$$\begin{aligned}15\,I_7 = {} & (1/3)\,(C_2 + C_4 + C_6 + C_8 + C_{10} + C_{12} + C_{14}) \\ & + (1/5)\,(C_4 + C_8 + C_{12}) \\ & + (1/7)\,(C_6 + C_{12}) \\ & + (1/11)\,(C_{10}) \\ & + (1/13)\,(C_{12}) \\ & - 2^{13} + 7\,, \\ & = 15\,.\end{aligned}$$

Hence we get, $I_7 = 1$, and the Bernoulli number is,

$$B_7 = 1 - (5/6 - 1) = 7/6\,.$$

An ingenious method for computing the Bernoulli numbers has been furnished the writer by F. J. Feinler as follows:*

Let us designate by $[a/b]$ the integral part of the fraction $a/b$. For example, we shall have $[24/7] = [3 + 3/7] = 3$. Designating by $p$ the succession of primes, $p = 2, 3, 5, 7, 11, 13, \cdots$, we compute the table given below (Table 1) for the symbol, $[k/(p-1)]$, $k = 1, 2, 3, 4, 5$, etc.

We next form a table (see Table 2) giving the exponents of the prime factors, $p$, of $k!$ .

If we designate by $k(p)$ the numbers in Table 2 which correspond to $k$, then we may write $k! = p^{k(p)}$. For example,

$$10! = 2^8\,3^4\,5^2\,7^1\,.$$

By means of Tables 1 and 2 it is now possible to compute values of the symbol,

$$M_n{}^k = p^{[k/(p-1)]}/\{p^{[n/(p-1)]} \cdot p^{(k-n+1)(p)}\}\,.$$

For example, we get from Tables 1 and 2 the following:

$$M_6{}^{11} = \frac{2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11}{(2^6 \cdot 3^3 \cdot 5 \cdot 7)\ (2^4 \cdot 3^2 \cdot 5)} = 22\,.$$

Some of these values are given below in Table 3.

*See also: A New Method for Calculating the Bernoulli Numbers. *Messenger of Mathematics*, vol. 55 (1926), pp. 40-44.

TABLE 1

A266742

| $k/p$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | |
| 2 | 2 | 1 | | | | | | | | | |
| 3 | 3 | 1 | | | | | | | | | |
| 4 | 4 | 2 | 1 | | | | | | | | |
| 5 | 5 | 2 | 1 | | | | | | | | |
| 6 | 6 | 3 | 1 | 1 | | | | | | | |
| 7 | 7 | 3 | 1 | 1 | | | | | | | |
| 8 | 8 | 4 | 2 | 1 | | | | | | | |
| 9 | 9 | 4 | 2 | 1 | | | | | | | |
| 10 | 10 | 5 | 2 | 1 | 1 | | | | | | |
| 11 | 11 | 5 | 2 | 1 | 1 | | | | | | |
| 12 | 12 | 6 | 3 | 2 | 1 | 1 | | | | | |
| 13 | 13 | 6 | 3 | 2 | 1 | 1 | | | | | |
| 14 | 14 | 7 | 3 | 2 | 1 | 1 | | | | | |
| 15 | 15 | 7 | 3 | 2 | 1 | 1 | | | | | |
| 16 | 16 | 8 | 4 | 2 | 1 | 1 | 1 | | | | |
| 17 | 17 | 8 | 4 | 2 | 1 | 1 | 1 | | | | |
| 18 | 18 | 9 | 4 | 3 | 1 | 1 | 1 | 1 | | | |
| 19 | 19 | 9 | 4 | 3 | 1 | 1 | 1 | 1 | | | |
| 20 | 20 | 10 | 5 | 3 | 2 | 1 | 1 | 1 | | | |
| 21 | 21 | 10 | 5 | 3 | 2 | 1 | 1 | 1 | | | |
| 22 | 22 | 11 | 5 | 3 | 2 | 1 | 1 | 1 | 1 | | |
| 23 | 23 | 11 | 5 | 3 | 2 | 1 | 1 | 1 | 1 | | |
| 24 | 24 | 12 | 6 | 4 | 2 | 2 | 1 | 1 | 1 | | |
| 25 | 25 | 12 | 6 | 4 | 2 | 2 | 1 | 1 | 1 | | |
| 26 | 26 | 13 | 6 | 4 | 2 | 2 | 1 | 1 | 1 | | |
| 27 | 27 | 13 | 6 | 4 | 2 | 2 | 1 | 1 | 1 | | |
| 28 | 28 | 14 | 7 | 4 | 2 | 2 | 1 | 1 | 1 | 1 | |
| 29 | 29 | 14 | 7 | 4 | 2 | 2 | 1 | 1 | 1 | 1 | |
| 30 | 30 | 15 | 7 | 5 | 3 | 2 | 1 | 1 | 1 | 1 | 1 |
| 31 | 31 | 15 | 7 | 5 | 3 | 2 | 1 | 1 | 1 | 1 | 1 |

The Bernoulli numbers can now be computed from the following formula:

$$B_k = c_{2k}/d_{2k} \ ,$$

where the numerator and denominator are given by,

$$d_{2k} = p^{[2k/(p-1)]}/p^{2k(p)} ,$$

$$c_{2k} = (-1)^k [M_0^{2k} c_0 - M_1^{2k} c_1 + M_2^{2k} c_2 - M_4^{2k} c_4 + M_6^{2k} c_6 - \cdots ] ,$$

the series terminating when the subscript is $2k - 2$.

TABLE 2

A115627

| $k/p$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | | | | | | | | | | |
| 3 | 1 | 1 | | | | | | | | | |
| 4 | 3 | 1 | | | | | | | | | |
| 5 | 3 | 1 | 1 | | | | | | | | |
| 6 | 4 | 2 | 1 | | | | | | | | |
| 7 | 4 | 2 | 1 | 1 | | | | | | | |
| 8 | 7 | 2 | 1 | 1 | | | | | | | |
| 9 | 7 | 4 | 1 | 1 | | | | | | | |
| 10 | 8 | 4 | 2 | 1 | | | | | | | |
| 11 | 8 | 4 | 2 | 1 | 1 | | | | | | |
| 12 | 10 | 5 | 2 | 1 | 1 | | | | | | |
| 13 | 10 | 5 | 2 | 1 | 1 | 1 | | | | | |
| 14 | 11 | 5 | 2 | 2 | 1 | 1 | | | | | |
| 15 | 11 | 6 | 3 | 2 | 1 | 1 | | | | | |
| 16 | 15 | 6 | 3 | 2 | 1 | 1 | | | | | |
| 17 | 15 | 6 | 3 | 2 | 1 | 1 | 1 | | | | |
| 18 | 16 | 8 | 3 | 2 | 1 | 1 | 1 | | | | |
| 19 | 16 | 8 | 3 | 2 | 1 | 1 | 1 | 1 | | | |
| 20 | 18 | 8 | 4 | 2 | 1 | 1 | 1 | 1 | | | |
| 21 | 18 | 9 | 4 | 3 | 1 | 1 | 1 | 1 | | | |
| 22 | 19 | 9 | 4 | 3 | 2 | 1 | 1 | 1 | | | |
| 23 | 19 | 9 | 4 | 3 | 2 | 1 | 1 | 1 | 1 | | |
| 24 | 22 | 10 | 4 | 3 | 2 | 1 | 1 | 1 | 1 | | |
| 25 | 22 | 10 | 6 | 3 | 2 | 1 | 1 | 1 | 1 | | |
| 26 | 23 | 10 | 6 | 3 | 2 | 2 | 1 | 1 | 1 | | |
| 27 | 23 | 13 | 6 | 3 | 2 | 2 | 1 | 1 | 1 | | |
| 28 | 25 | 13 | 6 | 4 | 2 | 2 | 1 | 1 | 1 | | |
| 29 | 25 | 13 | 6 | 4 | 2 | 2 | 1 | 1 | 1 | 1 | |
| 30 | 26 | 14 | 7 | 4 | 2 | 2 | 1 | 1 | 1 | 1 | |
| 31 | 26 | 14 | 7 | 4 | 2 | 2 | 1 | 1 | 1 | 1 | 1 |

TABLE 3

A266743

| k/n | 0 | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | | |
| 2 | 2 | 3 | 1 | | | | | | | | |
| 3 | 1 | 2 | 1 | | | | | | | | |
| 4 | 6 | 15 | 10 | 1 | | | | | | | |
| 5 | 2 | 6 | 5 | 1 | | | | | | | |
| 6 | 12 | 42 | 42 | 14 | 1 | | | | | | |
| 7 | 3 | 12 | 14 | 7 | 1 | | | | | | |
| 8 | 10 | 45 | 60 | 42 | 10 | 1 | | | | | |
| 9 | 2 | 10 | 15 | 14 | 5 | 1 | | | | | |
| 10 | 12 | 66 | 110 | 132 | 66 | 22 | 1 | | | | |
| 11 | 2 | 12 | 22 | 33 | 22 | 11 | 1 | | | | |
| 12 | 420 | 2730 | 5460 | 10010 | 8580 | 6006 | 910 | 1 | | | |
| 13 | 60 | 420 | 910 | 2002 | 2145 | 2002 | 455 | 1 | | | |
| 14 | 24 | 180 | 420 | 1092 | 1430 | 1716 | 546 | 2 | 1 | | |
| 15 | 3 | 24 | 60 | 182 | 286 | 429 | 182 | 1 | 1 | | |
| 16 | 90 | 765 | 2040 | 7140 | 13260 | 24310 | 13260 | 102 | 170 | 1 | |
| 17 | 10 | 90 | 255 | 1020 | 2210 | 4862 | 3315 | 34 | 85 | 1 | |
| 18 | 420 | 3990 | 11970 | 54264 | 135660 | 352716 | 293930 | 3876 | 13566 | 266 | 1 |
| 19 | 42 | 420 | 1330 | 6783 | 19380 | 58786 | 58786 | 969 | 4522 | 133 | 1 |
| 20 | 660 | 6930 | 23100 | 131670 | 426360 | 1492260 | 1763580 | 35530 | 213180 | 8778 | 110 |
| 21 | 60 | 660 | 2310 | 14630 | 53295 | 213180 | 293930 | 7106 | 53295 | 2926 | 55 |
| 22 | 360 | 4140 | 15180 | 106260 | 432630 | 1961256 | 3120180 | 89148 | 817190 | 57684 | 1518 |
| 23 | 30 | 360 | 1380 | 10626 | 48070 | 245157 | 445740 | 14858 | 163438 | 14421 | 506 |
| 24 | 3276 | 40950 | 163800 | 1381380 | 6906900 | 39369330 | 81124680 | 3120180 | 40562340 | 4374370 | 197340 |
| 25 | 252 | 3276 | 13650 | 125580 | 690690 | 4374370 | 10140585 | 445740 | 6760390 | 874874 | 49335 |

| k/n | 20 | 22 | 24 |
|---|---|---|---|
| 21 | 1 | | |
| 22 | 46 | 1 | |
| 23 | 23 | 1 | |
| 24 | 12558 | 910 | 1 |
| 25 | 4186 | 455 | 1 |

From these formulas we obtain the values given in Table 4.

As an example let us consider the computation of the seventh Bernoulli number. We shall have,

$$d_{14} = p^{[14/(p-1)]}/p^{14(p)} =$$

$$(2^{14} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)/\ (2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13) = 360;$$

$$c_{14} = (-1)^7\,(24 \cdot 1 - 180 \cdot 1 + 420 \cdot 1 - 1092 \cdot 1 + 1430 \cdot 2 - 1716 \cdot 3 + 546 \cdot 10 - 2 \cdot 1382)\ ,$$
$$= 420\ .$$

From these we thus obtain, $B_7 = 420/360 = 7/6$.

TABLE 4

A2443 A2444 A2443 A2444

| $k$ | $c_k$ | $d_k$ | $k$ | $c_k$ | $d_k$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 16 | 10851 | 1530 |
| [1 | 1 | 2] | 18 | 438670 | 7980 |
| 2 | 1 | 6 | 20 | 7333662 | 13860 |
| 4 | 1 | 30 | 22 | 51270780 | 8280 |
| 6 | 2 | 84 | 24 | 7090922730 | 81900 |
| 8 | 3 | 90 | 26 | 2155381956 | 1512 |
| 10 | 10 | 132 | 28 | 94997844116 | 3480 |
| 12 | 1382 | 5460 | 30 | 68926730208040 | 114576 |
| 14 | 420 | 360 | | | |

7. *The Bernoulli Polynomials of Higher Order.* We have already defined the Bernoulli polynomial of $m$th order and $n$th degree, $B_n^{(m)}(x)$, to be the coefficient of $t^n/n!$ in the development of the function $t^m e^{xt}/(e^t - 1)^m$.

These polynomials satisfy the equation,

$$B_m^{(n+1)}(x) = (1 - m/n)\,B_m^{(n)}(x) + (x - n)\,(m/n)\,B_{m-1}^{(n)}(x)\ .$$

They may be calculated for any positive integral value of the upper or lower index by means of the following equation:

$$B_m^{(n+1)}(x) = (m!/n!)\,\frac{d^{n-m}}{dx^{n-m}}\,\{(x-1)(x-2)\cdots(x-n)\}\ .$$