	35928	−5456	561	−35
37	−2 00384	40887	−5984	595
38	8 91280	−2 36280	46342	−6545
39	−32 21024	10 91167	−2 77134	52325
40	94 97036	−41 07312	13 26918	−3 23442
41	−224 82688	126 81636	−51 92990	16 03490
42	401 61120	−317 74380	167 47533	−65 13890
43	−414 84480	617 16600	−442 14280	218 93620
44	−361 86424	−782 24193	923 58926	−606 78695
45	3072 98784	−50 90228	−1355 97202	1351 99946
46	−8802 43728	3691 88424	593 61195	−2224 86005
47	16533 25728	−12371 95707	4101 59816	1768 83980
48	−19467 80098	25975 00664	−16807 76910	4002 05330
49	1768 95424	−36038 79642	39446 54024	−22000 37200
50	58519 50848	18569 64978	−62577 69870	58051 96040