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THE NUMBER OF TOPOLOGIES

Richard H. Warren

ABSTRACT. By means of T_0-identification spaces a formula is derived for the number of nonhomeomorphic topologies on a finite set. As a result of proving that special families of nonprincipal ultratopologies on an infinite set X have cardinality $2^{2^{|X|}}$, it follows that the number of nonhomeomorphic T_5-topologies on X is $2^{2^{|X|}}$.

1. **Introduction.** Motivated by Sharp's question [6, page 1347] to find a formula for the number of nonhomeomorphic topologies on a finite set, an investigation of T_0-identification spaces has led us to a new procedure for counting these topologies. Our procedure is to count in subclasses of T_0-topologies on partitions of the set. An earlier answer [1, Theorem 7(ii)] to Sharp's question sums over a combinatorial arrangement based on connected topologies. In the course of our development, a new proof is given for the known formula for the number of topologies on a finite set.

The family of nonprincipal ultratopologies on an infinite set X is partitioned into |X| classes and it is shown that each class contains $2^{2^{|X|}}$ nonhomeomorphic ultratopologies. Since a nonprincipal ultraspace is known to be a T_5-space, it then follows that the number of nonhomeomorphic T_5-topologies on X is $2^{2^{|X|}}$.

2. **T_0-identification spaces.** These spaces were originated by Stone [8] and were expounded by Thron [9, pages 91, 92] whose notation we use.

THEOREM 2.1. *If* X *is a set,* Y *is a partition of* X *and* $\mathcal{V}$ *is a* T_0*-topology on* Y, *then there is a unique topology* $\mathcal{T}$ *on* X *such that* $(Y,\mathcal{V})$ *is the* T_0*-identification space of* $(X,\mathcal{T})$.

PROOF. Since Y is a collection of disjoint subsets of X which covers X, for each $x \in X$ there is exactly one $D_x \in Y$ such that $x \in D_x$. Let $f: X \to Y$ by $f(x) = D_x$. By Theorem 10.10 in [9] the family $\mathcal{T} = \{ f^{-1}(B): B \in \mathcal{V}\}$ is the weakest topology on X such that f is continuous. We shall show that $(Y,\mathcal{V})$ is the T_0-identification of $(X,\mathcal{T})$.

Let $x,y \in X$. If $y \in D_x$ and $x \in f^{-1}(B)$ where $B \in \mathcal{V}$, then since $f^{-1}(B) = \cup\{D_x: D_x \in B\}$, it follows that $y \in f^{-1}(B)$, i.e., each member of D_x is in every open subset of X which contains x. On the other hand, if $y \notin D_x$, then $D_y \cap D_x = \emptyset$. Since $(Y,\mathcal{V})$ is T_0, there exists $B \in \mathcal{V}$ which contains D_y or D_x, but not both. Hence $f^{-1}(B)$ contains x or y, but not both. Therefore the members of Y are exactly the classes which are determined by the equivalence relation on X where $x \approx y$ iff $\overline{\{x\}} = \overline{\{y\}}$.

Let $\mathcal{U}$ be the quotient topology on Y determined by f. Since $\mathcal{U}$ is the strongest topology on Y such that f is continuous, $\mathcal{V} \subset \mathcal{U}$. If $G \in \mathcal{U}$, then $f^{-1}(G) \in \mathcal{T}$ and there is $B \in \mathcal{V}$ such that $f^{-1}(B) = f^{-1}(G)$. Since f is onto, $B = G$.

To see that $\mathcal{T}$ is unique, let $\mathcal{R}$ be a topology on X such that $(Y,\mathcal{V})$ is the T_0-identification of $(X,\mathcal{R})$. Since $\mathcal{T}$ is the weakest topology on X such that f is continuous, $\mathcal{T} \subset \mathcal{R}$ Suppose $S \in \mathcal{R}\backslash\mathcal{T}$. Since f is an open map, $f(S) \in \mathcal{V}$. So $f^{-1}(f(S)) \in \mathcal{T}$ and there is $t \in f^{-1}(f(S))\backslash S$. Now $t \in D_s$ for some $s \in S$. Thus s is a member of a set in $\mathcal{R}$ not containing t, which contradicts the equivalence relation $\approx$.

COROLLARY. *For any nonempty set there is a* 1-1 *correspondence between the family of all topologies on the set and the family of all* T_0*-topologies on partitions of the set.*

THEOREM 2.2. *Let* $\mathcal{T}$ *and* $\mathcal{S}$ *be topologies on* X*. Let* Y *(respectively,* Z*) be the* T_0*-identification space of* $(X,\mathcal{T})$ *(respectively* $(X,\mathcal{S})$*). Then* $(X,\mathcal{T})$ *and* $(X,\mathcal{S})$ *are homeomorphic iff there is a homeomorphism* k *from* Y *onto* Z *such that* $|k(D_x)| = |D_x|$ *for each* $D_x \in Y$.

PROOF. Let f (respectively g) be the T_0-identification map from $(X,\mathcal{T})$ (respectively $(X,\mathcal{S})$) onto Y (respectively Z). For $x \in X$, let D_x (respectively $[x]$) be the member of Y (respectively Z) containing x.

($\Rightarrow$) Let h be a homeomorphism from $(X,\mathcal{T})$ onto $(X,\mathcal{S})$. We shall show that $k = ghf^{-1}$ satisfies the theorem. It is easily verified that $h(D_x) = [h(x)]$ for each $x \in X$.

Since $g(h(f^{-1}(D_x))) = [h(x)]$, the map k is onto. To see that k is 1-1, let $k(D_x) = k(D_y)$. Then $[h(x)] = [h(y)]$, and therefore $\overline{h(x)} = \overline{h(y)}$. It follows that $\{\overline{x}\} = \{\overline{y}\}$ and thus $D_x = D_y$. Clearly, k and k^{-1} are continuous. Furthermore, h is a 1-1 map of D_x onto $[h(x)] = k(D_x)$, so that $|k(D_x)| = |D_x|$.

($\Leftarrow$) Since k is 1-1, onto and $|k(D_x)| = |D_x|$, there is a map h: $X \to X$ such that $h|_{D_x}$ is a 1-1 map onto $k(D_x)$ for each $D_x \in Y$. Clearly, h is a 1-1, onto map. If $G \in \mathcal{S}$, then $G = \cup\{[x] : x \in G\}$ and thus $h^{-1}(G) = f^{-1}(k^{-1}(g(G)))$. Therefore h is continuous. Similarly, h^{-1} is continuous.

3. **Finite case.** By S(n,k) we denote the Sterling numbers of the second kind. From [5, page 99] S(n,k) is the number of partitions of a set of n points into k pieces.

THEOREM 3.1. [2] *If τ_n is the number of topologies on a set of* n *points and if γ_k is the number of* T_0*-topologies on a set of* k *points, then*

$$\tau_n = \Sigma_{k=1}^{n} S(n,k)\gamma_k.$$

PROOF. Noting that a topology on a partition composed of k subsets is a topology on a set of k points, the formula is a consequence of the Corollary to Theorem 2.1.

Motivated by Theorem 2.2, partitions $\boldsymbol{L}$ and $\boldsymbol{P}$ of X are said to be *akin* if there is a map f: $\boldsymbol{L} \to \boldsymbol{P}$ which is 1-1, onto and $|f(A)| = |A|$ for each $A \in \boldsymbol{L}$. Also, $\boldsymbol{L}$ and $\boldsymbol{P}$ with topologies are called *p-homeomorphic* if there is a homeomorphism from $\boldsymbol{P}$ onto $\boldsymbol{L}$ such that $|f(A)| = |A|$ for each $A \in \boldsymbol{L}$. Then akin is an equivalence relation on the family of partitions of X, the set of equivalent classes $\widetilde{X}$ corresponds naturally with the set of unordered partitions of |X|, and p-homeomorphic partitions are akin. The following lemmas are easily proved by using a map from the definition of akin partitions.

LEMMA 3.1. *Akin partitions of a finite set have the same number of nonp-homeomorphic* T_0*-topologies.*

LEMMA 3.2. *Let* $\boldsymbol{L}$ *and* $\boldsymbol{P}$ *be akin partitions of a finite set. For each topology on* $\boldsymbol{L}$ *there is a topology on* $\boldsymbol{P}$ *such that the spaces are* p*-homeomorphic.*

THEOREM 3.2. *Let* $|X| = n$ *and let* α_n *be the number of nonhomeomorphic topologies on* X*. Then*

$$\alpha_n = \Sigma_{i \in \widetilde{X}} \lambda(n,i)$$

where $\lambda(n,i)$ *is the number of nonp-homeomorphic* T_0*-topologies on any partition in class* i.

PROOF. Consider a class i of akin partitions and a representative $\boldsymbol{L}$ from the class. By Lemma 3.1 the number $\lambda(n,i)$ is independent of the choice of representative.

By Theorems 2.1 and 2.2 the nonp-homeomorphic T_0-topologies on L correspond to unique topologies on X which are not homeomorphic. By Lemma 3.2 and Theorem 2.2 each T_0-topology on any partition akin to L corresponds to a topology on X which is homeomorphic to a topology on X corresponding to one on L, i.e., no new nonhomeomorphic topologies on X are formed from other members of class i. Since T_0-topologies on representatives from different akin classes cannot be p-homeomorphic, the topologies on X, to which they correspond, cannot be homeomorphic. Thus there is a 1-1 correspondence between a maximal family of nonhomeomorphic topologies on a finite set and the family of all nonp-homeomorphic T_0-topologies on a representative from each akin class.

EXAMPLE. We shall illustrate the counting procedure in Theorem 3.2 for n = 4. There are 5 equivalence classes of akin partitions of X. Let X = {a,b,c,d}.

Representative from class	λ(4,i)
{X}	1
{{a,b},{c,d}}	2
{{a},{b,c,d}}	3
{{a},{b},{c,d}}	11
{{a},{b},{c},{d}}	16
Total	$33 = \alpha_4$

4. Infinite case. If X is infinite, then there are $2^{2^{|X|}}$ topologies on X and $2^{2^{|X|}}$ nonprincipal ultratopologies on X [3]. A nonprincipal ultratopology is T_5 (i.e., T_1 and completely normal) and is denoted by $\tau(x,U)$ where $x \in X$ and U is a nonprincipal ultrafilter on X [7]. For each $x \in X$, we designate $\Theta_x = \{\tau(x,U)\colon U \text{ is nonprincipal}\}$.

THEOREM 4.1. *If* X *is infinite, then* Θ_x *contains* $2^{2^{|X|}}$ *nonhomeomorphic topologies.*

PROOF. If U and V are distinct nonprincipal ultrafilters on X, then there exists $R \in U \backslash V$. Therefore $R \cup \{x\} \in U \backslash V$, so that $R \cup \{x\} \in \tau(x,U) \backslash \tau(x,V)$. Since the number of nonprincipal ultrafilters on X is $2^{2^{|X|}}$, it follows that $|\Theta_x| = 2^{2^{|X|}}$.

If f: X → X is 1-1 and onto and if U is a nonprincipal ultrafilter on X, then f(U) =

$\{f(A)\colon A \in \mathcal{U}\}$ is a nonprincipal ultrafilter on X. Since there are at most $2^{|X|}$ such functions f, each topology $\tau(x,\mathcal{U})$ is homeomorphic to at most $2^{|X|}$ other topologies in Θ_X. From cardinal arithmetic it follows that there are $2^{2^{|X|}}$ nonhomeomorphic topologies in Θ_X.

COROLLARY. *The number of nonhomeomorphic* T_5*-topologies on an infinite set* X *is* $2^{2^{|X|}}$.

Hodel [4] has shown that the number of nonhomeomorphic metrizable topologies on X is $2^{|X|}$ and that the number of nonhomeomorphic connected paracompact Hausdorff topologies on X is $2^{2^{|X|}}$.

REFERENCES

1. K. K. Butler and G. Markowsky, *Enumeration of finite topologies,* Proc. of the Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Math., Winnipeg, 1973, 169-184.
2. J. W. Evans, F. Harary and M. S. Lynn, *On the computer enumeration of finite topologies,* Comm. ACM, 10(1967), 295-298.
3. O. Fröhlich, *Das Halbordnungssystem der topologischen Räume auf einer Menge,* Math. Ann., 156(1964), 79-95.
4. R. E. Hodel, *The number of metrizable spaces,* Fund. Math., to appear.
5. J. Riordan, *An Introduction to Combinatorial Analysis,* Wiley, New York, 1958.
6. H. Sharp, Jr., *Quasi-orderings and topologies on finite sets,* Proc. Amer. Math. Soc., 17(1966), 1344-1349.
7. A. K. Steiner, *On the lattice of topologies,* General Topology and its Relations to Modern Analysis and Algebra, III, Academic Press, New York, 1972, 411-415.
8. M. H. Stone, *Application of Boolean algebras to topology,* Mat. Sb., 1(1936), 765-771.
9. W. J. Thron, *Topological Structures,* Holt, Rinehart and Winston, New York, 1966.

University of Nebraska at Omaha
Omaha, Nebraska 68182

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