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ROTATABLE PARTITIONS

E. M. WRIGHT†

1. In what follows, all small latin letters denote non-negative rational integers or functions all of whose values are non-negative integers. By a *d-dimensional b-restricted partition* Y of n, where $d > 0$, $b > 0$, we understand a solution of the equation,

$$n = \sum_{x_1, x_2, \ldots, x_d \geqslant 0} y(x_1, x_2, \ldots, x_d), \tag{1}$$

where every $y \leqslant b$ and

$$y(x_1, x_2, \ldots, x_d) \geqslant y(x_1', x_2', \ldots, x_d'),$$

whenever $x_i \leqslant x_i'$ for all i. We may take $b = \infty$ when we shall call the partition *unrestricted*. The only other case of importance is that in which $b = 1$; such a partition we call a *unit partition* (more correctly, a partition into units). We write $q(d, b; n)$ for the number of d-dimensional b-restricted partitions of n. If we sum with respect to x_{d+1}, we see that

$$q(d+1, 1; n) = q(d, \infty; n). \tag{2}$$

2. We require the following lemma.

LEMMA. *Let p be a prime number, $d = p^t$ and T be a transformation such that T^d is the identity. If S is a finite set closed under T, then the number of members of S not invariant under T is divisible by p.*

Corresponding to every member s of S, we construct the set $\Sigma(s)$, *viz.*

$$s,\ Ts,\ T^2s,\ \ldots,\ T^{c-1}s,$$

where c is the least positive integer such that $T^c = s$. Then $c \leqslant d$; let us write $d = uc + r$, where $0 \leqslant r < c$. We have $s = T^d s = T^r (T^c)^u s = T^r s$ and so $r = 0$, by the definition of c. Hence $c | d$.

Clearly $\Sigma(s) \subset S$. Again, if $s' \subset \Sigma(s)$, then $\Sigma(s') = \Sigma(s)$; hence any two sets $\Sigma(s_1)$ and $\Sigma(s_2)$ either coincide or are disjoint. We have then all the members of S arranged in disjoint sets $\Sigma(s_1)$, $\Sigma(s_2)$, ...

If s is invariant under T, then $c = 1$. For all other s we have $c > 1$ and so, if $d = p^t$, $p | c$. Hence all the s not invariant under T are arranged in disjoint sets and the number of members in each of these sets is a multiple of p. Hence the total number of s not invariant under T is a multiple of p.

We now take the set S to be the set of d-dimensional b-restricted partitions Y of n, and T to be the transformation $Y' = TY$ such that

$$y'(x_1, x_2, \ldots, x_d) = y(x_2, x_3, \ldots, x_d, x_1).$$

The conditions of the lemma are clearly satisfied. Let us call any partition Y which is invariant under T, *i.e.*, one for which

$$y(x_1, x_2, \ldots, x_d) = y(x_2, x_3, \ldots, x_d, x_1)$$

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for all sets $x_1, x_2, \ldots, x_d$, a *rotatable* partition and let $q'(d, b; n)$ denote the number of d-dimensional b-restricted rotatable partitions of n. Then our lemma gives us at once the following theorem.

THEOREM. *If $d = p^t$, then*

$$q(d, b; n) \equiv q'(d, b, n) \pmod{p}.$$

We observe that nothing like (2) is true for the q'.

3. It has long been conjectured that the generating function for $q(d, \infty; n)$, *viz.*

$$Q_d = Q_d(X) = 1 + \sum_{n=1}^{\infty} q(d, \infty; n) X^n$$

is equal to

$$R_d = R_d(X) = \prod_{k=1}^{\infty} (1-X^k)^{-\binom{d+k-2}{k-1}} = 1 + r(d, n) X^n,$$

where $\binom{d+k-2}{k-1}$ takes the value 1 for $k = 1$ and otherwise denotes the usual binomial coefficient. For $d = 1$, this conjecture is true and its almost intuitive proof is due to Euler [4]. Macmahon [5] proved the conjecture true for $d = 2$, but neither his proof nor that of Chaundy [2] is at all simple. Attempts to produce a direct (*i.e.*, combinatorial) proof for $d = 2$ have not got very far. Cheema and Gordon [3] found a combinatorial proof that

$$(1-X)^{-1} \prod_{k=2}^{\infty} (1-X^k)^{-2} = 1 + \sum_{n=1}^{\infty} q(2, 2; n) X^n,$$

but it is not trivial and its further extension looks difficult. Recently [1] the conjecture has been disproved for $d = 3$, essentially by showing that $q(d, \infty; 6) \neq r(d, 6)$. The authors of [1] do not give the details of the calculation (which they describe as "more tedious"), but its nature is clear. They have also used a computer to calculate $q(d, 1; n)$ for $d \leq 8$ and a range of n.

The theorem of §2 enables me to give in §5 a very simple proof of the falsehood of the conjecture for $d = p$ or $d = p-1$. This theorem might also provide a test for any other conjecture. When d is a prime power, the theorem also provides a simple check of the accuracy of computed values of $q(d, 1; n)$ and $q(d, \infty; n)$ for fairly small values of n.

It is interesting to learn that $R_3(X)$ is not the generating function of $q(3, \infty; n)$ and it would be of some interest to have a more plausible conjecture as to what is the correct generating function. But the case of $q(2, \infty; n)$ shows that it is unlikely that any such conjecture would help us greatly to prove what is the generating function. In that case, our knowledge of the generating function has not enabled us to produce a simple proof or a direct, enumerative proof.

In [6] I showed that the generating function for the number of solutions of (1) for $d = 3$ subject to

$$y(0, 0, 0) \leq a, \quad y(1, 0, 0) \leq b, \quad y(0, 1, 0) \leq c \quad (b \leq a, c \leq a)$$

and

$$y(u, v, 0) = 0 \quad (u+v \geq 2)$$

is

$$\sum_{u=0}^{b} \sum_{v=0}^{c} \alpha(u, v) \xi_{a+u+v} \xi_{b-u} \xi_{c-v}$$

where

$$\xi_1 = \prod_{s=1}^{\infty} (1-X^s)^{-1}.$$

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Here $\alpha(u, v)$ is a polynomial in X whose term of lowest degree in X is of degree $\frac{1}{2}(u-v)^2+\frac{3}{2}(u+v)-1$. The $\alpha(u, v)$ can in theory be calculated from (increasingly elaborate) recurrence relations. In particular, $\alpha(u, v) = \alpha(v, u)$ and

$$\alpha(0, 0) = 1, \quad \alpha(0, 1) = \alpha(1, 0) = -X, \quad \alpha(0, v) = 0 \qquad (v \geqslant 2).$$

I am investigating the next step, in which we allow $y(1, 1, 0)$ to have positive values, but the work is not simple.

4. The values of $q'(d, 1; n)$ for the smaller values of d and n can be readily calculated by enumerating the rotatable unit partitions. This is particularly easy when d is a prime, the most interesting case from our point of view.

The values of $q'(d, \infty; n)$ can be deduced from those for $q'(d, 1; n')$ for $n' = 1, 2, \ldots, n$, since a rotatable unbounded partition of n can be dissected into suitable unit partitions of n', where $n = \Sigma n'$.

The values are given in the tables. If $p > 3$, where p is a prime, we have

$$q'(p, 1; 1) = q'(p, 1; p+1) = 1, \quad q'(p, 1; 2p+1) = \tfrac{1}{2}(p+1),$$

$$q'(p, 1; n) = 0 \qquad (2 \leqslant n \leqslant p,\ p+2 \leqslant n \leqslant 2p,\ 2p+2 \leqslant n \leqslant 3p).$$

$$q'(p, 1; 3p+1) = \tfrac{1}{8}(p^2-1)+1.$$

Also

$$q'(p, \infty; n) = 1 \qquad (1 \leqslant n \leqslant p).$$

$$q'(p, \infty; n) = 2 \qquad (p+1 \leqslant n \leqslant 2p).$$

$$q'(p, \infty; 2p+1) = \tfrac{1}{2}(p+5).$$

$$q'(p, \infty; n) = \tfrac{1}{2}(p+7) \qquad (2p+2 \leqslant n \leqslant 3p).$$

$$q'(p, \infty; 3p+1) = \tfrac{1}{8}(p+1)(p+3)+3.$$

Table of $q'(d, 1; n)$

$n \backslash d$	3	4	5	7	8	9	11
1	1	1	1	1	1	1	1
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	1	0	0	0	0	0	0
5	0	1	0	0	0	0	0
6	0	0	1	0	0	0	0
7	2	1	0	0	0	0	0
8	1	0	0	1	0	0	0
9	0	2	0	0	1	0	0
10	2	0	0	0	0	1	0
11	1	2	3	0	0	0	0
12	0	0	0	0	0	0	1
13	4	2	0	0	1	0	0
14	3	0	0	0	0	0	0
15	0	4	0	4	0	0	0
16	5	1	4	0	0	0	0

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Table of $q'(d, \infty; n)$

$n \backslash d$	3	4	5	7	8	9	11
1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1
4	2	1	1	1	1	1	1
5	2	2	1	1	1	1	1
6	2	2	2	1	1	1	1
7	4	3	2	1	1	1	1
8	6	3	2	2	1	1	1
9	6	5	2	2	2	1	1
10	8	6	2	2	2	2	1
11	11	8	5	2	2	2	1
12	13	9	6	2	2	2	2
13	17	11	6	2	3	2	2
14	24	14	6	2	3	2	2
15	28	19	6	6	3	2	2
16	36	22	10	7	3	2	2
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5. Here we compare $r(d. n)$ and $q(d, \infty; n) = q(d+1. 1; n)$ for two classes of d. We take p an odd prime. All congruences are to modulus p. We write $S_1(X) \sim S_2(X)$ to denote that

$$S_1(X) - S_2(X) = \sum_{n=0}^{\infty} a_n X^n \equiv \sum_{n=2p+1}^{\infty} a_n X^n.$$

First let us take $d = p$. From §4, we have

$$q(p. \infty; n) \equiv q'(p, \infty; n) = \begin{cases} 1 & (1 \leqslant n \leqslant p). \\ 2 & (p+1 \leqslant n \leqslant 2p). \end{cases}$$

Hence

$$Q_p = 1 + \sum_{n=1}^{\infty} q(p, \infty; n) \sim (1-X)^{-1}(1+X^{p+1}).$$

Again

$$\binom{p+k-2}{k-1} \equiv \begin{cases} 0 & (k \not\equiv 1) \\ 1 & (k \equiv 1) \end{cases}$$

and

$$(1-X^t)^p \equiv 1 - X^{pt}.$$

Hence

$$\begin{aligned} R_p &= \prod_{k=1}^{\infty} (1-X^k)^{-\binom{p+k-2}{k-1}} \\ &\sim (1-X)^{-1}(1-X^2)^{-p}(1-X^{p+1})^{-1} \\ &\sim (1-X)^{-1}(1-X^{2p})^{-1}(1-X^{p+1})^{-1} \\ &\sim (1-X)^{-1}(1+X^{2p})(1+X^{p+1}) \sim Q_p + X^{2p} \end{aligned}$$

and so

$$r(p, 2p) \not\equiv q(p, \infty; 2p).$$

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Next let $d = p-1$. We remark that $q(d, \infty; n) = q(p, 1; n)$ and that

$$q(p, 1; n) \equiv q'(p, 1; n) = \begin{cases} 1 & (n = 1,\ p+1), \\ 0 & (2 \leqslant n \leqslant p,\ p+2 \leqslant n \leqslant 2p), \end{cases}$$

so that

$$Q_d = Q_{p-1} \sim 1 + X + X^{p+1}.$$

We have

$$R_d = R_{p-1} = \prod_{k=1}^{\infty} (1-X^k)^{-\binom{p+k-3}{k-1}}.$$

It is easily seen that

$$\binom{p+k-3}{k-1} \equiv \begin{cases} 1 & (k \equiv 1), \\ -1 & (k \equiv 2), \\ 0 & (k \not\equiv 1, 2). \end{cases}$$

We have then

$$\begin{aligned} R_{p-1}(X) &\sim (1-X)^{-1}(1-X^2)^{-p+1}(1-X^{p+1})^{-1}(1-X^{p+2})^{-np+1} \\ &\sim (1+X)(1+X^{2p})(1+X^{p+1})(1-X^{p+2}) \\ &\sim (1+X)(1+X^{p+1}-X^{p+2}+X^{2p}) \\ &\sim 1+X+X^{p+1}-X^{p+3}+X^{2p} \end{aligned}$$

and so

$$r(p-1, n) \not\equiv q(p-1, \infty; n) \qquad (n = p+3, 2p).$$

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University of Aberdeen.