Scan A2083 etc

&

Wynne & Narayana A37254

Preprint

(it was published)

add to 6 sequences

f91

A2083 A5254 -5257 ~~A37354~~ A37254 preprint

TOURNAMENT CONFIGURATION, WEIGHTED VOTING, AND PARTITIONED CATALANS

by

B. E. Wynne and T. V. Narayana

1. Introduction Integer sequences can serve as powerful explicit descriptions of fundamental processes underlying organized activities. Seemingly unrelated decision processes may be perceived as mere variants of one another, once defined in terms of their essential sequences. As an example, two diverse sporting and/or business activities will be shown to involve the same recursively describable integer sequence.

One of the processes is key to determining the champion in a match play tournament. The second process considered is that of decision-making by voting. Following a brief exposition of each process and of the sequence they both utilize, some additional points of process similarity are noted as questions worthy of further investigation.

2. Match Play Tournaments As in [1], define T_n as the total number of ways in which a match play tournament among n entrants can be structured in order to determine a champion. Were no byes allowed, the number of entrants would be defined by $n = 2^t$, where t is the number of full match rounds in the tournament. To satisfy the practical general conditions we must allow any reasonable number of entrants, yet provide at least one match per round of the tournament. Therefore, redefine $n = 2^t + p$, with p ranging from 0 to 2^t-1. Here,

p is the number of players, if any, who may be accommodated in an incomplete "second flight." Letting k represent the number of rounds in such a tournament, we have; $\lceil (n-1)/2 \rceil \leq k \leq (n-1)$.

Capell and Narayana [1] showed that the number of different tournament match play arrangements possible for n entrants and a feasible number of match play rounds, k, is given by;

$$(1) \quad T_{(n,k)} = \sum_{i=\lceil (n-1)/2 \rceil}^{n-1} T_{(i,\ k-1)}$$

Though given in a slightly different form, their results of summing $T_{(n,k)}$ across the admissible range on k can be stated as:

$$(2) \quad T_n = \sum_{k=1}^{n-1} T_{(n,k)}$$ with slight notational change.

Capell and Narayana concluded by presenting the data given in Table 1. That data gives for n, the number of entrants, the number of tournaments or possible match-tree configurations, T_n, for all admissible k values.

- -

Insert Table 1. about here.

- -

3. Weighted Voting Procedure. A "Board of Directors" problem was posed in [2]. There, the challenge was to define an algorithm which would generate the minimal-sum set of weights for aggregating the votes of up to m participants such that under any behavior other than total abstention; (1) no tally of weighted votes could result in a tie, and (2) any group decision would always reflect the will of the majority of actual voters. An incomplete computational proof and an accompanying generation method satisfying the challenge have been submitted [3].

The essence of that methodology was to recursively compute a vector element increment, I_m, as follows;

$$(3) \quad I_m = 2(I_{m-1}) - \mathrm{mod}_2(m-1)\,(I_{\lceil m/2 \rceil - 1})$$

where, $m \geq 3$ and, $I_0 = 0$, $I_1 = I_2 = 1$.

Then, given the integer weighting elements, $W_{\cdot j}$, of the weights vector, $[W]_{m-1}$, the elements of the vector $[W]_m$ are given by:

$$(3) \quad W_{m,j} = W_{m-1,j} + I_m$$

where $j = 1,\ldots,m-1$ and, $W_{m,m} = I_m$.

Finally, denoting the sum of the weights vector elements as S_m, the data of Table 2. is generated for illustration. Comparison of the T_n values of Table 1. with the I_m values of Table 2. discloses the same sequence of numbers. That sequence is #297, p. 53 of [4]. This equivalence, for $m = (n-1)$, is perhaps more readily seen by restating T_{n+1} as in (3);

$$(4) \quad T_{n+1} = 2(T_n) - \mathrm{mod}(n-1)\,(T_{\lceil n/2 \rceil})$$

where $n \geq 3$ and, $T_1 = 0$, $T_2 = T_3 = 1$

This equivalence suggests (*) an inductive proof of the process for generating the required minimal-sum voting weights vector, $[W]_n = \{W_{n,1}, \ldots, W_{n,n}\}$. n henceforth replaces m.

---

* The simple proof is left to the reader.

Letting, $I_0 = 0,\ I_1 = I_2 = 1$

then $S_n = nI_n + (n-1)\,I_{n-1} + \ldots + 1$

from which $S_n - S_{n-1} = nI_n$.

Also, $W_{n,n-k} = I_n + I_{n-1} + \ldots + I_{n-k}$ for $k = 1, \ldots, n-1$

so that $\sum_{x=1}^{y} W_{n,x} < \sum_{x=n-y}^{n} W_{n,x}$ for $y=1, \ldots, \frac{n-1}{2}$ and, $n>2$

thus satisfying the three required conditions:

(a) any y+1 smallest weights' sum exceeds any y exclusive larger weights' sum taken from the same weights vector

(b) no subset of $[W]_n$ Equals any other non-intersecting subset

(c) S_n is minimal.

- -

Insert Table 2. about here.

- -

4. <u>Process Similarity Observations and Conjectures</u> It is tempting to pass off this equivalence of T_n and I_m sequences as a "fantastic combinatorial coincidence." However, to do so dismisses too casually the opportunity for discovery which is one of the values implicit in Sloane's original compilation of integer sequences [4]. If instead the process similarities are pursued, productive research directions may be opened.

One obvious conceptual shift is to liken the "declaration of a tournament champion" to the outcome of the other process, the "formulation of a group decision." As required, under such a shift, the unit difference between n and m vanishes. That is:

either, (1) assuming that each voter has a unique initial opinion and that the content of the final resolution adopted is not identical with any initial individual opinion, then there are actually m+1 "entrants" in the voting process.

or, (2) assuming that the tournament champion is analogous to the voting decision, then the n-1 non-champion tournament entrants are in one-to-one correspondence with the members of the voting body.

Based upon either interpretation of the conceptual shift discussed above, both processes can be thought of as being described by a "planar planted tree structure" which is trivalent, or limited to binary branching. The derivation of T_n is in fact an enumeration of such trees (a single root, n tips, and k branching levels, where k ranges from $\lceil (n-1)/2 \rceil$ to n-1). One interpretation [4, p. 19] of a prominent sequence, the Catalan numbers, is "the number of bifurcated rooted planar trees with n+1 endpoints"; a root plus n tips. The definition of C_n, the nth based Catalan is;

(5a) $$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$ a form of the central binomial numbers

(5b) $$C_n = \frac{C_{n-1}\,(4(n+1) - 10)}{n+1}$$ Euler's polygon triangularization form

Gardner [5] has pictorially shown Cayley's correspondence between Euler's polygon triangularization derivation of the Catalan numbers and the full sequence of planar planted trivalent trees. Investigation of polygon triangularization through order eight

(but order-nine is notation!)

demonstrates that the I_m sequence given here is a count of the number of ways in which an order-n polygon can be triangularized, provided that polygonal rotations (or planar tree reflections, whether full or partial) are disallowed.

However, a curious point of difference exists between the partitioning used (1) on trees counted as a subset of Euler's triangularization, and (2) those counted as match play tournament configurations. The latter counts all trees having k branching levels for a given value of n tips as one class. In contrast, the Euler subset method counts trees in a manner ambiguous as to k, but correct overall for an n forest. This ambiguity occurs if one attempts to designate which of the n+1 endpoints serves as the root. In general, one triangularized polygon can represent several different branching level values.

A caution is in order, having suggested that this Catalan subset tree method results in the same count as does that of counting the familiar (inverted) match play tournament trees. The latter is, for a given n value, conventionally constructed to define a minimum number of byes. Equivalence to the Euler subset is only achieved if the number of byes, b, configured into tournaments is allowed to range from the minimum up through b equal to $\frac{(n-1)(n-2)}{2}$.

Despite the fact that the Catalan numbers also have been interpreted [4, p. 19] directly in terms of tie vote domination, the planar planted tree alternative applies better to the Board of Directors problem. The intuitive basis for this is to conceive of the voting process of a small decision-making body as a sequential aggregation of coalitions. In some instances a board decision is formed "rapidly", just as a tournament with a minimum of byes may be said to progress rapidly. In other instances some individuals may fail to form or join any coalition until late in the board decision process. This is analogous to a sports tournament with byes approaching the maximum. Preserving prior terminology, it may be said that the kth coalition in effect determines the board's decision.

Two conjectures remain to be made in the context of the Board of Directors problem. First, the S_n sequence of Table 1 does not appear in Sloane's collection. However, Riordan [6, p. 215] discusses what he terms reciprocal central factorial numbers as being perhaps as general as the dual class of Stirling numbers. Riordan's polynomials are closely linked to T_n and $T_{(n,k)}$ as used here. Thus maybe the S_n sequence is relevant to Riordan's ideas.

Second, if every board member votes, each individual may be said to be equally powerful in the sense of determining the board's decision. However, intriguing power imbalances are introduced by the unequal integer weights whenever one or more abstentions occur. Perhaps further "pruning" of the reduced set of Catalan trees, I_n and/or its related sequence, S_n, has applicability for this issue of intermember power dynamics?

TABLE 1.

Match Tournament Entrants and Configurations

Entrants, n	2	3	4	5	6	7	8	9	10	11
Configurations, T_n	1	1	2	3	6	11	22	42	84	165

A2083

~~for~~ Knockout tournaments

5254

TABLE 2.

Minimal-sum, Non-distorting, Tie-avoiding Integer Vote Weights

Δ = A37354

members, m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
totals, S_m	1	3	9	21	51	117	271	607	1363	3013	6643	14491	31495	67965	146115
column vectors of vote weights, W_m (5257) (5255)	1	2	4	7	13	24	46	88	172	337	667	1321	2629	5234	10444
(5256)		1	3	6	12	23	45	87	171	336	666	1320	2628	5233	10443
			2	5	11	22	44	86	170	335	665	1319	2627	5232	10442
				3	9	20	42	84	168	333	663	1317	2625	5230	10440
					6	17	39	81	165	330	660	1314	2622	5227	10437
						11	33	75	159	324	654	1308	2616	5221	10431
							22	64	148	313	643	1297	2605	5210	10420
								42	126	291	621	1275	2583	5188	10398
									84	249	579	1233	2541	5146	10356
										165	495	1149	2457	5062	10272
											330	984	2292	4897	10107
												654	1962	4567	9777
													1308	3913	9123
														2605	7815
															5210

NOTE

The underlined values along the diagonal of vector elements are the I_m values.

References

[1] P. Capell and T.V. Narayana, "On Knock-out Tournaments", Canadian Mathematical Bulletin 13 (1), 1970, pp. 105-9.

[2] D.L. Silverman, "The Board of Directors' Problem", Journal of Recreational Mathematics 8 (3), 1975-76, p. 234.

[3] B.E. Wynne, "INTWAT, STREAK and Weights Vectors", submitted May 15, 1976 to Journal of Recreational Mathematics.

[4] N.J.A. Sloane, A Handbook of Integer Sequences, Academic Press, New York, (1973).

[5] M. Gardner, "Mathematical Games", Scientific American 234 (6), June, 1976, p. 120.

[6] J. Riordan, Combinatorial Identities, John Wiley & Sons, Inc., New York, (1968).