Please enter 2

Bound! RRSE 21 (1896)

of 9. Similarly, when we come to the 7th order the square of a 15-termed expression turns up, and this we must now count as containing, not 15^2 terms, but $15^2 - C_{15,2}$. The numbers of unlike terms are thus—

for the 5th order $1+4\cdot1+6\cdot3+4\cdot3^2+(3^2\cdot5-3)$ $= 101$

,, 6th ,, $1+5\cdot1+10\cdot3+10\cdot3^2+5(3^2\cdot5-3)+(15^2-5\cdot3)$ $= 546$

,, 7th ,, $1+6\cdot1+15\cdot3+20\cdot3^2+15(3^2\cdot5-3)+6(15^2-5\cdot3)+(15^2\cdot7-105-6\cdot5\cdot3)= 3502$

. . . .

12. As a bordered skew determinant of the nth order can be expressed as a sum of $n-1$ such determinants of the $(n-1)$th order, together with a skew determinant of the latter order, and as the number of unlike terms in a skew determinant is known,* it is clear that we have a ready means of verifying the figures just obtained.

Please enter 2771

Denoting the number of unlike terms in a skew determinant of the nth order by S_n we have *

1, 2, $S_3 = 4$, $S_4 = 13$, $S_5 = 41$, $S_6 = 226$, $S_7 = 1072$, . . .

type 1 – see over

so that, if $(BS)_n$ be used in a similar way in connection with bordered skew determinants, we have

2
6

$$
\begin{aligned}
(BS)_4 &= S_3 + 3(BS)_3 = 4 + 3\cdot6 = 22\,,\\
(BS)_5 &= S_4 + 4(BS)_4 = 13 + 4\cdot22 = 101\,,\\
(BS)_6 &= S_5 + 5(BS)_5 = 41 + 5\cdot101 = 546\,,\\
(BS)_7 &= S_6 + 6(BS)_6 = 226 + 6\cdot546 = 3502\,,
\end{aligned}
$$

. . . .

2772 Please enter

exactly as in the preceding section.

It should also be noted that as a consequence of this

$$(BS)_8 = S_7 + 7S_6 + 7\cdot6S_5 + \ldots + 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\,.$$

13. The most important special case of the theorem is got by putting $k_1, k_2, k_3, \ldots = h_1, h_2, h_3, \ldots$, for then the bordered determinant becomes itself a skew determinant. A notable change takes place also in the development, the two Pfaffians in every product, even where in the general theorem they are of different

* Cunningham, Allan, "An Investigation of the Number of Constituent Elements, and Minors of a Determinant," *Journ. of Science*, iv. (1874) pp. 212–228.

orders, becoming equal. For example, when the order is the 5th the product

$$
\begin{vmatrix} a_1 & a_2 & a_3 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix} \cdot \begin{vmatrix} m_1 & h_1 & h_2 & h_3 & h_4 \\ & h_1 & h_2 & h_3 & h_4 \\ & & a_1 & a_2 & a_3 \\ & & & \beta_1 & \beta_2 \\ & & & & \gamma_1 \end{vmatrix},
$$

on account of the identity of the first two frame-lines of the second Pfaffian, becomes

$$
m_1 \begin{vmatrix} a_1 & a_2 & a_2 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix}^2 ;
$$

and the full statement of the identity is

$$
\begin{vmatrix} m_1 & h_1 & h_2 & h_3 & h_4 \\ -h_1 & m_2 & a_1 & a_2 & a_3 \\ -h_2 & -a_1 & m_3 & \beta_1 & \beta_2 \\ -h_3 & -a_2 & -\beta_1 & m_4 & \gamma_1 \\ -h_4 & -a_3 & -\beta_2 & -\gamma_1 & m_5 \end{vmatrix} = m_1 \begin{vmatrix} a_1 & a_2 & a_3 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix}^2 + \sum m_2 \begin{vmatrix} h_2 & h_3 & h_4 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix}^2
$$
$$
+ m_1 \sum m_2 m_3 \cdot \gamma_1^2 + \sum m_2 m_3 m_4 \cdot h_4^2 + m_1 m_2 m_3 m_4 m_5\,,
$$

or

$$
= \sum m_1 \begin{vmatrix} a_1 & a_2 & a_3 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix}^2 + \sum m_1 m_2 m_3 \cdot \gamma_1^2 + m_1 m_2 m_3 m_4 m_5\,,
$$

if we bear in mind the wider sphere to which Σ now refers.

This special case is, as implied in § 1, that first dealt with by Cayley, being the subject of his second paper on Skew Determinants.*

14. A consideration of this expansion of a skew determinant suffices to determine the number of unlike terms in such a determinant. For example, for the 5th order we clearly have the number of unlike terms

$$
\begin{aligned}
&= C_{5,1}(3^2 - C_{3,1}) + C_{5,3} + C_{5,5}\,,\\
&= 30 \qquad\qquad\quad + 10 \;\; + 1\,,\\
&= 41\,.
\end{aligned}
$$

* Cayley, A., "Sur les déterminants gauches," *Crelle's Journ.*, xxxviii. pp. 93–96; or *Collected Math. Papers*, i. pp. 410–413.

PRSE 21 (1896)

And, generally, the number of unlike terms in a skew determinant of the nth order is *

2771

$$1 + C_{n,2} + C_{n,4} \cdot \frac{3(1+3)}{2} + C_{n,6}\frac{3\cdot 5(1+3\cdot 5)}{2}$$
$$+ C_{n,8}\frac{3\cdot 5\cdot 7(1+3\cdot 5\cdot 7)}{2} + \cdots\cdot$$

15. If we specialise further, by making $m_1 = m_2 = m_3 = m_4 = m_5 = 1$, the expansion takes the form of a sum of squares.

If, on the other hand, we make each of the m's equal to 0, the expansion in the case of odd orders entirely disappears; and, in the case of even orders, reduces to one term. This is the "very special case" referred to in §2, and used in §7 in proving the general theorem. It is the subject of Cayley's fourth paper on Skew Determinants.†

It would be well, however, to combine these two special cases and others in one statement, viz., that, *if the values of the diagonal elements of a skew determinant be confined to* 0 *or* 1, *the determinant is expressible as a sum of squares of Pfaffians.* For example, in the case of the 5th order, if $m_1 = 0$, and $m_2 = m_3 = m_4 = m_5 = 1$, we have

$$\left|\begin{array}{ccccc} \cdot & h_1 & h_2 & h_3 & h_4 \\ -h_1 & 1 & a_1 & a_2 & a_3 \\ -h_2 & -a_1 & 1 & \beta_1 & \beta_2 \\ -h_3 & -a_2 & -\beta_1 & 1 & \gamma_1 \\ -h_4 & -a_3 & -\beta_2 & -\gamma_1 & 1 \end{array}\right| = \left|\begin{array}{ccc} h_2 & h_3 & h_4 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{array}\right|^2 + \left|\begin{array}{ccc} h_1 & h_3 & h_4 \\ & a_2 & a_3 \\ & & \gamma_1 \end{array}\right|^2 + \left|\begin{array}{ccc} h_1 & h_2 & h_4 \\ & a_1 & a_3 \\ & & \beta_2 \end{array}\right|^2 + \left|\begin{array}{ccc} h_1 & h_2 & h_3 \\ & a_1 & a_2 \\ & & \beta_1 \end{array}\right|^2$$
$$+ h_4{}^2 + a_3{}^2 + \beta_2{}^2 + h_1{}^2 .$$

16. Before proceeding to our next theorem, the nature and notation of the elements of product-determinants require to be recalled to mind. The elements of the determinant which is the product of $|\,a_1b_2c_3\,|$ and $|\,a_1\beta_2\gamma_3\,|$ are of the form,

$$a_1a_1 + a_2\beta_1 + a_3\gamma_1 ,$$
$$\text{or } (a_1a_2a_3)(a_1\beta_1\gamma_1),$$

* *Cf.* Cunningham's paper, above referred to, p. 225.

† Cayley, A., "Théorème sur les déterminants gauches," *Crelle's Journ.*, lv. pp. 277, 278; or *Collected Math. Papers*, iv. pp. 72, 73.

or, for compactness' sake,

$$\frac{a_1a_2a_3}{a_1\beta_1\gamma_1} .$$

In the next place, the product $|\,a_1b_2c_3\,|\cdot|\,a_1\beta_2\gamma_3\,|\cdot|\,x_1y_2z_3\,|$ has elements of the form

$$x_1\frac{a_1a_2a_3}{a_1\beta_1\gamma_1} + y_1\frac{a_1a_2a_3}{a_2\beta_2\gamma_2} + z_1\frac{a_1a_2a_3}{a_3\beta_3\gamma_3},$$

$$\text{or}\qquad \begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ \hline a_1 & \beta_1 & \gamma_1 & x_1 \\ a_2 & \beta_2 & \gamma_2 & y_1 \\ a_3 & \beta_3 & \gamma_3 & z_1 \end{array} ,$$

the final expansion of the latter expression being easily obtainable on remembering that there is a term corresponding to every element in the square array, and that this term is the product of that element, β_2 say, and the two outside elements a_2 and z_1 standing in the same column or row with it.

Further than these two cases it is not necessary at present to go. The succeeding expressions of the same kind will be found in a paper published in *Transactions* of the Society, where also an exposition of their properties is given.*

17. Now, the first three instances of our new theorem are—

$$m\left|\begin{array}{cc} 1 & a_1 \\ -a_1 & 1 \end{array}\right| + \begin{array}{cc|c} h_1 & h_2 & \\ \hline 1 & a_1 & k_1 \\ -a_1 & 1 & k_2 \end{array} = m + \frac{h_1\ h_2}{k_1\ k_2} + a_1\left|\begin{array}{ccc} m & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{array}\right| ,$$

$$\left|\begin{array}{ccc} 1 & a_1 & a_2 \\ -a_1 & 1 & \beta_1 \\ -a_2 & -\beta_1 & 1 \end{array}\right| + \begin{array}{ccc|c} h_1 & h_2 & h_3 & \\ \hline 1 & a_1 & a_2 & k_1 \\ -a_1 & 1 & \beta_1 & k_2 \\ -a_2 & -\beta_1 & 1 & k_3 \end{array} = m + \frac{h_1h_2h_3}{k_1k_2k_3} + \Sigma a_1\left|\begin{array}{ccc} m & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{array}\right| ,$$

$$\left|\begin{array}{cccc} 1 & a_1 & a_2 & a_3 \\ -a_1 & 1 & \beta_1 & \beta_2 \\ -a_2 & -\beta_1 & 1 & \gamma_1 \\ -a_3 & -\beta_2 & -\gamma_1 & 1 \end{array}\right| + \begin{array}{cccc|c} h_1 & h_2 & h_3 & h_4 & \\ \hline 1 & a_1 & a_2 & a_3 & k_1 \\ -a_1 & 1 & \beta_1 & \beta_2 & k_2 \\ -a_2 & -\beta_1 & 1 & \gamma_1 & k_3 \\ -a_3 & -\beta_2 & -\gamma_1 & 1 & k_4 \end{array} = m + \frac{h_1h_2h_3h_4}{k_1k_2k_3k_4}\Sigma + a_1\left|\begin{array}{ccc} m & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{array}\right|$$
$$+ m\left|\begin{array}{ccc} a_1 & a_2 & a_3 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{array}\right|^2 .$$

* "On Bipartite Functions," *Trans. R.S.E.*, xxxii. pp. 461–481.