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From: Oren Patashnik <op@Neon.Stanford.EDU>
To: njas@research.att.com
Subject: Sequences
Message-Id: <CMM.0.88.628116644.op@Neon.Stanford.EDU>
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Philippe Flajolet sent this to sci.math:

> Here is a list of the F_n for n=2..20 as computed by Maple
> [0, 0, 2, 14, 90, 646, 5242, 47622, 479306, 5296790, 63779034, 831283558,
> 11661506218, 175203184374, 2806878055610, 47767457130566, 860568917787402,
> 16362838542699862, 327460573946510746]

This appears, apparently, as your sequence 818, and when halved, as 1871, but Philippe's has more terms than either 818 or 1871. (And, incidentally, 1871 goes four terms past 818. I know I've noticed this discrepancy for several other pairs of sequences---for instance sequence 1729, which is essentially sequence 799 doubled, has lots more terms than 799. But I suppose it's too much of a pain to actually prove that such pairs of sequences are truly related in the way they obviously seem to be to make it worth trying to keep such pairs consistent.)

Anyway, in case you want more information on Philippe's sequence, I've included his full message below.

--Oren

Path: neon!shelby!decwrl!wuarchive!usc!samsung!uunet!mcsun!inria!seti!margaux!flajolet
From: flajolet@margaux.uucp (Philippe Flajolet)
Newsgroups: sci.math
Subject: Combinatorics, Permutations, Complexity
Keywords: generating function, counting, combinatorial search
Message-ID: <319@seti.inria.fr>
Date: 26 Nov 89 14:23:54 GMT
Sender: news@seti.inria.fr
Lines: 119

Dr Ilan Vardi happened to be passing by Inria the other day. He mentioned an interesting problem about ''contiguous'' elements in permutations which has been recently posed by Owen L. Astrachan <ola@cs.duke.edu> on the net, though it has not reached our machine yet. I propose elements of a solution that could be extended to solve the original problem if necessary.

One way of viewing the problem is:
''Given an n x n chessboard, in how many ways can you place n pawns, one per row, one per column in such a way that no two pawns are attacking?''

Say that position i in a permutation s=s_1 s_2 ... s_n is a ''contiguity'' if |s_{i-1}-s_i|=1. Also we may distinguish between Upward (U) and Downward (D) contiguities depending on the sign of s_{i-1}-s_i. The original problem asks for the number of permutations of n having k contiguities (possibly considering also circular permutations).

I shall mostly show how to find the number of perms of n that have 0 contiguity. The rest follows by similar methods (see exercises section). The proof involves: (i) enumerating permutations (of n) with k ''distinguished'' contiguities; (ii) using inclusion-exclusion; (iii) generating function fudges.

1. Let U and D be as above and '*' denote a position in a perm which is not distinguished. I.e., we don't care whether a '*' carries a contiguity or not. For instance

UUD*UU***UUU*DD**DD***

is a possible pattern of a perm of n=26 with k=12 distinguished contiguities. (Sequences UD and DU are forbidden!)
Clearly, there are f_{n,k} such patterns, with generating function

f(z,u)= sum f_{n,k} u^k z^n
= 1/(1-z) [1/(1-2uz^2/(1-uz)(1-z))].

Hint: 1/(1-A)=1+A+A^2+... represents all sequences of A objects.
Use: '*' -> z; D,U -> zu.

2. A perm of n with k distinguished contiguities is obtained by selecting a pattern of k contiguities (cf #1), and filling in the (n-k) *-positions by a permutation of size (n-k). (Hint: Use non standard analysis!) The number is therefore:

D_{n,k}= f_{n,k} (n-k)!

[Such permutations have >=k contiguities altogether out of which k are distinguished, being ''marked'' in some way.]

3. By inclusion exclusion, the number F_n of contiguity--free perms of n is

sum_{k=0}^{n-1} (-1)^k D_{n,k}

Hint: With <c,k>={c choose k}, observe that

<c,0> - <c,1> + <c,2> - <c,3> ...

is 0 or 1 depending on whether c>=1 or c=0. If a perm has "c" contiguities the number of times it appears as a k-distinguished perm is <c,k>. QED

Example.The list of the f_{6,k} is [1,10,32,38,16,2].
Thus F_6 = 1*6!-10*5!+32*4!-38*3!+16*2!-2*1! = 90. [It checks!!!]

4. Using #2, #3,an ordinary (and divergent) generating function of the F_n is obtained from f(z,u) as

F(z) =sum F_n z^n = int_0^infty f(zt,-1/t) e^(-t) t dt.

via the integral form of n!

n! = int_0^infty e^(-t) t^n dt.

Whence:

F(z) = sum_{n>=0} n! z^n (1-z)^n/(1+z)^n.

%%

Here is a list of the F_n for n=2..20 as computed by Maple
[0, 0, 2, 14, 90, 646, 5242, 47622, 479306, 5296790, 63779034, 831283558,
11661506218, 175203184374, 2806878055610, 47767457130566, 860568917787402,
16362838542699862, 327460573946510746]
using a program which checks our maths.

```
F:=proc(n)
        f:=1/(1-z)*1/(1-2*u*z^2/(1-z)/(1-u*z));
        taylor(",z=0,n+2);
        fzu:=convert(map(expand,"),polynom);
        for nn from 2 to n do
                fnk:=coeff(fzu,z,nn);
        map(proc(x) (-1)^degree(x)*GAMMA(nn+1-degree(x))*coeffs(x) end,fnk);
                FF[nn]:=";
        od;
convert(FF,list); sort(");
end;
```

The computation requires 30 seconds on a Sun 3/60. Its complexity seems to be slightly subexponential. :-)

%%%

Exercises.

1. Prove that [We use <a,b> as abbrev for binomial coeffs]:

F_n = sum_{m+k+l=n} m! <m,k> <m-1,l> (-1)^{k+l}.

2. Show that F_n is asymptotic to (n! * e^(-2)).
3. Find the number F_{n,r} of perms of n with exactly r contiguities,

showing that

F_{n,r} = sum_{k>=0} (-1)^k {k+r choose r} D_{n,k+r} .

4. Find an asymptotic form. Show that
the distribution of the number of contiguities in a random
perm of n is asymptotically Poisson(2).

5. How does this generalize to circular perms?
[I have been too lazy to do it!].

6. Show that the F_n are ''P-recursive'' (in the sense or Stanley) or
holonomic (in the sense of Zeilberger).
I.e., there are polynomials p_0, p_1, ..., p_d such that

sum F_{n+j} p_j(n) = 0

7. Deduce that the F_n can be computed on a bignum machine in space
O(1) and time O(n). Compute F_{1000} in a matter of minutes on
your Mac and F_{10000} in a matter of hours on your Sun.

8. Does the exponential generating function of the F_n have a closed form?

Conclusion: A good theorem is worth a dozen optimizing compilers.

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Generalization. Based on a discussion with Dr. Vardi.
Same chess problem on an elliptic curve? What happens if
the curve has complex multiplication?
