X

Milne-Thomson

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Putting $x = 1$ and dividing by t^n, we have

$$(1) \qquad \left[\frac{\log(1+t)}{t}\right]^n = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n+\nu+1)}(1)$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \frac{n}{n+\nu} B_\nu^{(n+\nu)},$$

using 6·3 (4).

In particular, for $n = 1$,

$$(2) \qquad \frac{\log(1+t)}{t} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{(\nu+1)!} B_\nu^{(\nu+1)}.$$

Again, integrating $(1+t)^{x-1}$ with respect to x from x to $x+1$, n times in succession, we have from 6·11 (9),

$$\frac{(1+t)^{x-1}\, t^n}{[\log(1+t)]^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(\nu-n+1)}(x).$$

Putting $x = 0$, we have

$$(3) \qquad \frac{t^n}{(1+t)[\log(1+t)]^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(\nu-n+1)},$$

and in particular, for $n = 1$,

$$(4) \qquad \frac{t}{(1+t)\log(1+t)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(\nu)},$$

which is the generating function of the numbers $B_\nu^{(\nu)}$.

Again putting $x = 1$, we have

$$(5) \qquad \left[\frac{t}{\log(1+t)}\right]^n = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(\nu-n+1)}(1),$$

which shews that (1) also holds when n is negative.

In particular, for $n = 1$,

$$(6) \qquad \frac{t}{\log(1+t)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(\nu)}(1),$$

which is the generating function of the numbers $B_\nu^{(\nu)}(1)$.

Using 6·3 (4), we have from (6),

$$(7) \qquad \frac{t}{\log(1+t)} = 1 + \tfrac{1}{2}t - \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} \frac{B_\nu^{(\nu-1)}}{\nu-1}.$$

Please enter numerators as seq

$\frac{2657}{2790}$

1, 5, 9, 251, ...

We give a list of ten of the numbers $B_\nu^{(\nu)}$:

$$B_1^{(1)} = -\tfrac{1}{2}, \qquad B_6^{(6)} = \tfrac{19087}{84},$$
$$B_2^{(2)} = \tfrac{5}{6}, \qquad B_7^{(7)} = -\tfrac{36799}{24},$$
$$B_3^{(3)} = -\tfrac{9}{4}, \qquad B_8^{(8)} = \tfrac{1070017}{90},$$
$$B_4^{(4)} = \tfrac{251}{30}, \qquad B_9^{(9)} = -\tfrac{2082753}{20},$$
$$B_5^{(5)} = -\tfrac{475}{12}, \qquad B_{10}^{(10)} = \tfrac{134211265}{132}.$$

6·5. Bernoulli's Polynomials of the First Order. We shall write $B_\nu(x)$ instead of $B_\nu^{(1)}(x)$, the order unity being understood. Thus from 6·1 (2), we have

(1) $$\frac{t\,e^{xt}}{e^t-1} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu(x),$$

as the generating function of the polynomials and

(2) $$\frac{t}{e^t-1} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu,$$

as the generating function of Bernoulli's numbers, B_ν, of the first order.

From 6·11, we have the following properties:

(3) $$B_\nu(x) \doteqdot (B+x)^\nu.$$

(4) $$(B+1)^\nu - B_\nu \doteqdot 0, \quad \nu = 2, 3, 4, \dots .$$

(5) $$\frac{d}{dx} B_\nu(x) = \nu\, B_{\nu-1}(x).$$

(6) $$\int_a^x B_\nu(t)\,dt = \frac{1}{\nu+1}[B_{\nu+1}(x) - B_{\nu+1}(a)].$$

(7) $$\Delta B_\nu(x) = \nu\, x^{\nu-1}.$$

(8) $$B_\nu(1-x) = (-1)^\nu B_\nu(x), \text{ from 6·2.}$$

The first seven polynomials are given in the following list:

$$B_0(x) = 1,$$
$$B_1(x) = x - \tfrac{1}{2},$$
$$B_2(x) = x^2 - x + \tfrac{1}{6},$$

$$B_3(x) = x(x-1)(x-\tfrac{1}{2}) = x^3 - \tfrac{3}{2}x^2 +$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30},$$
$$B_5(x) = x(x-1)(x-\tfrac{1}{2})(x^2-x-\tfrac{1}{3}) =$$
$$B_6(x) = x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{42}.$$

We have also for the values of the first se

B_0	B_1	B_2	B_3	B
1	$-\tfrac{1}{2}$	$\tfrac{1}{6}$	0	$-$

6·501. A Summation Problem. T

We have by 6·5 (6) and (7),

$$\int_s^{s+1} B_\nu(x)\,dx = \frac{1}{\nu+1}[B_{\nu+1}(s+1$$
$$= s^\nu.$$

Thus

$$\sum_{s=1}^{n} s^\nu = \int_0^{n+1} B_\nu(x)\,dx = \frac{1}{\nu+1}[B_{\nu+}$$

For example, if $\nu = 3$,

$$\sum_{s=1}^{n} s^3 = \tfrac{1}{4}[B_4(n+1) - B_4]$$
$$= \tfrac{1}{4}[(n+1)^4 - 2(n+1)^3 +$$
$$= [\tfrac{1}{2}n(n+1)]^2.$$

The method can clearly be applied if the s be a polynomial in s.

6·51. Bernoulli's Numbers of the F from 6·5 (2),

(1) $$\frac{t}{2} + \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu = \frac{t}{2} \cdot \frac{e^t+}{e^t-}$$

The function on the right is even, since the function unaltered. It follows t no odd powers of t, and hence

$$B_{2\mu+1} = 0, \quad \mu > 0,$$
$$B_1 = -\tfrac{1}{2}.$$