PLMS 1884 vol 15

On the Function which denotes the difference between the number of $(4m+1)$*-divisors and the number of* $(4m+3)$*-divisors of a Number.* By J. W. L. Glaisher, M.A., F.R.S.

[*Read Feb.* 14*th*, 1884.]

1. The excess of the number of the divisors of a number n which have the form $4m+1$ over the number of divisors which have the form $4m+3$, is a quantity which occurs in researches connected with the Theory of Numbers, and also as coefficient in certain systems of q-series in Elliptic Functions.

If we denote this quantity by $E(n)$, so that

$$E(n) = \left.\begin{array}{l}\text{number of divisors of } n \text{ of the form } 4m+1 \\ - \quad \text{,,} \qquad\qquad \text{,,} \qquad\qquad \text{,,} \qquad 4m+3\end{array}\right\},$$

then it is obvious that, if $n = 2^p r$, where r is uneven (so that r is the greatest uneven divisor of n), we have $E(n) = E(r)$.

It is also easy to see that, if n be a number of the form $4m+3$, then $E(n) = 0$, for to every divisor of the form $4m+1$ there must correspond a conjugate divisor of the form $4m+3$. It can be shown also that $E(n)$ cannot be negative (see § 2).

2. In § 40 of the *Fundamenta Nova*, Jacobi states that, if $n = 2^p uv$, where u is an uneven number having all its prime factors of the form $4m+1$, and v an uneven number having all its prime factors of the form $4m+3$, then $E(n) = 0$ unless v is a square number, in which case

$$E(n) = \phi(u),$$

where $\phi(u)$ denotes the number of the divisors of u.

This important theorem may be proved in the following manner. The case of n uneven need alone be considered, as the uneven divisors of $2^p uv$ are evidently identical with those of uv.

As already remarked, the theorem is obviously true if n is of the form $4m+3$, for the product of two factors, both of which are of the form $4m+1$ or $4m+3$, is of the form $4m+1$; so that, in the case of a number of the form $4m+3$, there corresponds to every divisor of the form $4m+1$ a conjugate divisor of the form $4m+3$; and the number of divisors of the one form is therefore equal to the number of divisors of the other.

If n is of the form $4m+1$, suppose, first, that it $= a^\alpha b^\beta c^\gamma \dots$, where $a, b, c, \dots$ are all prime factors of the form $4m+3$. Then the divisors of n are the terms in the developed expression obtained by multiplying out the factors in the product

$$(1-a+a^2-\dots\pm a^\alpha)(1-b+b^2\dots\pm b^\beta)(1-c+c^2\dots\pm c^\gamma)\dots$$

The sign of each term in the developed expression is positive in the case of a divisor of the form $4m+1$, and negative in the case of a divisor of the form $4m+3$. Thus $E(n)$ is equal to the value of this product when $a, b, c, \ldots$ are all replaced by unity; whence it follows that $E(n) = 0$, unless $\alpha, \beta, \gamma, \ldots$ are all even, in which case $E(n) = 1$. Next, suppose that $n = a^\alpha b^\beta c^\gamma \ldots r^\rho s^\sigma t^\tau \ldots$ where $a, b, c, \ldots$ are, as before, prime factors of the form $4m+3$, and where $r, s, t, \ldots$ are prime factors of the form $4n+1$. Then, reasoning as above, we see that $E(n)$ is equal to the value of

$$(1-a+a^2 \ldots \pm a^\alpha)(1-b+b^2 \ldots \pm b^\beta) \ldots$$
$$\times (1+r+r^2 \ldots + r^\rho)(1+s+s^2 \ldots + s^\sigma) \ldots,$$

when $a, b, \ldots r, s, \ldots$ are all replaced by unity.

Denoting, as above, by $\phi(p)$ the number of divisors of p, we have therefore
$$E(n) = E(a^\alpha b^\beta c^\gamma \ldots) \times \phi(r^\rho s^\sigma t^\tau \ldots),$$
which, by means of the result found in the first case,
$$= \phi(r^\rho s^\sigma t^\tau \ldots) \quad \text{or} \quad 0,$$
according as $\alpha, \beta, \gamma, \ldots$ are all even, or are not all even.

It has thus been shown that $E(n) = 0$ unless all the prime factors of n, which are of the form $4m+3$, occur with even exponents; in which case, if $n = 2^p u v^2$, all the prime factors of u being of the form $4m+1$, and all the prime factors of v of the form $4m+3$, then $E(n) = \phi(u)$. We see also that $E(n)$ cannot ever be negative.

3. It follows from the preceding investigation that, if $n = n_1 n_2 n_3 \ldots$, where $n_1, n_2, n_3, \ldots$ are any relatively prime numbers, then
$$E(n) = E(n_1)\, E(n_2)\, E(n_3) \ldots$$
It is evident that, if p be a prime of the form $4m+1$, then
$$E(p^r) = r+1,$$
and that, if p be a prime of the form $4m+3$, then
$$E(p^r) = 1 \quad \text{or} \quad 0,$$
according as r is even or uneven.

Also
$$E(2^r) = 1.$$

By means of these formulæ we may write down at once the value of $E(n)$, when n has been resolved into its prime factors. For example, since
$$495000 = 2^3 \times 3^2 \times 5^4 \times 11,$$
we have
$$E(495000) = 1 \times 1 \times 5 \times 2 = 10.$$

4. The following table, which was calculated in the manner just explained, contains the values of $E(n)$ for all the values of n, up to $n = 1000$, for which $E(n)$ is not equal to zero.

TABLE OF THE VALUES OF $E(n)$ FROM $n = 1$ TO $n = 1000$.

n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$
1	1	136	2	293	2	464	2	640	2	820	4
2	1	137	2	296	2	466	2	641	2	821	2
4	1	144	1	298	2	468	2	648	1	829	2
5	2	145	4	305	4	477	2	650	6	832	2
8	1	146	2	306	2	481	4	653	2	833	2
9	1	148	2	313	2	482	2	656	2	841	3
10	2	149	2	314	2	484	1	657	2	842	2
13	2	153	2	317	2	485	4	661	2	845	6
16	1	157	2	320	2	488	2	666	2	848	2
17	2	160	2	324	1	490	2	673	2	850	6
18	1	162	1	325	6	493	4	674	2	853	2
20	2	164	2	328	2	500	4	676	3	857	2
25	3	169	3	333	2	505	4	677	2	865	4
26	2	170	4	337	2	509	2	680	4	866	2
29	2	173	2	338	3	512	1	685	4	872	2
32	1	178	2	340	4	514	2	689	4	873	2
34	2	180	2	346	2	520	4	692	2	877	2
36	1	181	2	349	2	521	2	697	4	881	2
37	2	185	4	353	2	522	2	698	2	882	1
40	2	193	2	356	2	529	1	701	2	884	4
41	2	194	2	360	2	530	4	706	2	890	4
45	2	196	1	361	1	533	4	709	2	898	2
49	1	197	2	362	2	538	2	712	2	900	3
50	3	200	3	365	4	541	2	720	2	901	4
52	2	202	2	369	2	544	2	722	1	904	2
53	2	205	4	370	4	545	4	724	2	905	4
58	2	208	2	373	2	548	2	725	6	909	2
61	2	212	2	377	4	549	2	729	1	914	2
64	1	218	2	386	2	554	2	730	4	916	2
65	4	221	4	388	2	557	2	733	2	925	6
68	2	225	3	389	2	562	2	738	2	928	2
72	1	226	2	392	1	565	4	740	4	929	2
73	2	229	2	394	2	569	2	745	4	932	2
74	2	232	2	397	2	576	1	746	2	936	2
80	2	233	2	400	3	577	2	754	4	937	2
81	1	234	2	401	2	578	3	757	2	941	2
82	2	241	2	404	2	580	4	761	2	949	4
85	4	242	1	405	2	584	2	765	4	953	2
89	2	244	2	409	2	585	4	769	2	954	2
90	2	245	2	410	4	586	2	772	2	961	1
97	2	250	4	416	2	592	2	773	2	962	4
98	1	256	1	421	2	593	2	776	2	964	2
100	3	257	2	424	2	596	2	778	2	965	4
101	2	260	4	425	6	601	2	784	1	968	1
104	2	261	2	433	2	605	2	785	4	970	4
106	2	265	4	436	2	610	4	788	2	976	2
109	2	269	2	441	1	612	2	793	4	977	2
113	2	272	2	442	4	613	2	794	2	980	2
116	2	274	2	445	4	617	2	797	2	981	2
117	2	277	2	449	2	625	5	800	3	985	4
121	1	281	2	450	3	626	2	801	2	986	4
122	2	288	1	452	2	628	2	808	2	997	2
125	4	289	3	457	2	629	4	809	2		
128	1	290	4	458	2	634	2	810	2		
130	4	292	2	461	2	637	2	818	2		

The number of arguments for which $E(n)$ is not zero, and the sum of the values of $E(n)$ for each hundred numbers, are as follows :—

	Number of arguments.	Sum of values.
0—99	42	76
100—199	36	79
200—299	35	82
300—399	31	74
400—499	32	80
500—599	32	80
600—699	31	81
700—799	30	75
800—899	28	73
900—999	30	79
Total 0—999	327	779

5. The following two formulæ serve to express $E(n)$ linearly in terms of the E's of numbers less than n.

I.

If n be any uneven number, then

$$E(n) - 2E(n-4) + 2E(n-16) - 2E(n-36) + \&c.$$
$$= 0 \text{ or } (-1)^{\frac{1}{2}(\sqrt{n}-1)} \times \sqrt{n},$$

according as n is not, or is, a square number.

Every term in this formula is zero if n is of the form $4m+3$, so that no generality is lost by restricting n to the form $4m+1$.

II.

If n be any number,

$$E(n) - E(n-1) - E(n-3) + E(n-6) + E(n-10) - \&c.$$
$$= 0 \text{ or } (-1)^n \times \tfrac{1}{4}\{(-1)^{\frac{1}{2}[\sqrt{(8n+1)}-1]} \times \sqrt{(8n+1)} - 1\},$$

according as n is not, or is, a triangular number.

The numbers 1, 3, 6, 10, ..., which occur in the second formula, are the triangular numbers, given by the formula $\frac{1}{2}r(r+1)$, and, if n be itself a triangular number, the last term is $E(n-n) = 0$. The signs of the terms after the first are negative and positive in pairs, the terms involving even triangular numbers in the argument having the positive sign, and those involving uneven triangular numbers having the negative sign.

Both formulæ are to be continued up to the term preceding the first term in which the argument becomes negative, *i.e.*, a term with negative argument is to be treated as zero. It may be noticed that, A2654

Excess divisor function

1, 1, 0, 1, 2, 0, 0, 1, 1, 2, 0, 0, 2,

0, 0, 1, 2, 1, 0, 2

0 0 0 0 3 2 0 0 2 0 0 1

omit ~~N~~26 : N1 is better

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