**A002283 and some continued fraction expansions**

**Peter Bala, Sep 27 2015**

A002283 is the repdigit sequence $[0, 9, 99, 999, 9999, ...]$. The sequence terms are given by

$$a(n) = 10^n - 1.$$

For $n >= 1$, the simple continued fraction expansion of $\sqrt{a(2n)}$ has period 2

$$\sqrt{10^{2n}-1} = 10^n - 1 + \cfrac{1}{1 + \cfrac{1}{2(10^n-1) + \cfrac{1}{1 + \cfrac{1}{2(10^n-1) + \cdots}}}}$$

Written in flat notation

$$\sqrt{10^{2n}-1} = [10^n - 1; 1, 2(10^n - 1), 1, 2(10^n - 1), ...].$$

Note the occurrence of the large partial quotients $2(10^n - 1)$.

The simple continued fraction expansion of $\sqrt{a(2n+1)}$ has a more complicated structure. The initial terms in the expansion appear to depend on the parity of $n$. Empirically, for $n \geq 2$, the first 20 or so partial quotients seem to be given by the formulas

$$\begin{aligned}\sqrt{10^{4n+1}-1} &= [100^{4n+1} - 0.5 \times 10^{4n+1}, 2, 1, 2, 0.05 \times 10^{4n+1} - 1, 1, 2, 52, 1, \frac{10^{4n} - 304}{202}, \\ &\quad 1, 1, 10, 2, 1, 28, 1, 1, 11, 2, 4, ...]\end{aligned}$$

$$\begin{aligned}\sqrt{10^{4n-1}-1} &= [100^{4n-1} - 0.5 \times 10^{4n-1}, 2, 1, 2, 0.05 \times 10^{4n-1} - 1, 1, 2, 52, 1, \frac{10^{4n-2} - 302}{202}, \\ &\quad 1, 1, 18, 2, 16, 1, 17, 8, 12, ...]\end{aligned}$$

Again, as $n$ increases, we see the occurrence of large partial quotients.

A theorem of Kuzmin in the measure theory of continued fractions says that for a random real number $\alpha$, the probability that some given partial quotient of $\alpha$ is equal to a positive integer $k$ is given by

$$\frac{1}{\log(2)}\left(\log\left(1 + \frac{1}{k}\right) - \log\left(1 + \frac{1}{k+1}\right)\right).$$

For example, almost all real numbers have 41.5% of their partial quotients equal to 1, 17% of their partial quotients equal to 2, 9.3% of their partial quotients equal to 3 and so on. The probability that some given partial quotient of a random real number is equal to $10^6$ is $1.4 \text{x} 10^{-12}$. Thus large partial quotients as in the expansions of $\sqrt{a(2n)}$ and $\sqrt{a(2n+1)}$ above are the exception in continued fraction expansions.

Empirically, we also observe unexpectedly large partial quotients early on in the continued fraction expansions of the $m^{th}$ roots of the numbers $a(mn)$ for $m = 3, 4, 5, ...$ . For example,

$$\sqrt[3]{a(9)} = [999; 1, 2999998, 1, 998, 1, 4499998, 1, 798, 1, 5357141, 1, 5, 1, 13, 2, 2, 1, 5999999, 3, 1, 1, ...]$$

$$\sqrt[5]{a(20)} = [9999; 1, 49999999999999998, 1, 4998, 1, 99999999999999998, 1, 3332, 3, 15151515151515151, 5, 1, 1, 194, 3, 1, 5, 1, 2, 378787878787878, ...]$$

$$\sqrt[7]{a(28)} = [9999; 1, 6999999999999999999999998, 1, 3332, 3, 1749999999999999999999999, 1, 1, 1, 2050, 1, 1, 1, 1, 2, 134615384615384615384614, 1, 259998, 1, ...]$$

Let's examine the expansions of the cube root of $a(3n)$ in a bit more detail. A little numeric experimentation suggests the early terms in the continued fraction expansion of the numbers $\sqrt[3]{a(3n)}$ depend on the value of $n$ mod 6. For example, consider the continued fraction expansion of numbers of the form $\sqrt[3]{10^{18n} - 1}$. The first three cases are

$\sqrt[3]{10^{18} - 1} =$

[999999; 1, 2999999999998, 1, 999998, 1, 4499999999998, 1, 799998, 1, 5357142857141, 1, 5, 1, 14284, 1, 5, 1, 5999999999999, 7, 12986, 1, 6, 1, 2, 3, 53946053945, 1, 3, 11, 1, 2, 3496, 6, 4, 1, 1, 2, 57692307691, 1, 10, 1, 10, 1, 3289, 1, 10, 4, 1, 13,...]

$\sqrt[3]{10^{36} - 1} =$

[999999999999; 1, 2999999999999999999999998, 1, 999999999998, 1, 4499999999999999999999998, 1, 799999999998, 1, 5357142857142857142857141, 1, 5, 1, 14285714284, 1, 5, 1, 5999999999999999999999999, 7, 12987012986, 1, 6, 1, 2, 3, 53946053946053946053945, 1, 3, 11, 1, 2, 3496503496, 6, 4, 1, 1, 2, 57692307692307692307691, 1, 10, 1, 10, 1, 3290826819, 3, 9, 1, 12, 6,...]

$\sqrt[3]{10^{54} - 1} =$

[999999999999999999; 1, 2999999999999999999999999999999999998, 1, 999999999999999998, 1, 4499999999999999999999999999999999998, 1, 799999999999999998, 1, 5357142857142857142857142857142857141, 1, 5, 1, 14285714285714284, 1, 5, 1, 5999999999999999999999999999999999999, 7, 12987012987012986, 1, 6, 1, 2, 3, 53946053946053946053946053946053945, 1, 3, 11, 1, 2, 3496503496503496, 6, 4, 1, 1, 2, 57692307692307692307692307692307691, 1, 10, 1, 10, 1, 3290826820238584, 35, 4, 3, 4, 1,...]

Notice the pattern of small partial quotients is identical in each case, up to the block $1, 10, 1, 10, 1$, and corresponding large partial quotients seem to follow a predictable pattern . Further calculation leads to the conjecture that the simple continued fraction expansion of $\sqrt[3]{10^{18n}-1}$ begins

$$[10^{6n}-1; 1, 3\times 10^{12n}-2, 1, 10^{6n}-2, 1, 4.5\times 10^{12n}-2, 1, 0.8\times 10^{6n}-2, 1, \frac{37.5\times 10^{12n}-13}{7},$$

$$1, 5, 1, \frac{0.1\times 10^{6n}-12}{7}, 1, 5, 1, 6\times 10^{12n}-1, 7, \frac{10^{6n}-78}{77}, 1, 6, 1, 2, 3, \frac{54\times 10^{12n}-1055}{1001},$$

$$1, 3, 11, 1, 2, \frac{10^{6n}-144}{286}, 6, 4, 1, 1, 2, \frac{3\times 10^{12n}-68}{52}, 1, 10, 1, 10, 1, ...].$$

We have checked this result up to $n = 50$. Similar results seem to hold for the simple continued fraction expansions of the numbers $\sqrt[3]{10^{18n+3k}-1}$ for $k = 1, 2, 3, 4, 5$.

Calculation suggests there will be analogous results for higher roots. For example, it appears that the simple continued fraction expansion of $\sqrt[4]{10^{24n}-1}$ begins

$$[10^{6n}-1; 1, 4\times 10^{18n}-2, 1, \frac{2\times 10^{6n}-5}{3}, 1, 1, 1, 0.8\times 10^{18n}-1, 3, \frac{10}{21}(10^{6n}-1),$$

$$7, \frac{4}{21}(10^{18n}-1), 21, \frac{10}{231}(10^{6n}-1)-1, 1, 229, 1, \frac{4\times 10^{18n}-3865}{2145}, 1, 4, 6, 3, 1, 6, 1,$$

$$\frac{2}{1001}(10^{6n}-1)-1, 1, 7, 1, 1, 1, 3, 1, 2, 2, 1, ...].$$

We have checked this result up to $n = 50$.

The behaviour of the sequence $a(n) = 10^n - 1$ of having untypically large partial quotients early in the continued fraction expansion of the $m^{th}$ roots of $a(mn)$ is shared with other sequences such as $10^n + 1$, $10^{2n} + 10^n + 1$, $10^{2n} - 10^n + 1$ and their ratios. See A000533, A066138 and A168624.

A particularly nice example is the continued fraction expansion of the numbers $\sqrt[3]{10^{3n}+1}$, which appear to start with six large (and predictable) partial quotients. For example, the continued fraction expansion of $\sqrt[3]{10^{30}+1}$ begins

[10000000000; 300000000000000000000, 10000000000, 450000000000000000000, 8000000000, 535714285714285714285, 1, 2, 1, 1, 142857142, 3, 2, 599999999999999999999, 1, 1, ...].