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Reprinted from The American Mathematical Monthly
Vol. XXXVI, No. 2, February, 1929

ON STERN'S DIATOMIC SERIES

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The purpose of the present paper is to extend the investigation of the following "Diatomic Series" studied by Stern.[1]

0	1																1
1	1								2								1
2	1				3				2				3				1
3	1		4		3		5		2		5		3		4		1
4	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
	.		.		.		.		.		.		.		.		.
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

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Each line of this table is formed from the preceeding one by inserting between consecutive elements their sum. The lines of this table, of which there are infinitely many, are numbered as indicated on the left.

Stern has proved a number of interesting facts concerning this series among which are the following:

1. The number of terms in the n-th line is 2^n+1, and their sum is 3^n+1.

2. The number of terms in the table down to and including the n-th line is $2^{n+1}+n$ and their sum is $\frac{1}{2}(3^{n+1}+1)+n$.

[1] Journal für Mathematik, vol. 55, page 193.

3. The average value of the terms in the n-th line is nearly $(3/2)^n$ which is twice the average value of the terms in the whole table down to and including the n-th line.

4. The table is symmetric: the k-th term on the nth line is equal to the (2^n+2-k)th term.

5. The terms which appear in the n-th line as the sums of their two adjacent terms are called *dyads* of the n-th order. There are 2^{n-1} dyads of the n-th order and $2^{n-1}+1$ non-dyads. The dyads occupy the positions of even rank in the line.

6. In the sequence of terms (a, b, c), $(a+c)/b$ is an integer.

7. If the sequence (a, b, c) appears in the n-th line the dyad b occurs in the $(n-k)$th line where $k=(a+c-b)/2b$.

8. Two consecutive terms have no common factor.

9. The sequence (a, b) can occur but once in the table.

10. If a and b are relatively prime the sequence (a, b) appears in the line whose number is one less than the sum of the quotients appearing in the expansion of a/b in a regular continued fraction.

11. The number k cannot appear as a dyad in the n-th line if $n \geqq k$.

12. The number of times an element k appears in the $(k-1)$st and all succeeding lines is Euler's $\phi(k)$.

13. The number p is a prime if and only if it appears $(p-1)$ times in the $(p-1)$st line.

We proceed to a closer study of the series. A term of the series is determined by its line and its rank in that line. We shall exhibit an algorithm for finding the value of the term in the k-th line and of rank R.

Theorem 1. *If a number has the rank R_n in the n-th line it appears directly below in the $(n+k)$th line with the rank:*

$$R_{n+k} = 2^k(R_n - 1) + 1. \tag{1}$$

In fact this formula is easily seen to satisfy the necessary recurrence: $R_{n+k} = 2R_{n+k-1} - 1$, and to have the proper initial value for $k=0$. If R_n is equal to one, we see that the rank of one is always one. It is easy to see that the number m appears in the $(m-1)$st line as a dyad of rank 2. Applying the preceding theorem with $R=2$ we see that the rank of m in the n-th line is $2^{n-m+1}+1$. This being true for all values of $m \leqq n+1$, it follows that the $(m-1)$st line contains all the natural numbers from 1 to m in descending order, the integer l appearing at the rank $2^{m-l}+1$. Thus the 4-th line contains the numbers 5, 4, 3, 2, 1 with the ranks of 2, 3, 5, 9, 17 respectively.

If r_1 and r_2 are consecutive entries in the n-th line and $r_1 > r_2$, then r_1 is a dyad. In the line above there are the two entries (r_1-r_2, r_2). If $r_1-r_2 > r_2$, r_2 is again a non-dyad and in the $(n-2)$nd line there will appear the sequence (r_1-2r_2, r_2). But if on the other hand $r_1-r_2 < r_2$, then r_2 is a dyad and the $(n-2)$nd line will reveal the sequence $(r_1-r_2, 2r_2-r_1)$.

In fact if

$$
\begin{aligned}
r_1 &= q_1 r_2 + r_3, \\
r_2 &= q_2 r_3 + r_4, \\
r_3 &= q_3 r_4 + r_5, \\
&\cdot\ \cdot\ \cdot\ \cdot \\
&\cdot\ \cdot\ \cdot\ \cdot \\
r_{m-2} &= q_{m-2} r_{m-1} + r_m,
\end{aligned} \tag{2}
$$

we pass upwards q_1 lines to the top of the column of r_2 where the sequence (r_3, r_2) occurs. Taking the column of non-dyads r_3 we move upwards q_2 lines to the sequence (r_3, r_4) and so on until we at last reach the sequence (r_{m-1}, r_m) or else (r_m, r_{m-1}) according as m is even or odd. But r_1 and r_2, being consecutive terms, are prime to each other so that we at length run into the edge of the table at $(r_{m-1}, 1)$ or $(1, r_{m-1})$.

From the above equations we can write r_1/r_2 as a continued fraction:

$$r_1/r_2 = [q_1, q_2, q_3, \cdots, q_{m-2}, r_{m-1}].$$

Since the sequences $(r_{m-1}, 1)$ or $(1, r_{m-1})$ are on the $(r_{m-1}-1)$st line, it follows that (r_1, r_2) occurs on the n-th line where

$$n = q_1 + q_2 + q_3 + \cdots r_{m-1} - 1,$$

which is in fact Stern's result 10.

If instead of choosing a sequence (r_1, r_2) we select values of $q_1, q_2, q_3 \cdots r_{m-1}$ such that their sum is $n+1$, and calculate the corresponding continued fraction we will get a sequence (r_1, r_2) on the n-th line; for the equations (2) may be solved backwards for the r_i knowing the q_i.

Since there are 2^n+1 terms in the n-th line we can form 2^n fractions r_1/r_2. Since the table is symmetric for every fraction r_1/r_2 we have a corresponding fraction r_2/r_1. The quotients in the expansion of these two fractions are identical, except that in the case of the proper fraction, the set of quotients is preceded by zero. If we change the complete quotient $1/(r_{m-2})$, which is never unity, to $1/[r_{m-1}-1+(1/1)]$ in the expansion of the proper fraction and disregard the zero quotient q_0, the number of quotients and their sum will remain unchanged. Since the sequence (r_1, r_2) occurs but once in the table we have by this device a set of quotients corresponding to each fraction in the line with no two sets identical. The number of these sets is 2^n which gives at once the proof of the theorem in the theory of partitions that the number of ways of expressing $n+1$ as the sum of positive integers is 2^n, sums differing only in the order of their terms being counted as distinct.

If r_1/r_2 expands with an odd (even) number of quotients, r_2/r_1 has an even (odd) number of quotients. In other words for every expression of $n+1$ as a sum of an odd number of integers, there corresponds one and only one set of even number of integers whose sum is $n+1$. This establishes the theorem that the numbers of ways of expressing $n+1$ as a sum of an odd or even number of integers are the same and are equal to 2^{n-1}.

By finding all the representations of $n+1$ as the sum of integers we can calculate all the sequences (r_1, r_2) on the n-th line. The question now arises how to distribute these sequences on the line. In a particular case the distribution can be effected by tentative methods using the facts 4, 5, 6 and others. But we shall develop a formula which assigns to any number (r_1) in the sequence (r_1, r_2) a definite rank. A given sequence (r_1, r_2) appears on the left or right side of the table according as the number of quotients in the expansion of r_1/r_2 is odd or even. We can always suppose that the given sequence (r_1, r_2) is such that the continued fraction for r_1/r_2 has an even number of quotients and is such that $r_1 > r_2$.

For if r_1/r_2 has an odd number of quotients and is thus on the right side of the table, the fraction r_2/r_1 on the left has an even number. If $r_2 > r_1$ then the adjacent sequence $(r_2, r_2 - r_1)$ can be taken in lieu of the given sequence. When we find the rank of r_1 on the left side, the rank of r_1 on the right can be easily calculated. For example if we wish to find the position of 85 in the sequence (85, 16) we expand the continued fraction $85/16 = [5, 3, 5]$. Thus (85, 16) is on the 12th line on the right hand side. If we know the rank of 85 in (16, 85) which is on the left side of the table we can answer our question by subtracting this rank from $2^{12}+2$. But (16, 85) is followed by $(85, 85-16)$ or (85, 69). We find that[1] $85/69 = [1, 4, 3, 5]$ which has in fact an even number of quotients. Referring to equations (2), the rank of a dyad r_{m-1} in the $(r_{m-1}-1)$st line is 2. According to our Theorem 1, the rank of r_{m-1}, q_{m-2} lines down is $2^{q_{m-2}}+1$ and that of its right hand neighbour r_{m-2} is $2^{q_{m-2}}$. Then q_{m-3} lines farther down the number r_{m-2} has the rank $2^{q_{m-3}}(2^{q_{m-2}}-1)+1$, and its right neighbour r_{m-3} has the rank

$$2^{q_{m-3}+q_{m-2}} - 2^{q_{m-3}} + 2.$$

Similarily the rank of r_{m-4}, q_{m-4} lines farther down is

$$2^{q_{m-4}+q_{m-3}+q_{m-2}} - 2^{q_{m-4}+q_{m-3}} + 2^{q_{m-4}}.$$

Finally the rank of r_1 on the right side of the line n is

$$R = 2^{q_1+q_2+q_3+\cdots+q_{m-2}} - 2^{q_1+q_2+\cdots+q_{m-3}} + \cdots - 2^{q_1+q_2} + 2^{q_1}. \tag{3}$$

We shall next consider the problem of finding the number r_1 which has a given rank R in a given line n. If $R > 2^{n-1}$ we can consider instead the corresponding rank 2^n+2-R which is less than 2^{n-1}. If R is odd, the number r_1 is a non-dyad in the n-th line and we can follow r_1 back to the $(n-k)$th line where it becomes a dyad, and hence has an even rank. By Theorem 1, k is the largest power of 2 in $R-1$ and the rank of r_1 in the $(n-k)$th line is $R_1 = (R-1)2^{-k}+1$. Thus we have only to consider the cases in which R is even and less than 2^{n-1}. Let now r_2 be the left neighbour of r_1 and let

$$r_1/r_2 = [q_1, q_2\ \ q_3 \cdots q_{n-2}, r_{m-1}]$$

[1] The labor of making the second expansion is obviated by noting that if $r_1/r_2 = [q_1 q_2 q_3 \cdots]$, then $r_1/(r_1-r_2) = [1, q_1-1, q_2 \cdots]$ or $[q_2+1, q_3, q_4 \cdots]$, according as $r_1 > 2r_2$ or $r_1 < 2r_2$.

The rank of r_1 is given by equation (3). We have only to determine the quotients q_i. The first quotient q_1 is simply the highest power of 2 in the given R, q_2 is the highest power of 2 in $R\cdot 2^{-q_1}-1$, q_3 is the highest power of 2 in $(R\cdot 2^{-q_1}-1)2^{-q_2}+1$, and so on.

In this way we can determine the successive partial quotients in the expansion of r_1/r_2. Thus far we have not taken into account the number of the line n. The complete quotient r_{m-1} is determined as the difference between $n+1$ and the sum of the partial quotients. Thus we see that the complete quotient r_{m-1} is a function of the line n for a given rank R. It follows from the theory of continued fractions that the numbers which occupy the same rank in successive lines are in arithmetical progression whose common difference is the numerator of the penultimate convergent. The value of r_1 for a given line can now be readily computed from the continued fraction expansion of r_1/r_2.

Example: What number on the hundredth line has the rank of one million?

$$
\begin{array}{llll}
2^{q_1}(2^{q_2+q_3+\cdots q_{m-2}} & -\cdots+1) = 1000000 & & q_1 = 6\\
2^{q_2}(2^{q_3+\cdots q_{m-2}} & -\cdots-1) = 15624 & & q_2 = 3\\
2^{q_3}(2^{q_4+\cdots q_{m-2}} & -\cdots+1) = 1954 & & q_3 = 1\\
2^{q_4}(2^{q_5+\cdots q_{m-2}} & -\cdots-1) = 976 & & q_4 = 4\\
2^{q_5}(2^{q_6+\cdots q_{m-2}} & -\cdots+1) = 62 & & q_5 = 1\\
2^{q_6}(2^{q_7+\cdots q_{m-2}} & -\cdots-1) = 30 & & q_6 = 1\\
2^{q_7}(& \cdots\cdots\cdots+1) = 16 & & \underline{q_7 = 4}\\
 & & & \sum_{i=1}^{7} q_i = 20
\end{array}
$$

$$r_{m-1} = r_8 = 101 - 20 = 81, \qquad r_1/r_2 = [6,3,1,4,1,1,4,81].$$

Calculating the value of r_1/r_2, we have $r_1 = 97139$.

In the preceding discussion we have found answers to our problems in terms of certain algorithms. We shall pass on to the consideration of series of dyads whose values are given by a definite formula.

Theorem 2. *If the sequence* (r_1, r_2) *occurs in the n-th line with* $r_1 > r_2$ *and the rank of* r_1 *being R, the smallest dyad occuring between the terms* r_1 *and* r_2 *on the line* $n+k$ *is* r_1+kr_2 *and its rank is* 2^kR.

Consider the portion of the table in which we are interested.

n	r_1								
$n+1$	r_1				r_1+r_2				r_2
$n+2$	r_1		$2r_1+r_2$		r_1+r_2		r_1+2r_2		r_2
$n+3$	r_1	$3r_1+r_2$	$2r_1+r_2$	$3r_1+2r_2$	r_1+r_2	$2r_1+3r_2$	r_1+2r_2	r_1+3r_2	r_2

In the line $n+2$, the smallest dyad between r_1 and r_2 is r_1+2r_2, since $r_1 > r_2$.

Applying the same reasoning to the sequence (r_1+r_2, r_2), the smallest dyad on the $(n+3)$rd line is r_1+3r_2, etc. The rank of each minimum dyad at each step is one less than that of r_2. By Theorem 1, after k moves, the rank of r_1+kr_2 is 2^kR, which is the theorem.

In general if (r_1, r_1+r_2, r_2) be any three consecutive numbers in the $(n+1)$st line, the largest dyad in the $(n+2)$nd line between r_1 and r_2 is obtained by starting from the dyad r_1+r_2 and moving down one line and towards its largest neighbour. Thus if $r_1>r_2$, we would move to the dyad $2r_1+r_2$. Since $r+r_2>r_1$ we would next move down to the right to the dyad $3r_1+2r_2$.

We shall define by a zig-zag move one which is continually descending and changing its direction from left to right and from right to left etc. at each line. A right (left) zig-zag move starts down towards the right (left). Continually applying the above reasoning we have the theorem:

Theorem 3. *If the sequence (r_1, r_2) for $r_1>r_2$ appears in the n-th line the right zig-zag move passes over dyads which are greater than any other elements between r_1 and r_2 in any line.*

Corollary. *In any line $n>1$ there are two equal terms which are larger than all the other terms on the line.*

This follows at once from considering the sequence (1, 3, 2) and the symmetry of the table. We shall return to these maximum dyads later. We next consider the rank of any dyad after k steps of a right zig-zag move starting with a term of rank R_0. The first step brings us down towards the right to r_1+r_2, a dyad of rank $R_1=2R_0$ by Theorem 1. The next move takes us down to the left to a dyad of rank $R_2=2R_1-2$, and after k steps we stop on the dyad in question of rank:

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(4) $$R_k = 2R_{k-1} - [1 + (-1)^k].$$ A128209

The solution of this difference equation gives us:

$$R_k = \tfrac{1}{3}\{2^k + 2 + [1 + (-1)^{k+1}]\} + 2^k(R_0 - 1).$$

It is easy to verify that this solution satisfies the required recurrence (4) and that for $k=0$ it has the proper value R_0. For a left zig-zag move the recurrence is:

$$R_k = 2R_{k-1} - [1 + (-1)^{k+1}],$$

the solution of which is seen to be

$$R_k = \tfrac{1}{3}\{-2^k + 4 - [1 + (-1)^{k+1}]\} + 2^k(R_0 - 1).$$

These zig-zag moves have another important property namely: the dyads passed over are such that any one is equal to the sum of the preceding two dyads. For in the above diagram

$$3r_1 + 2r_2 = (2r_1 + r_2) + (r_1 + r_2) \text{ for the left move,}$$

and

$$2r_1 + 3r_2 = (r_1 + 2r_2) + (r_1 + r_2) \text{ for the right move.}$$

The property is proved by induction.

We have then to deal with sets of dyads which satisfy the difference equation:

$$W_{n+2} = W_{n+1} + W_n.$$

The general theory of the recurring series of the second order has been considered at length by Lucas.[1] The series W_n are determined when one assigns definite values to W_0 and W_1. If $W_0=0$. $W_1=1$ the series W_n is the celebrated Fibonacci or Pisano series:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610,\ \cdots$$

and the value of the n-th term is:

$$U_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}\,.$$

The series is fundamental in our discussion. If W_0 and W_1 assume other values than (0, 1) it can be shown that

$$W_n = W_1U_n + W_0U_{n-1},$$

where U_n is the n-th term of the Fibonacci series. Making use of this fact we can write down the value of the dyad occuring at the end of k steps of a right or left zig-zag move. It is only necessary to know the first two dyads W_0 and W_1.

The maximum dyad on each line deserves special attention. Taking the lines $n=0$, 1 we have for $W_0=1$, $W_1=2$,

$$W_n = 2U_n + U_{n-1} = U_{n+2}.$$

Hence we can state:

Theorem 4: *In any line n the largest dyads have the common value*

$$U_{n+2} = [(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}]/2^{n+2}\sqrt{5}\ ;$$

and their ranks are

$$R_n = \tfrac{1}{3}(2^r + 2 + [1 + (-1)^{n+1}])$$

and

$$2^n + 2 - R_n = \tfrac{1}{3}(2^{n+1} + 4 - [1 + (-1)^{n+1}]).$$

Again if we start with one in the second line and make a right zig-zag move we pass over the dyads 1, 3, 4, 7, 11, 18, $\cdots$. Here $W_0=1$, $W_1=3$. $W_n=3U_n + U_{n-1}$. Substituting the expressions derived for U_n and U_{n-1} we have

$$W_n = 3\left[\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}\right] + \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^{n+1} + (1-\sqrt{5})^{n+1}}{2^{n+1}}\,.$$

[1] American Journal of Mathematics, vol. 1, pp. 184–240, 289–321.

These numbers are what Lucas terms V_n. We can state the following theorem:

Theorem 5: *In any line $n>2$ the largest dyads in the first and last quarters of the line n have the common value*;

$$\frac{(1+\sqrt{5})^{n+1}+(1-\sqrt{5})^{n+1}}{2^{n+1}}$$

and the ranks

$$R=\tfrac{1}{3}\{2^{n-1}+2+[1+(-1)^n]\} \text{ and } \tfrac{1}{3}\{5\cdot 2^{n-1}+4-[1+(-1)^n].$$

Any number of such theorems may be written out.

In general, if the values W_0 and W_1 are the first two dyads in a right zig-zag move where W_0 is in the n-th line with the rank R, there is on the $(n+k)$th line the dyad

$$(5)\quad W_k=W_1\left(\frac{(1+\sqrt{5})^k-(1-\sqrt{5})^k}{2^k\sqrt{5}}\right)+W_0\left(\frac{(1+\sqrt{5})^{k-1}-(1-\sqrt{5})^{k-1}}{2^{k-1}\sqrt{5}}\right)$$

whose rank is

$$R_k=\tfrac{1}{3}\{2^k+2+[1+(-1)^{k+1}]\}+2^k(R_0-1).$$

The same value is found l lines farther down with the rank

$$(6)\qquad R_{k+l}=\frac{2^l}{3}\{2^k(3R_0-2)-1+[1+(-1)^{k+1}]\}+1.$$

Let m be any line with $m>n$. Let $m-n$ be represented as the sum of two integers $k+l$ in all the $m-n$ ways. Then there exist $(m-n+1)$ elements on the m-th line whose values are obtained by putting the values of k in (5) and whose ranks are obtained by putting the values of l and k in (6).

It may be remarked that

$$W_k/W_{k-1}=[1,1,1,1,1,\cdots 1,q_{k+1},q_{k+2}\cdots q_{m-2}].$$

It is hopeless to try to account for every dyad in the table by the zig-zag moves, since there exist sequences (r_1, r_2) in the n-th line with all possible arrangements of quotients whose sum is $n+1$. We might define other moves to give other types of continued fraction expansions such as 1, 2, 1, 2, 1, 2, $\cdots$ or 1, 2, 3, 1, 2, 3, $\cdots$ and an enormous variety of other simple types. But we would still be at a loss to account for the infinitude of non periodic expansions occuring in the majority of cases.

Example: The numbers $r_1=2960276935825111$, $r_2=1679421121698828$ being selected at random, to determine whether the sequence (r_1, r_2) appears in the array and, if so, where. We find that

$$r_1/r_2=[1,1,3,4,1,2,7,1,7,7,1,5,10,7,1,3,1,2,1,10,1,3,1,4,5,9,3,1,6,16].$$

Therefore since r_1 and r_2 are prime to each other and the sum of the quotients is 124, the sequence appears in the 123rd line and on the left side of the middle since there is an even number of quotients. In fact the rank is

$$R = 2^{108} - 2^{102} + 2^{101} - 2^{98} + 2^{89} + \cdots + 2^{5} - 2^{2} + 2$$
$$= 321666940077382478983219549565470.$$

The number of terms on this line is

$$2^{123} + 1 = 10633823966279326983230456482242756609.$$

So the sequence (r_1, r_2) is about 1/33058 of the way across the line. The average value of the terms on this line is

$$(3/2)^{123} = 456273098478477544048,$$

which is 15 million times larger than the number r_1. The largest term on this line is

$$U_{125} = 59425114757512643212875125$$

whose least rank is

$$R = 35446079887597756610768188274142522 04.$$