A triangle for calculating A002438.

Peter Bala, April 24, 2017

Oste and van der Jeugt [1, Section 7] show that a continued fraction of the form

$$\cfrac{1}{1 - xd_0 - \cfrac{xh_1}{1 - xd_1 - \cfrac{xh_2}{1 - xd_2 - \cfrac{xh_3}{1 - xd_3 - \cdots}}}} \tag{1}$$

is the generating function for 2-Motzkin paths weighted by the integers d_i and h_i. This combinatorial interpretation allows one to rapidly calculate the terms of a sequence whose generating function can be expressed as a continued fraction of the form (1). The results are conveniently displayed in the form of a lower triangular array, where the $d's$ occur as multiplication factors along diagonals of the array and the $h's$ as horizontal multiplication factors along rows of the array. In the particular case of A002438, the generating function can be expressed as the continued fraction

$$x/(1-(3^2-2^2)x/(1-6^2x/(1-(9^2-2^2)x/(1-12^2x/(1-...)))))$$

So in this case the $d's$ are all zero and the horizontal multiplication factors are given by the formulas $h_{2n} = (3n)^2, h_{2n-1} = (6n-3)^2 - 2^2$. A002438 is the leading diagonal of the following lower triangular array:

1									
↓									
1	— x5 —>	**5**							
↓		↓							
1	— x36 —>	**41**	— x5 —>	**205**					
↓		↓		↓					
1	— x77 —>	**118**	— x36—>	**4453**	— x5 —>	**22265**			
↓		↓		↓		↓			
1	— x144—>	**262**	— x77—>	**24627**	— x36 —>	**908837**	— x5 —>	**4544185**	
↓		↓		↓		↓		↓	$\ddots$

References

[1] R. Oste and J. Van der Jeugt, Motzkin paths, Motzkin polynomials and recurrence relations, Electronic Journal of Combinatorics 22(2) (2015), #P2.8. Section 7