Refining the Stern Diatomic Sequence

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Abstract

We refine the celebrated Stern Diatomic Sequence $\{b(n)\}_{n\geq 0}$, in which $b(n)$ is the number of partitions of n into powers of 2 for which each part has multiplicity 1 or 2, by studying the sequence $\{b(n,k)\}_{n,k\geq 0}$, in which $b(n,k)$ counts the partitions of n into powers of 2 in which exactly k parts have multiplicity 2, the remaining parts being of multiplicity 1. We find closed formulas for the $b(n,k)$ as well as for various of their associated generating functions. Relationships with Lucas polynomials and other number theoretic functions are discussed.

1 Introduction

The Stern diatomic sequence, $\{b(n)\}_{n=0}^{\infty}$, which begins as[1]

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, \ldots, \tag{1}$$

has many magical properties, among which are the following.

1. The sequence $\{b(n+1)/b(n)\}_{n\geq 0}$ assumes every positive rational value, each once and only once.

2. For all $n \geq 0$, $b(n)$ and $b(n+1)$ are relatively prime, so that the consecutive ratios are all *reduced* fractions.

3. $b(n)$ is the number of partitions of the integer n into powers of 2, in which no power of 2 is used more than twice.

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[1]Some authors take $b(0) = 0$ and continue as in (1), which complicates the partition connection and the generating function, while having no obvious advantages.

The sequence was discovered by M. A. Stern [6] in 1858, and it has been studied by many authors since. Its relation to partitions into powers of 2 was discussed by Reznick [5]. Calkin and Wilf [1] derived the properties of the sequence from a presentation in the form of a binary tree, with a rational number sprouting at each node.

In [2], Mansour and Bates generalized the Calkin-Wilf tree to a q-tree, and showed that their tree defines a generating function for a refinement of the Stern sequence, namely to a new sequence $\{b(n,k)\}_{n,k\geq 0}$ which counts the partitions of n into powers of 2, no part used more than twice, and exactly k parts being used exactly twice. Here we study further properties of the $b(n,k)$'s, and of their generating polynomials, which we call $B_n(t) = \sum_k b(n,k)t^k$.

We will first, in order to illustrate the method, derive a fairly explicit formula for the original Stern sequence $b(n)$, and second, we'll use the same method to find an explicit expression for $b(n,k)$. We will then discuss which properties of the original Stern sequence carry over to the generalization $\{B_n(t)\}$.

2 A formula for $b(n)$

Theorem 1 *Let $a(n)$ be the sum of the binary digits of n. For each $i = 0, 1, \ldots, 5$ let*

$$s_i = |\{r : 0 \leq r \leq n \text{ and } a(r) - a(n-r) \equiv i \bmod 6\}|.$$

Then $b(n) = s_0 + s_1 - s_3 - s_4$.

We start with the generating function

$$B(x) \stackrel{\text{def}}{=} \sum_{n\geq 0} b(n)x^n = \prod_{j\geq 0}\left(1 + x^{2^j} + x^{2^{j+1}}\right).$$

Now since

$$1 + u + u^2 = (1-\omega u)(1-\overline{\omega}u), \qquad (\omega = e^{\frac{2\pi i}{3}})$$

we have

$$B(x) = \prod_{j\geq 0}(1-\omega x^{2^j})\prod_{\ell\geq 0}(1-\overline{\omega}x^{2^\ell}).$$

However, in view of the uniqueness of the binary expansion we have

$$\prod_{j\geq 0}(1-zx^{2^j}) = \sum_{n\geq 0}(-z)^{a(n)}x^n,$$

from which

$$B(x) = \sum_{r\geq 0}(-\omega)^{a(r)}x^r\sum_{s\geq 0}(-\overline{\omega})^{a(s)}x^s. \tag{2}$$

The coefficient of x^n here is, since $\overline{\omega} = 1/\omega$,

$$\begin{aligned}
\sum_{r+s=n} (-\omega)^{a(r)-a(s)} &= \sum_{j=0}^{5} s_j(-\omega)^j = s_0 - s_1\omega + s_2\omega^2 - s_3\omega^3 + s_4\omega^4 - s_5\omega^5 \\
&= (s_0 - s_3) + (s_4 - s_1)\omega + (s_2 - s_5)\omega^5 \qquad (\text{since } \omega^3 = 1) \\
&= (s_0 - s_3) + (s_4 - s_1)(\omega + \overline{\omega}) = s_0 + s_1 - s_3 - s_4,
\end{aligned}$$

as required. □

Theorem 1 was first proved by Northshield [3], where it appears as Proposition 4.4. We remark that Theorem 1, when regarded as a method of computing the sequence $\{b(n)_{n\geq 0}\}\}$, is distinctly inferior to the well known divide-and-conquer recurrence for the sequence, which is eq. (8) below with $t := 1$.

3 A formula for $b(n, k)$

We now will find a formula, that appears to be new, for $b(n, k)$, the number of partitions of n into powers of 2, which have exactly k different powers of 2 of multiplicity 2, the remaining parts being of multiplicity 1. A short table of this sequence is shown below.

$k\backslash n$	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1	0	0	1	0	2	1	1	0	3
2	0	0	0	0	0	0	1	0	0

Theorem 2 *For $b(n, k)$, the number of partitions of n into powers of 2 which have exactly k different powers of 2 of multiplicity 2, the remaining parts being of multiplicity 1, we have the formula*

$$b(n, k) = \sum_{\ell+s=n} \phi(k - a(\ell), a(\ell) - a(s)),$$

in which ϕ is given by (5) below, and $a(m)$ is the sum of the binary digits of m.

The generating function is obviously

$$B(x, t) \stackrel{\text{def}}{=} \sum_{n,k\geq 0} b(n, k)x^n t^k = \prod_{j\geq 0}(1 + x^{2^j} + tx^{2^{j+1}}), \tag{3}$$

and we will treat it in the same way that the result of the previous section was obtained.

First,

$$1 + u + tu^2 = \left(1 - \frac{u}{r_+}\right)\left(1 - \frac{u}{r_-}\right),$$

in which

$$r_\pm = r_\pm(t) = \frac{-1 \pm \sqrt{1-4t}}{2t}. \tag{4}$$

Therefore, since $r_- r_+ = 1/t$,

$$B(x,t) = \prod_{j \geq 0} (1 - tr_+ x^{2^j})(1 - tr_- x^{2^j}),$$

from which, as in (2) above, we obtain

$$B(x,t) = \sum_{\ell \geq 0} (-tr_+)^{a(\ell)} x^\ell \sum_{m \geq 0} (-tr_-)^{a(m)} x^m.$$

Thus the coefficient of x^n is

$$\sum_{\ell+m=n} t^{a(\ell)} (-r_+)^{a(\ell)-a(m)}.$$

Next we have the well known series

$$\left(\frac{1-\sqrt{1-4t}}{2}\right)^m = \sum_{n \geq 0} \phi(n,m) t^{n+m},$$

where, for all integer m, we have

$$\phi(n,m) = \begin{cases} 0, & \text{if } \ n < 0; \\ 1, & \text{if } m = n = 0; \\ \frac{m(n+m+1)^{\overline{n-1}}}{n!}, & \text{otherwise,} \end{cases} \tag{5}$$

in which $z^{\overline{r}}$ is the rising factorial.

It follows that

$$\begin{aligned} b(n,k) &= [t^k] \sum_{\ell+s=n} (-r_+)^{a(\ell)-a(s)} t^{a(\ell)} \\ &= [t^k] \sum_{\ell+s=n} \sum_{\mu \geq 0} \phi(\mu, a(\ell)-a(s)) t^{\mu+a(\ell)} \\ &= \sum_{\ell+s=n} \phi(k-a(\ell), a(\ell)-a(s)), \end{aligned} \tag{6}$$

as required. □

4 Some interesting polynomials

Now we study the polynomials

$$B_n(t) = \sum_{k\geq 0} b(n,k)t^k. \qquad (n = 0, 1, 2, 3, \dots)$$

These series, of course, terminate, and in fact it is easy to see that $B_n(t)$ is of degree at most $\lfloor \log(1 + n/2) \rfloor$.

If we multiply (6) by t^k and sum over $k \geq 0$ we find a quite explicit formula, viz.

$$B_n(t) = \sum_{\ell=0}^{n} t^{a(\ell)} \left(\frac{1+\sqrt{1-4t}}{2}\right)^{a(n-\ell)-a(\ell)}. \tag{7}$$

in which $a(m)$ is, as before, the sum of the bits of m.

For example,

$$\begin{aligned} B_4(t) &= \sum_{\ell=0}^{n} t^{a(\ell)} \left(\frac{1+\sqrt{1-4t}}{2}\right)^{a(n-\ell)-a(\ell)} \\ &= t^0 \left(\frac{1+\sqrt{1-4t}}{2}\right) + t\left(\frac{1+\sqrt{1-4t}}{2}\right) + t + t^2 \left(\frac{1+\sqrt{1-4t}}{2}\right)^{-1} + t\left(\frac{1+\sqrt{1-4t}}{2}\right)^{-1} \\ &= 1 + 2t, \end{aligned}$$

In the form (7) it might not be obvious that $B_n(t)$ is a polynomial, but note that the right-hand side of equation (7) is invariant under the automorphism defined by $\sqrt{1-4t} \mapsto -\sqrt{1-4t}$. Another representation of $B_n(t)$, which clearly shows its polynomial character, is given by Theorem 4 below.

5 A divide-and-conquer recurrence for the polynomials

The generating function $B(x,t)$ in (3) evidently satisfies

$$(1 + x + tx^2)B(x^2, t) = B(x, t),$$

which is to say that

$$(1 + x + tx^2)\sum_n B_n(t)x^{2n} = \sum_n B_n(t)x^n.$$

If we match the coefficients of like powers of x on both sides, we find the recurrence

$$\begin{aligned} B_{2n}(t) &= B_n(t) + tB_{n-1}(t), \quad &(n = 1, 2, 3, \dots) \\ B_{2n+1}(t) &= B_n(t), &(n = 0, 1, 2, \dots) \end{aligned} \tag{8}$$

with $B_0(t) = 1$. This sequence of polynomials[2] begins as

$$1\,,1\,,t+1\,,1\,,2t+1\,,t+1\,,t^2+t+1\,,1\,,3t+1\,,2t+1\,,2t^2+2t+1\,,t+1\,,2t^2+2t+1\,,\ldots.$$

From the recurrence (8) we obtain recurrences for the coefficients $b(n,k)$, namely

$$\begin{aligned} b(2n,k) &= b(n,k)+b(n-1,k-1), \qquad (n\geq 1; k\geq 1) && (9)\\ b(2n+1,k) &= b(n,k). \qquad (n\geq 0; k\geq 0) && (10)\end{aligned}$$

Now define the "vertical" generating functions $C_k(x) = \sum_{n\geq 0} b(n,k)x^n$. Multiply (9) by x^{2n} and sum over $n \geq 1$, then add to the result of multiplying (10) by x^{2n+1} and summing over $n \geq 0$. The result is that

$$C_k(x) = (1+x)C_k(x^2) + x^2C_{k-1}(x^2). \qquad (k\geq 1; C_0(x) = 1/(1-x)) \tag{11}$$

Now in general, the solution of the functional equation

$$G(x) = (1+x)G(x^2) + F(x) \qquad (F(0) = 0)$$

is seen, by iteration, to be

$$G(x) = \frac{1}{1-x}\sum_{j\geq 0}(1-x^{2^j})F(x^{2^j}).$$

If we apply this to the present case (11), we find a recurrence formula for the vertical generating functions $\{C_k(x)\}_{k=0}^{\infty}$, viz.

$$C_k(x) = \sum_{j\geq 0}\frac{1-x^{2^j}}{1-x}x^{2^{j+1}}C_{k-1}(x^{2^j+1}). \qquad (k\geq 1;\ C_0 = 1/(1-x)) \tag{12}$$

With $k = 1$ we obtain

$$C_1(x) = \frac{1}{1-x}\sum_{j=1}^{\infty}\frac{x^{2^j}}{1+x^{2^{j-1}}}. \tag{13}$$

The coefficient of x^n in the power series expansion of $C_1(x)$, i.e., $b(n,1)$, *is the number of 0's in the binary expansion of* n (see [4], sequence #`A080791`) We state this as

Theorem 3 *The number of partitions of n into powers of 2 in which exactly one part has multiplicity 2, the others being of multiplicity 1, is equal to the number of 0 digits in the binary expansion of n.*

[2]These are the $f(n;q)$ of [2].

Proof #1. (Recurrence) In eq. (11) put $k = 1$, and compare the coefficients of x^n on both sides, to obtain

$$b(n, 1) = b(n/2, 1) + b((n - 1)/2, 1) + 1,$$

with the usual convention that $b(u, 1) = 0$ unless u is an integer. If $z(n)$ is the number of 0 digits in the binary expansion of n, it is easy to check that $z(n)$ satisfies this same recurrence with the same initial values. □

Proof #2. (Bijection) Suppose that 2^j does not appear in the binary expansion of $n > 2^j$. Let k be the least integer for which $k > j$ and 2^k does appear. Then in the binary expansion of n, replace 2^k with $2^j + 2^j + 2^{j+1} + 2^{j+2} + ... + 2^{k-1}$. □

An alternative method for computing $C_1(x)$ is to use the generating function

$$B(x, t) \stackrel{\text{def}}{=} \sum_{n\geq 0} B_n(t)x^n = \prod_{j\geq 0} \left(1 + x^{2^j} + tx^{2^{j+1}}\right).$$

Applying $\frac{\partial}{\partial t}$ and setting $t = 0$ yields equation (13). Similarly, applying $\frac{1}{2}\frac{\partial^2}{\partial t^2}$ and setting $t = 0$ yields

$$C_2(x) = \frac{1}{2(1-x)} \left[\left(\sum_{i\geq 0} \frac{x^{2^{i+1}}}{1 + x^{2^i}} \right)^2 - \sum_{i\geq 0} \frac{x^{2^{i+2}}}{\left(1 + x^{2^i}\right)^2} \right].$$

Similar but more complicated formulas clearly exist for every $C_k(x)$.

As a final remark, note that if we put $t = 1$ in (7) we recover another proof of Theorem 1.

6 Lucas polynomials

The *Lucas polynomials* $L_n(t)$ can be defined as

$$L_n(-t) = u_+^n + u_-^n. \tag{14}$$

For instance,

$$\begin{aligned}
L_1(t) &= 1 \\
L_2(t) &= 1 + 2t \\
L_3(t) &= 1 + 3t \\
L_4(t) &= 1 + 4t + 2t^2 \\
L_5(t) &= 1 + 5t + 5t^3 \\
L_6(t) &= 1 + 6t + 9t^2 + 2t^3 \\
L_7(t) &= 1 + 7t + 14t^2 + 7t^3 \\
L_8(t) &= 1 + 8t + 20t^2 + 16t^3 + 2t^4.
\end{aligned}$$

In particular, $L_n(1) = L_n$, a Lucas number. The Lucas polynomials have many equivalent definitions. For instance, the coefficient of t^k in $L_n(x)$ is the number of k-edge matchings of an n-cycle. We also have the recurrence $L_{n+1}(t) = L_n(t) + tL_{n-1}(t)$, $n \geq 2$.

Equation (14) allows us to write a formula for $B_n(t)$ in terms of the Lucas polynomials. We write $M_n(t) = L_n(-t)$ and omit the argument t for simplicity.

Theorem 4 *We have*

$$B_n = \begin{cases} M_{a(n)} + \displaystyle\sum_{k=1}^{(n-1)/2} \left(M_{a(k)} M_{a(n-k)} - M_{a(k)+a(n-k)}\right), & n \text{ odd} \\ M_{a(n)} + \displaystyle\sum_{k=1}^{(n-2)/2} \left(M_{a(k)} M_{a(n-k)} - M_{a(k)+a(n-k)}\right) & \\ \quad + \frac{1}{2}\left(M_{a(n/2)}^2 - M_{2a(n/2)}\right), & n \text{ even.} \end{cases}$$

For instance,

$$\begin{aligned} B_6 &= M_2 + (M_2 M_1 - M_3) + (M_1^2 - M_2) + \frac{1}{2}(M_2^2 - M_4) \\ &= (1 - 2t) + t + 2t + \frac{1}{2}(2t^2) \\ &= 1 + t + t^2. \end{aligned}$$

Proof of Theorem 4. Note that

$$a^i b^j + a^j b^i = (a^i + b^i)(a^j + b^j) - (a^{i+j} + b^{i+j}). \tag{15}$$

In particular $a^i b^i = \frac{1}{2}\left((a^i + b^i)^2 - (a^{2i} + b^{2i})\right)$. If we collect the terms of the rightmost sum of equation (7) indexed by ℓ and $n - \ell$, apply the identity (15), and use equation (14), the proof follows. $\square$

Note that Theorem 4 expresses $B_n(t)$ as a nonnegative integer linear combination of the polynomials M_k, $M_i M_j - M_{i+j}$ $(i \neq j)$, and $\frac{1}{2}(M_k^2 - M_{2k})$. If we set $t = 1$ in Theorem 4 then we get a formula for $b(n)$. The sequence $M_1(1)$, $M_2(1)$, $\ldots$ is periodic of period 6, and the resulting formula for $b(n)$ is equivalent to Theorem 1.

7 Generalizations of the tree of fractions

The Stern sequence, as we remarked earlier, has the property that the ratios b_n/b_{n+1} of consecutive members run through the set of all positive reduced rational numbers, each occurring just once. In this section we look at the extent to which this property generalizes. That is, for fixed integer $t \geq 1$, what can be said about the set $S(t)$ of values that are assumed by the ratios $\{B_n(t)/B_{n+1}(t)\}_{n\geq 0}$ of consecutive members?

7.1 Which fractions appear in the sequence?

First, since all $B_n(t)$ are $\equiv 1 \bmod t$, $S(t)$ can contain only fractions p/q for which $p \equiv q \equiv 1 \bmod t$. But it can't contain all of these, for, according to the recurrence (8), we have

$$\frac{B_{2n}(t)}{B_{2n+1}(t)} = 1 + t\frac{B_{n-1}(t)}{B_n(t)} > 1,$$

and

$$\frac{B_{2n+1}(t)}{B_{2n+2}(t)} = \frac{B_n(t)}{B_{n+1}(t) + tB_n(t)} < 1/t.$$

Thus, if $t > 1$, the ratios of consecutive polynomials can assume only rational values p/q such that $p \equiv q \equiv 1 \bmod t$, and p/q lies outside of the interval $[1/t, 1]$. We can, however, recover some of the structure of the Stern case $t = 1$, in general.

7.2 Are consecutive values relatively prime?

Theorem 5 *For fixed integer $t \geq 1$ and for all $n \geq 0$, $B_n(t)$ and $B_{n+1}(t)$ are relatively prime.*

Proof. From the recurrence (8) we have that any common divisor d of $B_{2n}(t)$ and $B_{2n+1}(t)$ must divide $B_n(t)$ and $tB_{n-1}(t)$. But d must be relatively prime to t, for if p is a prime that divides d and t, then in view of

$$B_n(t) = 1 + \sum_{k\geq 1} b(n,k)t^k,$$

the left side is 0 mod p and right side is 1 mod p, a contradiction. Thus $\gcd(B_{2n}(t), B_{2n+1}(t)) = \gcd(B_n(t), B_{n-1}(t))$.

On the other hand, the recurrences

$$\begin{aligned} B_{2n}(t) &= B_n(t) + tB_{n-1}(t), \quad &(n = 1, 2, 3, \dots) \\ B_{2n-1}(t) &= B_{n-1}(t), \quad &(n = 0, 1, 2, \dots) \end{aligned} \tag{16}$$

imply that every common divisor d of $B_{2n}(t)$ and $B_{2n-1}(t)$ must divide $B_{n-1}(t)$ and $B_n(t)$ and conversely. Thus $\gcd(B_{2n}(t), B_{2n-1}(t)) = \gcd(B_n(t), B_{n-1}(t))$. Therefore the gcd of every consecutive pair is 1. □

7.3 Can a pair of consecutive values recur?

We can also recover the uniqueness of each ordered pair of consecutive values.

Theorem 6 *Each reduced positive rational number p/q can occur at most once in the sequence $\{B_n(t)/B_{n+1}(t)\}_{n=0}^{\infty}$.*

Proof. Suppose that p/q occurs twice in the sequence. Say, to fix ideas, that

$$\frac{B_{2j}(t)}{B_{2j+1}(t)} = \frac{p}{q} = \frac{B_{2i}(t)}{B_{2i+1}(t)},$$

with $j < i$ and j chosen to be minimal among all such indices. Then from the recurrence (8) we have

$$B_j(t) + tB_{j-1}(t) = p; B_j(t) = q; B_i(t) + tB_{i-1}(t) = p; B_i(t) = q.$$

Therefore t divides $p - q$ and

$$B_{j-1}(t) = (p - q)/t; B_j(t) = q; B_{i-1}(t) = (p - q)/t; B_i(t) = q.$$

Hence the ratio of consecutive members is also repeated not only at the indices $2j$ and $2i$, but also at $j - 1$ and $i - 1$, which contradicts the minimality of the chosen j. $\square$

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