MES ~~[illegible]~~ 31 (1901)



# TABLE OF THE EXCESS OF THE NUMBER OF (8k+1)- AND (8k+3)-DIVISORS OF A NUMBER OVER THE NUMBER OF (8k+5)- AND (8k+7)-DIVISORS.

By *J. W. L. Glaisher.*

§ 1. This paper contains a table, extending from $n=1$ to $n=1000$, of the function $J(n)$, where $J(n)$ denotes the excess of the number of divisors of $n$ which have the forms $8k+1$ and $8k+3$ over the number of divisors which have the forms $8k+5$ and $8k+7$. The table is exactly similar to the corresponding table of $H(n)$ on pp. 67–72.

The quantity $2J(n)$, when $n$ is uneven, is equal to the number of representations of any number $n$ by the form $x^2+2y^2$. It also occurs as coefficient in certain $q$-series in Elliptic Functions (§§ 4, 5).

§ 2. The table was calculated from the formula*:—if

$$n = 2^h a^\alpha b^\beta c^\gamma \ldots r^\rho s^\sigma t^\tau \ldots,$$

where $a, b, c, \ldots$ are primes of the forms $8k+1$ or $8k+3$ and $r, s, t, \ldots$ are primes of the forms $8k+5$ or $8k+7$, then $J(n)$ is zero unless all the exponents $\rho, \sigma, \tau, \ldots$ are even, and, if all are even, then

$$J(n) = (\alpha+1)(\beta+1)(\gamma+1)\ldots$$

(the exponents of the primes of the forms $8k+1$ or $8k+3$ alone appearing in this product).

The table was verified by means of the formula

$$\Sigma_1^n J(s) = -\rho\psi(\rho) + j_n + I(n) + I\left(\frac{n}{3}\right) - I\left(\frac{n}{5}\right) - I\left(\frac{n}{7}\right)$$

$$+ I\left(\frac{n}{9}\right) + I\left(\frac{n}{11}\right) - I\left(\frac{n}{13}\right) - \ldots,$$

where $\rho$ is the number next below the square root of $n$, if $n$ is not a square, and is equal to the square root of $n$, if $n$ is a square, $\psi(\rho) = 1, 1, 2, 2, 1, 1, 0, 0$ according as $\rho \equiv 1, 2, 3,$

* This formula may easily be derived from the general principles explained in *Proc. Lond. Math. Soc.*, Vol. XXI., p. 186 (note).



4, 5, 6, 7, 0, mod. 8, $I(r)$ denotes the greatest integer contained in $r$ if $r$ is fractional and is equal to $r$ if $r$ is an integer, and $j_n$ is derived from the $\rho$ numbers

$$I(n),\quad I\left(\frac{n}{2}\right),\quad I\left(\frac{n}{3}\right),\quad \ldots,\quad I\left(\frac{n}{\rho}\right)$$

by counting those which $\equiv 1, 2, 5$, or 6, mod. 8, as 1 each, those which $\equiv 3$ or 4, mod. 8, as 2 each, and ignoring those which $\equiv 7$ or 0, mod. 8. The series

$$I(n) + I\left(\frac{n}{3}\right) - I\left(\frac{n}{5}\right) - \ldots$$

is to be continued so long as the denominators do not exceed $\rho$.

§ 3. By means of this formula* the value of $\Sigma_1^n J(s)$ was calculated for $n=100, 200, \ldots, 1000$. The steps and results of the calculation are shown below.

| $n$ | $\rho$ | $\rho\psi(\rho)$ | $j_n$ | sum of positive terms in series | sum of negative terms in series | $\Sigma_1^n J(s)$ |
|---|---|---|---|---|---|---|
| 100 | 10 | 10 | 13 | 144 | 34 | 113 |
| 200 | 14 | 14 | 13 | 306 | 83 | 222 |
| 300 | 17 | 17 | 22 | 477 | 145 | 337 |
| 400 | 20 | 40 | 20 | 657 | 193 | 444 |
| 500 | 22 | 22 | 23 | 821 | 265 | 557 |
| 600 | 24 | 0 | 26 | 986 | 345 | 667 |
| 700 | 26 | 26 | 30 | 1178 | 402 | 780 |
| 800 | 28 | 56 | 22 | 1376 | 460 | 882 |
| 900 | 30 | 30 | 29 | 1519 | 519 | 999 |
| 1000 | 31 | 0 | 30 | 1721 | 640 | 1111 |

* This formula occurs in Vol. XXXIII., p. 207 of the *Quarterly Journal* (not yet published). The corresponding formula relating to $H(n)$ referred to on p. 64 of this volume of the *Messenger* occurs on p. 198 of the same paper.



f Glaisher fil.

MES 31 (1901)

2325

These values of $\Sigma_1^n J(n)$ show that the sum of the values of $J(n)$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| from | $n=1$ | to | $n=100$, | both | inclusive, | is | 113, |
| ,, | 101 | ,, | 200, | ,, | ,, | ,, | 108, |
| ,, | 201 | ,, | 300, | ,, | ,, | ,, | 115, |
| ,, | 301 | ,, | 400, | ,, | ,, | ,, | 107, |
| ,, | 401 | ,, | 500, | ,, | ,, | ,, | 113, |
| ,, | 501 | ,, | 600, | ,, | ,, | ,, | 110, |
| ,, | 601 | ,, | 700, | ,, | ,, | ,, | 113, |
| ,, | 701 | ,, | 800, | ,, | ,, | ,, | 102, |
| ,, | 801 | ,, | 900, | ,, | ,, | ,, | 117, |
| ,, | 901 | ,, | 1000, | ,, | ,, | ,, | 112. |

These numbers agree with those obtained by counting from the table of $J(n)$ (pp. 86–91).

It is evident from the expression for $J(n)$ in § 2 that, if $n_1$, $n_2$ are any two numbers prime to each other, then $J(n_1 n_2) = J(n_1)\,J(n_2)$.

§ 4. From Elliptic Functions we have, putting $\rho = \dfrac{2K}{\pi}$,

$$\rho^{\frac{1}{2}} = 1 + 2q + 2q^4 + 2q^9 + \&c.,$$

$$k^{\frac{1}{2}}\rho^{\frac{1}{2}} = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + \&c.,$$

and therefore, changing $q$ into $q^2$,

$$\frac{(1+k')^{\frac{1}{2}}}{\sqrt{2}}\,\rho^{\frac{1}{2}} = 1 + 2q^{2\cdot 1} + 2q^{2\cdot 4} + 2q^{2\cdot 9} + \&c.,$$

$$\frac{(1-k')^{\frac{1}{2}}}{\sqrt{2}}\,\rho^{\frac{1}{2}} = 2q^{\frac{1}{2}} + 2q^{\frac{9}{2}} + 2q^{\frac{25}{2}} + \&c.;$$

whence, by multiplication,

$$\frac{(1+k')^{\frac{1}{2}}}{\sqrt{2}}\,\rho = (1 + 2q + 2q^4 + 2q^9 + \&c.)\,(2q^{2\cdot 1} + 2q^{2\cdot 4} + 2q^{2\cdot 9} + \&c.),$$

$$\frac{(1-k')^{\frac{1}{2}}}{\sqrt{2}}\,\rho = (2q^{\frac{1}{2}} + 2q^{\frac{9}{2}} + 2q^{\frac{25}{2}} + \&c.)\,(1 + 2q + 2q^4 + 2q^9 + \&c.).$$



Since $2J(n)$ is the number of representations of $n$ by the form $x^2+2y^2$, the products on the right-hand side are respectively equal to

$$1 + 2J(1)\,q + 2J(2)\,q^2 + 2J(3)\,q^3 + \&c.,$$

and

$$2J(1)\,q^{\frac{1}{2}} + 2J(3)\,q^{\frac{3}{2}} + 2J(5)\,q^{\frac{5}{2}} + \&c.,$$

so that we find

$$(1+k')^{\frac{1}{2}}\rho = \sqrt{2}\,\{1 + 2\Sigma_1^\infty J(n)\,q^n\},$$

$$(1-k')^{\frac{1}{2}}\rho = 2\sqrt{2}\,\Sigma_1^\infty J(m)\,q^{\frac{1}{2}m},$$

where $n$ is any number and $m$ any uneven number.

§ 5. By making use only of the definition of $J(n)$ as the excess of the number of divisors of $n$ of the forms $8k+1$, 3 over the number of divisors of $n$ of the forms $8k+5$, 7, we may derive these formulæ directly form the $q$-series for $k\rho\,\mathrm{sn}\,\rho x$, &c.*

We thus find, putting $x = \frac{1}{4}\pi$,

$$k\rho\,\mathrm{sn}\,\tfrac{1}{2}K = k\rho\,\mathrm{cd}\,\tfrac{1}{2}K = 2\sqrt{2}\,\Sigma_1^\infty J(m)\,q^{\frac{1}{2}m},$$

$$kk'\rho\,\mathrm{sd}\,\tfrac{1}{2}k = k\rho\,\mathrm{cn}\,\tfrac{1}{2}K = 2\sqrt{2}\,\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}J(m)\,q^{\frac{1}{2}m},$$

$$\rho\,\mathrm{ns}\,\tfrac{1}{2}K = \rho\,\mathrm{dc}\,\tfrac{1}{2}K = \sqrt{2}\,\{1 + 2\Sigma_1^\infty J(n)\,q^n\},$$

$$\rho\,\mathrm{ds}\,\tfrac{1}{2}K = k'\rho\,\mathrm{nc}\,\tfrac{1}{2}K = \sqrt{2}\,\{1 + 2\Sigma_1^\infty (-1)^n J(n)\,q^n\}.$$

Since

$$\mathrm{sn}\,\tfrac{1}{2}K = \frac{1}{(1+k')^{\frac{1}{2}}},\quad \mathrm{cn}\,\tfrac{1}{2}K = \frac{k'^{\frac{1}{2}}}{(1+k')^{\frac{1}{2}}},\quad \mathrm{dn}\,\tfrac{1}{2}K = k'^{\frac{1}{2}},$$

it is evident that the first and third of these formulæ are the same as those obtained in the previous section.†

* It is convenient to use the $q$-series in the form given on p. 8 of Vol. XVIII. of the *Messenger*.

† By equating the $q$-series for $k\rho\,\mathrm{sn}\,\frac{1}{2}K$ or $\rho\,\mathrm{ns}\,\frac{1}{2}K$ to the $q$-products obtained for $(1-k')^{\frac{1}{2}}\rho$ or $(1+k')^{\frac{1}{2}}\rho$ in the previous section we obtain a proof by Elliptic Functions that the number of representations of any number $n$ by the form $x^2+2y^2$ is $2J(n)$.

2325



Table of $J(n)$ from $n=1$ to $n=1000$.

| $n$ | $J(n)$ | $n$ | $J(n)$ | $n$ | $J(n)$ | $n$ | $J(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 46 | 0 | 91 | 0 | 136 | 2 |
| 2 | 1 | 47 | 0 | 92 | 0 | 137 | 2 |
| 3 | 2 | 48 | 2 | 93 | 0 | 138 | 0 |
| 4 | 1 | 49 | 1 | 94 | 0 | 139 | 2 |
| 5 | 0 | 50 | 1 | 95 | 0 | 140 | 0 |
| 6 | 2 | 51 | 4 | 96 | 2 | 141 | 0 |
| 7 | 0 | 52 | 0 | 97 | 2 | 142 | 0 |
| 8 | 1 | 53 | 0 | 98 | 1 | 143 | 0 |
| 9 | 3 | 54 | 4 | 99 | 6 | 144 | 3 |
| 10 | 0 | 55 | 0 | 100 | 1 | 145 | 0 |
| 11 | 2 | 56 | 0 | 101 | 0 | 146 | 2 |
| 12 | 2 | 57 | 4 | 102 | 4 | 147 | 2 |
| 13 | 0 | 58 | 0 | 103 | 0 | 148 | 0 |
| 14 | 0 | 59 | 2 | 104 | 0 | 149 | 0 |
| 15 | 0 | 60 | 0 | 105 | 0 | 150 | 2 |
| 16 | 1 | 61 | 0 | 106 | 0 | 151 | 0 |
| 17 | 2 | 62 | 0 | 107 | 2 | 152 | 2 |
| 18 | 3 | 63 | 0 | 108 | 4 | 153 | 6 |
| 19 | 2 | 64 | 1 | 109 | 0 | 154 | 0 |
| 20 | 0 | 65 | 0 | 110 | 0 | 155 | 0 |
| 21 | 0 | 66 | 4 | 111 | 0 | 156 | 0 |
| 22 | 2 | 67 | 2 | 112 | 0 | 157 | 0 |
| 23 | 0 | 68 | 2 | 113 | 2 | 158 | 0 |
| 24 | 2 | 69 | 0 | 114 | 4 | 159 | 0 |
| 25 | 1 | 70 | 0 | 115 | 0 | 160 | 0 |
| 26 | 0 | 71 | 0 | 116 | 0 | 161 | 0 |
| 27 | 4 | 72 | 3 | 117 | 0 | 162 | 5 |
| 28 | 0 | 73 | 2 | 118 | 2 | 163 | 2 |
| 29 | 0 | 74 | 0 | 119 | 0 | 164 | 2 |
| 30 | 0 | 75 | 2 | 120 | 0 | 165 | 0 |
| 31 | 0 | 76 | 2 | 121 | 3 | 166 | 2 |
| 32 | 1 | 77 | 0 | 122 | 0 | 167 | 0 |
| 33 | 4 | 78 | 0 | 123 | 4 | 168 | 0 |
| 34 | 2 | 79 | 0 | 124 | 0 | 169 | 1 |
| 35 | 0 | 80 | 0 | 125 | 0 | 170 | 0 |
| 36 | 3 | 81 | 5 | 126 | 0 | 171 | 6 |
| 37 | 0 | 82 | 2 | 127 | 0 | 172 | 2 |
| 38 | 2 | 83 | 2 | 128 | 1 | 173 | 0 |
| 39 | 0 | 84 | 0 | 129 | 4 | 174 | 0 |
| 40 | 0 | 85 | 0 | 130 | 0 | 175 | 0 |
| 41 | 2 | 86 | 2 | 131 | 2 | 176 | 2 |
| 42 | 0 | 87 | 0 | 132 | 4 | 177 | 4 |
| 43 | 2 | 88 | 2 | 133 | 0 | 178 | 2 |
| 44 | 2 | 89 | 2 | 134 | 2 | 179 | 2 |
| 45 | 0 | 90 | 0 | 135 | 0 | 180 | 0 |



| $n$ | $J(n)$ | $n$ | $J(n)$ | $n$ | $J(n)$ | $n$ | $J(n)$ |
|---|---|---|---|---|---|---|---|
| 181 | 0 | 226 | 2 | 271 | 0 | 316 | 0 |
| 182 | 0 | 227 | 2 | 272 | 2 | 317 | 0 |
| 183 | 0 | 228 | 4 | 273 | 0 | 318 | 0 |
| 184 | 0 | 229 | 0 | 274 | 2 | 319 | 0 |
| 185 | 0 | 230 | 0 | 275 | 2 | 320 | 0 |
| 186 | 0 | 231 | 0 | 276 | 0 | 321 | 4 |
| 187 | 4 | 232 | 0 | 277 | 0 | 322 | 0 |
| 188 | 0 | 233 | 2 | 278 | 2 | 323 | 4 |
| 189 | 0 | 234 | 0 | 279 | 0 | 324 | 5 |
| 190 | 0 | 235 | 0 | 280 | 0 | 325 | 0 |
| 191 | 0 | 236 | 2 | 281 | 2 | 326 | 2 |
| 192 | 2 | 237 | 0 | 282 | 0 | 327 | 0 |
| 193 | 2 | 238 | 0 | 283 | 2 | 328 | 2 |
| 194 | 2 | 239 | 0 | 284 | 0 | 329 | 0 |
| 195 | 0 | 240 | 0 | 285 | 0 | 330 | 0 |
| 196 | 1 | 241 | 2 | 286 | 0 | 331 | 2 |
| 197 | 0 | 242 | 3 | 287 | 0 | 332 | 2 |
| 198 | 6 | 243 | 6 | 288 | 3 | 333 | 0 |
| 199 | 0 | 244 | 0 | 289 | 3 | 334 | 0 |
| 200 | 1 | 245 | 0 | 290 | 0 | 335 | 0 |
| 201 | 4 | 246 | 4 | 291 | 4 | 336 | 0 |
| 202 | 0 | 247 | 0 | 292 | 2 | 337 | 2 |
| 203 | 0 | 248 | 0 | 293 | 0 | 338 | 1 |
| 204 | 4 | 249 | 4 | 294 | 2 | 339 | 4 |
| 205 | 0 | 250 | 0 | 295 | 0 | 340 | 0 |
| 206 | 0 | 251 | 2 | 296 | 0 | 341 | 0 |
| 207 | 0 | 252 | 0 | 297 | 8 | 342 | 6 |
| 208 | 0 | 253 | 0 | 298 | 0 | 343 | 0 |
| 209 | 4 | 254 | 0 | 299 | 0 | 344 | 2 |
| 210 | 0 | 255 | 0 | 300 | 2 | 345 | 0 |
| 211 | 2 | 256 | 1 | 301 | 0 | 346 | 0 |
| 212 | 0 | 257 | 2 | 302 | 0 | 347 | 2 |
| 213 | 0 | 258 | 4 | 303 | 0 | 348 | 0 |
| 214 | 2 | 259 | 0 | 304 | 2 | 349 | 0 |
| 215 | 0 | 260 | 0 | 305 | 0 | 350 | 0 |
| 216 | 4 | 261 | 0 | 306 | 6 | 351 | 0 |
| 217 | 0 | 262 | 2 | 307 | 2 | 352 | 2 |
| 218 | 0 | 263 | 0 | 308 | 0 | 353 | 2 |
| 219 | 4 | 264 | 4 | 309 | 0 | 354 | 4 |
| 220 | 0 | 265 | 0 | 310 | 0 | 355 | 0 |
| 221 | 0 | 266 | 0 | 311 | 0 | 356 | 2 |
| 222 | 0 | 267 | 4 | 312 | 0 | 357 | 0 |
| 223 | 0 | 268 | 2 | 313 | 2 | 358 | 2 |
| 224 | 0 | 269 | 0 | 314 | 0 | 359 | 0 |
| 225 | 3 | 270 | 0 | 315 | 0 | 360 | 0 |