where $S_k = (n_1, n_2, \ldots, n_k)$ runs through the sets of positive integers such that $n_1 < n_2 < \cdots < n_k$.

EXAMPLE 1 [2]. Suppose $a_n = x^n$ where $0 \leqq x < 1$. We obtain for Euler's product,

$$\left(\prod_{n=1}^{\infty} (1-x^n)\right)^{-1} = 1 + \sum_{k=1}^{\infty} \sum_{S_k} \frac{x^{n_1+n_2+\cdots+n_k}}{(1-x^{n_1})(1-x^{n_2})\cdots(1-x^{n_k})}.$$

EXAMPLE 2. Let $\Pi_1, \Pi_2, \Pi_3, \ldots$ be an enumeration of primes greater than or equal to 2 and s be any number greater than 1. Let $a_n = 1/\Pi_n^s$. We have for the Riemann zeta-function

$$\prod_{n=1}^{\infty} \frac{1}{(1-\Pi_n^{-s})} = 1 + \sum_{k=1}^{\infty} \sum_{S_k} \frac{1}{(\Pi_{n_1}^s - 1)(\Pi_{n_2}^s - 1)\cdots(\Pi_{n_k}^s - 1)}.$$

EXAMPLE 3 [4, p. 245]. Let $\{\rho_n\}$ be a sequence of increasing positive numbers and let r be greater than zero. Let $a_n = r/(r+\rho_n)$. Define the entire function $f(z) = \Pi_{n=1}^{\infty}(1 + z/\rho_n)$ with power series $\Sigma_{k=0}^{\infty} c_k z^k$. Then we get

$$f(r) = \prod_{n=1}^{\infty}\left(1+\frac{r}{\rho_n}\right) = 1 + \sum_{k=1}^{\infty} r^k \sum_{S_k} (\rho_{n_1}\rho_{n_2}\cdots\rho_{n_k})^{-1}$$

and $c_k = \Sigma_{S_k} (\rho_{n_1}\rho_{n_2}\cdots\rho_{n_k})^{-1}$.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

fail ✓

WHEN ARE PERMUTATIONS ADDITIVE?

A. KOTZIG AND P. J. LAUFER

Suppose $n = 2k+1$ is odd and call an n-vector a **permutation** if its n coordinates comprise the set $\{0, \pm 1, \pm 2, \ldots, \pm k\}$. The vector $f = (-k, -k+1, -k+2, \ldots, k)$ is called the **fundamental permutation**. A permutation p is said to be a **σ-permutation** if $p + f$ is also a permutation. Denote by S_n the set of σ-permutations.

1. *How big is S_n?*

S_n is never empty since $(0, 1, 2, \ldots, k, -k, -k+1, \ldots, -1)$ is a σ-permutation. For $n > 1$, $|S_n|$ appears to be even, but we cannot prove this by observing that the negative reversal of a σ-permutation is also a σ-permutation, since some σ-permutations, such as

Vol 85

Neil Sequence #666

$$(1, 3, -2, 0, 2, -3, -1) \quad \text{and} \quad (2, -1, 3, 0, -3, 1, -2)$$

are their own negative reversals.

Two σ-permutations $\boldsymbol{u}, \boldsymbol{v}$, are called an **additive pair** (α-pair) if $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{f}$ is a permutation.

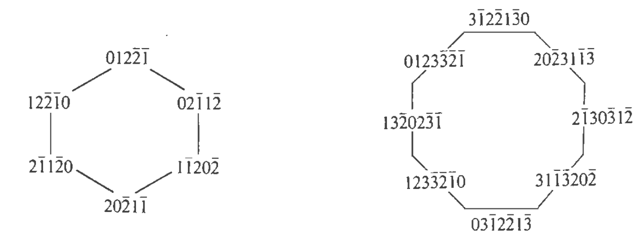
2. *For what values of n are there α-pairs, and how many α-pairs are there?*

Here are the answers to the questions for the first few values of n:

$n=$	1	3	5	7	9	11	13
$\lvert S_n\rvert=$	1	2	6	28	244	2544	35600
# of α-pairs	0	0	6	8	0	0	?

why? Colin

For each n we can draw a graph with the σ-permutations as vertices, joining two by an edge just if they form an α-pair. It is easy to see that no σ-permutation forms an α-pair with itself (except in the trivial case $n = 1$) so for $n > 1$ the graph has no loops. Here are the graphs for $n = 5$ and 7,



except that 20 isolated vertices are not shown in the latter.

We can make an α-pair as the product of two smaller ones. For example the product of

$$\begin{matrix} 012\bar{2}\bar{1} \\ 12\bar{2}\bar{1}0 \end{matrix} \quad \text{and} \quad \begin{matrix} 0123\bar{3}\bar{2}\bar{1} \\ 13\bar{2}02\bar{3}\bar{1} \end{matrix} \quad \text{is}$$

$$0\,1\,2\,\bar{2}\,\bar{1}\;\; 5\;\, 6\;\, 7\;\, 3\;\, 4\;\, 10\;\, 11\;\, 12\;\, 8\;\, 9\;\, 15\;\, 16\;\, 17\;\, 13\;\, 14\;\, \overline{15}\;\, \overline{14}\;\, \overline{13}\;\, \overline{17}\;\, \overline{16}\;\, \overline{10}\;\, \bar{9}\;\, \bar{8}\;\, \overline{12}\;\, \overline{11}\;\, \bar{5}\,\bar{4}\,\bar{3}\,\bar{7}\,\bar{6}$$

$$6\,7\,3\,4\,5\;\; 16\;\, 17\;\, 13\;\, 14\;\, 15\;\, \bar{9}\;\, \bar{8}\;\, \overline{12}\;\, \overline{11}\;\, \overline{10}\;\, 1\;\, 2\;\, \bar{2}\;\, \bar{1}\;\, 0\;\, 11\;\, 12\;\, 8\;\, 9\;\, 10\;\, \overline{14}\;\, \overline{13}\;\, \overline{17}\;\, \overline{16}\;\, \overline{15}\;\, \bar{4}\,\bar{3}\,\bar{7}\,\bar{6}\,\bar{5}$$

so that a partial answer to Question 2 is:

all n of the form $5^a 7^b$.

Are there any others?

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July 25, 1978

Dr. A. Kotzig
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Dear Dr. Kotzig:

Your $|S_N|$ is Sequence 666 in The Handbook of Integer Sequences (NJAS, Academic Press, N.Y. 1973). The problem you describe was considered by Bennett and Potts, J. Australian Math. Soc., 7 (1967) pp. 23- . They give

$$|S_{15}| = 659632$$

but assert that the problem is exceedingly difficult.

Yours sincerely,

Colin Mallows

C. L. Mallows

Neil Sloane

N. J. A. Sloane

MH-1215-CLM / 1216-NJAS-rb

Copy to
R. P. Guy